Exercise answers: EOS 250

February 27, 2024

2. (Notes on volume integrals) The tetrahedron is bounded by $0 < x < 1$, $0 < y < x$ and $0 < z < 1 - x$ (draw the tetrahedron to see this, or go through the lecture note procedure for calculating formulas for the planes that define the faces of the tetrahedron). So

$$
\begin{aligned}
\int_V \rho \, \mathrm{d}V &= \int_0^1 \int_0^x \int_0^{1-x} 1 + xy + z \, \mathrm{d}z \, \mathrm{d}y \, \mathrm{d}x \\
&= \int_0^1 \int_0^x (1-x)(1+xy) + (1-x)^2/2 \, \mathrm{d}y \, \mathrm{d}x \\
&= \int_0^1 (1-x)x + (1-x)x^3/2 + x(1-x)^2/2 \, \mathrm{d}x \\
&= \left[x^2/2 - x^3/3 + x^4/8 - x^5/10 - x(1-x)^3/6 - (1-x)^4/24\right]_0^1 \\
&= \frac{1}{2} - \frac{1}{3} + \frac{1}{8} - \frac{1}{10} + \frac{1}{24} \\
&\frac{7}{30}
\end{aligned}
$$

3. (Notes on volume integrals) From $\int \rho \, \mathrm{d}V = \sum \rho \Delta V$, we get $\int 1 \, \mathrm{d}V = \sum \Delta V =$ the volume V (you're chopping the volume V, chopping it into little rectangular prisms, calculating the volume of each prism, and adding). For the tetrahedron in the previous example, base area $= 1/2$, height $= 1$, so volume $= 1/6$. Computing the

integral,

$$
\begin{aligned}
\int_V 1\,\mathrm{d}V &= \int_0^1 \int_0^x \int_0^{1-x} 1\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^1 \int_0^x 1 - x\,\mathrm{d}y\,\mathrm{d}x \\
&= \int_0^1 x(1-x)\,\mathrm{d}x \\
&= \left[x^2/2 - x^3/3\right]_0^1 \\
&= \frac{1}{6}
\end{aligned}
$$

as required.
Also $\int_V \rho_0\,\mathrm{d}V = \rho_0 \int_V 1\,\mathrm{d}V = \rho_0 V$. This is the high school formula for mass — which is therefore just a special case of the non-high school formula.

6. (Notes on surface integrals) We have $\mathbf{q} = y\mathbf{i} - x\mathbf{j} + z\mathbf{k}$ and the surface given by $z = h(x,y) = \cos(x)\cos(y)$, its projection bounded by $0 < x < \pi/2$, $0, y < \pi/2$. We therefore get

$$
\frac{\partial h}{\partial x} = -\sin(x)\cos(y), \qquad \frac{\partial h}{\partial y} = -\cos(x)\sin(y).
$$

while

$$
q_x(x,y,h(x,y)) = y, \qquad q_y(x,y,h(x,y)) = -x, \qquad q_z(x,y,h(x,y)) = z = \cos(x)\cos(y)
$$

For reference, note that by integration by parts

$$
\begin{aligned}
\int u\cos(u)\,\mathrm{d}u =& u\sin(u) - \int \sin(u)\,\mathrm{d}u \\
=& u\sin(u) + \cos(u)
\end{aligned}
$$

Hence the surface integral is

$$
\begin{aligned}
\int_S \mathbf{q}\cdot\hat{\mathbf{n}}\,\mathrm{d}S =& \int_0^{\pi/2}\int_0^{\pi/2} \cos(x)\cos(y) + y\sin(x)\cos(y) - x\cos(x)\sin(y)\,\mathrm{d}y\,\mathrm{d}x \\
=& \int_0^{\pi/2} [\cos(x)\sin(y) + \sin(x)y\sin(y) + \sin(x)\cos(y) + x\cos(x)\cos(y)]_{y=0}^{y=\pi/2}\,\mathrm{d}x \\
=& \int_0^{\pi/2} \cos(x) + \frac{\pi}{2}\sin(x) - \sin(x) - x\cos(x)\,\mathrm{d}x \\
=& \left[\sin(x) - \left(\frac{\pi}{2} - 1\right)\cos(x) - x\sin(x) - \cos(x)\right]_{x=0}^{x=\pi/2} \\
=& 1 + \left(\frac{\pi}{2} - 1\right) - \frac{\pi}{2} + 1 \\
=& 1
\end{aligned}
$$

(Note that this could have been done more easily, by symmetry under exchanging x and y you can tell that

$$\int_0^{\pi/2}\int_0^{\pi/2} \sin(x) y \cos(y)\,\mathrm{d}y\,\mathrm{d}x = \int_0^{\pi/2}\int_0^{\pi/2} x\cos(x)\sin(y)\,\mathrm{d}y\,\mathrm{d}x$$

and therefore the integral is more simply

$$\int_S \mathbf{q}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \int_0^{\pi/2}\int_0^{\pi/2} \cos(x)\cos(y)\,\mathrm{d}y\,\mathrm{d}x = 1.$$