Differential Equations

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Overview

These notes cover the following:

- Introduction to formulating physical problems as differential equations
- The concept of constitutive relations
- Separation of variables and integrating factors
- Initial conditions
- Visualizing solutions
- Deriving differential equations from scratch: a continuum physics example

Mechanics

We have previously seen how motion can be described using the tools of calculus: velocity can be defined as the derivative of position with respect to time, and position can be calculated from velocity by integrating. There is, however, no physics in these statements: we have simply used differentiation to extend the *definition* of velocity to the case of non-steady motion, and used the fundamental theorem of calculus to reverse that differentiation to find displacement from velocity. We have not said anything about why an object has a particular velocity, or how its velocity changes over time.

Another way of viewing this is that we could measure the motion of an object as its position $x(t)$ as a function of time and compute its velocity from the position data, or possible measure the velocity $v(t)$ and compute position from velocity, but this by itself does not tell us anything about what makes the object behave the way it does. A *model* for the motion of the object is an attempt to describe mathematically the physics driving the motion; the model is therefore a set of equations. It has two purposes. First, it allows us to *test* whether our understanding of the physics involved is correct. This is done by solving the equations (i.e., by solving the model) and comparing its predictions against data. If the model fails to predict the data, then something in the model is wrong: it may be missing something important, or it may not describe correctly some aspect of the physics that is included in the model.

To construct a model for motion, we need the laws of mechanics. You will know these as Newton's laws, the second of which you probably know as force being equal to mass times acceleration. You may also know acceleration as

$$\text{acceleration} = \frac{\text{change in velocity}}{\text{time elapsed,}}$$

which is analogous to velocity being displacement (or change in position) over time elapsed. Once more, we look only at an object that is forced to move along a straight line, so we do not have to deal with acceleration as a vector. If the initial point in time is t_0 and we wait until t_1 to measure the change in velocity, we get

$$a = \frac{v(t_1) - v(t_0)}{t_1 - t_0},$$

which is analogous to how we initially we tried to define velocity through

$$v = \frac{x(t_1) - x(t_0)}{t_1 - t_0}.$$

We saw how this definition of velocity is problematic when velocity is not constant, and how we can get around that difficulty if we require the time elapsed $t_1 - t_0$ to be small. It makes sense to do the same to the definition of acceleration, and to look at change in velocity over a short time interval $\delta t = t_1 - t_0$. We therefore take the limit

$$a(t_0) = \lim_{\delta t \to 0} \frac{v(t_0 + \delta t) - v(t_0)}{\delta t},$$

so acceleration a is the derivative of $v(t)$.

Newton's second law therefore says

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = F. \tag{1}$$

This is however not much use unless we know what F is. The other two Newton's laws are not much help to decide this. Newton's first law is really a special case of the second: if there is no force, velocity does not change, which is the same as saying that $\mathrm{d}v/\,\mathrm{d}t = 0$ if $F = 0$. Newton's third law meanwhile says that forces between objects are equal and opposite, but it does not say anything about the size or direction of these forces.

This is because the force acting on an object, or the forces acting between two interacting objects, depend on the particulars of the situation we are looking at. Two objects connected by a spring will experience different forces than two electrically charged objects that attract or repel each other, and these forces will differ from the forces experienced by an object falling through the atmosphere subject to aerodynmamic drag.

Newton's second law (1) must therefore be supplemented by some additional statement, or equation, that defines F. In the area of physics that we will mostly focus on later in the course (namely *continuum physics*), an additional statement of this kind is often called a *constitutive relation*.

Newton's second law always holds, at least if we ignore more exotic situations in which quantum mechanics or relativity become relevant. By contrast, the constitutive relation for F is problem-specific. In some cases, for instance when electric charges repel or attract each other, the form of F is given by physical laws that hold just as generally as Newton's second law (in this case, Maxwell's equations for electrical and magnetic fields). More often, as with the case of the force acting between objects connected by springs, or with the drag on an object falling through the atmosphere, the constitutive relation for F will be empirical. For instance, Hooke's law states that the force generated by extending a spring is proportional to the extension of the spring. This is 'true' in the sense that it is confirmed by experiment, but only up to a point: stretch the spring far enough so it deforms irreversibly, and Hooke's law no longer holds. Hooke's law also involves a 'spring constant', which must generally be found by experiment. A constitutive relation that does not hold universally and is based on experiment is called an *empirical* relation, and has a lesser status as a 'law' than for instance Newton's law.

Note 1 *Newton's third law is actually best understood if we combine it with the second law and restate it as saying that 'forces conserve momentum'. Take two objects, A and B, that exert forces on each other. Let the force exerted by B on A be F_{AB}, and F_{BA} be the force exerted by A on B. Newton's third law states that*

$$F_{BA} = -F_{AB}.$$

Suppose that the two objects are a closed system*, meaning that they experience no forces from elsewhere. Then Newton's second law applied to A and B in turn says*

$$m_A \frac{\mathrm{d}v_A}{\mathrm{d}t} = F_{AB},$$
$$m_B \frac{\mathrm{d}v_B}{\mathrm{d}t} = F_{AB},$$

where m_A and m_B are the masses of the two objects, and v_A and v_B their velocities. Adding the two equations gives

$$m_A \frac{\mathrm{d}v_A}{\mathrm{d}t} + m_B \frac{\mathrm{d}v_B}{\mathrm{d}t} = F_{AB} + F_{BA} = 0$$

because $F_{BA} = -F_{AB}$. But the momentum of of the two objects is $m_A v_A$ and $m_B v_B$, respectively. The total momentum p in the system is the sum of momenta,

$$p = m_A v_A + m_B v_B.$$

With constant masses (mass is also conserved) we have

$$\frac{\mathrm{d}p}{\mathrm{d}t} = m_A \frac{\mathrm{d}v_A}{\mathrm{d}t} + m_B \frac{\mathrm{d}v_B}{\mathrm{d}t} = 0,$$

so p is constant.

Newton's second and third laws together therefore state that momentum is conserved. Conservation laws are fundamental principles in physics that must not be violated, and any successful theory must satisfy them. Conservation laws alone do not allow the evolution of a system to predicted, however: as we have seen, Newton's laws do not tell us about the force F other than that it has an equal and opposite counterpart. Constitutive relations are therefore of equal importance to conservation laws in allowing predictions to be made. This will be a recurring theme in this course.

Let us return to the case of a body falling under the action of gravity and aerodynamic drag. If x is measured vertically upwards from the ground, then v is also positive when the body moves upwards. F is therefore positive when pointing up. We expect the force to be the sum of gravity and drag. The gravitational force is $-mg$. Drag should act in the opposite direction to the velocity (so it slows the object down) but we know little else about it. For now, assume that the magnitidue of drag is proportional to speed. In that case we can write

$$F = -mg - Cv, \tag{2}$$

C being a drag coefficient.

Note 2 *There is an important lesson here: models are always simplifications of how nature really works. In the present case, the simplification of drag being proportional to velocity v is probably the most questionable. In reality, drag arises from complicated interactions between the falling object and the flow of air around it. Ultimately, there is no end to refinements that could be made to any single model — it is impossible in practice to capture every detail of the physics that drives some process, and trying to achieve that impossible goal will only lead to an intractable and therefore useless model. The level of sophistication in a good model*[1] *depends on its purpose (how accurately do you need an answer) and the ability to capture all the physics necessary to achieve that purpose (do we know enough about the physics involved to achieve the accuracy we want). Often, a relatively simple model like (2) can provide the most insight.*

[1] A good quote to remember — due to the statistician George Box – is that 'all models are wrong, but some are useful'

If we substitute this form of F into Newton's second law (1), we get

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = -mg - Cv. \tag{3}$$

This is a *differential equation*: rather than telling us how big the velocity v is, the equation tells us something about the rate of change $\mathrm{d}v/\,\mathrm{d}t$.

It is worth defining some notation here:

1. t is the *independent* variable, while v is the *dependent* variable, the 'unknown'. We want to find a solution that gives velocity as a function of time $v(t)$. In other words, we want to be able to predict the velocity at some point t in the future.

2. m, g and C are neither dependent nor independent variables, but *parameters*. They are values of fixed physical quantities that may differ between different settings, but not during the motion of a single object. That is, if we repeat the experiment of throwing an object on a different planet, g might be different. Similarly, if we repeat experiment of throwing an object with an object of a different shape or mass, C or m might be different. They will not vary while a given object falls in a given location. We denote parameters like m, g and C by letters rather than numbers because it makes our solution more generally applicable, and we can see what effect changing the parameters has on the solution.

To solve (3), we cannot simply integrate left- and right-hand sides as they are, because v on the right-hand side is unknown. Instead, we use a technique known as separation of variables. This requires the following steps: first we need to move everything involving the dependent variable on the right to the left by dividing (not adding or subtracting). We do this by dividing both sides by $mg + Cv$:

$$\frac{m}{mg + Cv}\frac{\mathrm{d}v}{\mathrm{d}t} = -1 \tag{4}$$

Now we can integrate both sides with respect to t:

$$\int \frac{m}{mg + Cv}\frac{\mathrm{d}v}{\mathrm{d}t}\,\mathrm{d}t = \int -1\,\mathrm{d}t$$

Having removed everything containing v from the right, the integral on the right is straightforward to do. On the left, the integrand is the product of a function of v with $\mathrm{d}v/rdt$, and we can use the change-of-variable formula

$$\int F(v)\frac{\mathrm{d}v}{\mathrm{d}t}\,\mathrm{d}t = \int F(v)\,\mathrm{d}v.$$

In other words,

$$\int \frac{m}{mg + Cv}\frac{\mathrm{d}v}{\mathrm{d}t}\,\mathrm{d}t = \int \frac{m}{mg + Cv}\,\mathrm{d}v$$

and so

$$\int \frac{m}{mg + Cv}\,\mathrm{d}v = \int -1\,\mathrm{d}t$$

Evaluating the integrals gives

$$\frac{m}{C}\log\left(\frac{mg}{C} + v\right) = -t + K,$$

where K is a constant of integration. Note that we only need one constant of integration; if we had one on each side, we could move the one on the left of the equality to the right, and K would simply be the difference of the two constants of integration.[2]

Now all we have to do is solve for v: Rearrange

$$\log\left(\frac{mg}{C} + v\right) = -\frac{Ct}{m} + \frac{CK}{m}.$$

Exponentiate

$$\frac{mg}{C} + v = \exp\left(-\frac{Ct}{m} + \frac{CK}{m}\right) = \exp\left(\frac{CK}{m}\right)\exp\left(-\frac{Ct}{m}\right),$$

and rearrange again,

$$v(t) = \exp\left(\frac{CK}{m}\right)\exp\left(-\frac{Ct}{m}\right) - \frac{mg}{C}. \tag{5}$$

This is a *general solution* of the differential equation (3). (5) satisfies (3) for any choice of the constant of integration K. Nothing in that equation will constrain K further. This is however a problem. We do not have a unique solution unless we know K: given a time t and values for the parameters m, g and C, we cannot say what value velocity v takes unless we also know K.

In order to fix K, we need additional physical information, not contained in Newton's second law or the formula for F in equation (2). In the case of an object falling in the atmosphere, the most obvious piece of physical information that could help us fix would be the velocity the object starts out with at $t = 0$. This is known as an *initial condition.*

[2]Explicitly, if we had

$$\frac{m}{C}\log\left(\frac{mg}{C} + v\right) + K_1 = -t + K_2,$$

we could rearrange to given

$$\frac{m}{C}\log\left(\frac{mg}{C} + v\right) = -t + K_2 - K_1,$$

and we would simply get $K = K_2 - K_1$.

Suppose we know that $v = v_0$ at $t = 0$, where v_0 is a fixed value (another *parameter* describing the problem. The way to determine the constant of integration is to substitute $v(0) = v_0$ into the general equation (5), where $v(0)$ of course means $v(t)$ at $t = 0$. This gives

$$v_0 = \exp\left(\frac{CK}{m}\right)\exp(0) - \frac{mg}{C} = \exp\left(\frac{CK}{m}\right) - \frac{mg}{C}$$

This means

$$\exp\left(\frac{CK}{m}\right) = v_0 + \frac{mg}{C}.$$

We could go on to solve for K, but this is not necessary as K only appears in the combination $\exp(CK/m)$ in (5). Substituting gives

$$v(t) = \left(v_0 + \frac{mg}{C}\right)\exp\left(-\frac{Ct}{m}\right) - \frac{mg}{C}. \tag{6}$$

This finally is our solution.

Note 3 *It is often worth checking you have not made a mistake. To do this, substitute v on both sides of* (3) *and check the two sides equal each other. Also substitute $t = 0$ and check that $v = v_0$ when you do so.*

Remember that we essentially invented the drag term in the force formula $F = -mg - Cv$ from little more than intuition, and we certainly have no way of knowing what C should be for a given object. There is therefore no guarantee that the *actual* velocity of a falling object will satisfy the formula (6). This is again where direct measurement (the source of those *empirical* constitutive relations) becomes important: we could measure the actual motion of a falling object and compare it with the predictions of (6). An obvious first way to compare data and prediction would be to plot both on the same graph.

Visualizing a solution

Deriving a formula for a solution is all very well, but it does not help you *understand* the solution. To do that, you almost certainly need to visualize the solution — that is, to sketch the graph of v against t.

There are a number of steps you need to take in order to sketch a solution, especially if you do not have numerical values of the parameters to fall back on. The concepts below should have been covered in basic calculus, but you should remember the following:

1. You need to identify intercepts. Intercepts on the vertical (in this case, v-) axis are easy to find: you simply set $t = 0$ in the formula for $v(t)$.

2. Intercepts on the horizontal (in this case, t-) axis are harder to find: you need to find values of t for which $v(t) = 0$

3. You need to identify if the function has any singularities. Is there a value of t at which $v(t)$ becomes infinite? If yes, does v go to positive or negative infinity as that value of t is approached from above or below?

4. You need to determine whether the solution is defined for all values of the independent variable. For instance, $y = \sqrt{x}$ is not defined for $x < 0$, unless you are willing to admit solutions in imaginary numbers (beyond the scope of this course, and not relevant to the kinds of plots we are talking about here).

5. You need to identify asymptotes. As $t \to \pm\infty$, does the graph of $v(t)$ approach a simple curve like a straight line? Note that an asymptote does not need to be a horizontal or vertical straight line. For instance, the graph of the function $f(x) = \exp(-x) + x^2$ asymptotes to the parabola $y = x^2$ when $x \to +\infty$. (It does not do the same as $x \to -\infty$.)

6. You need to find minima or maxima of the function. This involves finding the derivative $\mathrm{d}v/\mathrm{d}t$ as a function of t, and solving for values of t for which $\mathrm{d}v/\mathrm{d}t = 0$. Points at which $\mathrm{d}v/\mathrm{d}t = 0$ are called *stationary points*. To determine whether they are local maxima or minima, you can look at the second derivative. If $\mathrm{d}^2v/\mathrm{d}t^2 > 0$, a stationary point is a local minimum, and a local maximum if $\mathrm{d}^2v/\mathrm{d}t^2 < 0$. You do not always need to take the second derivative; sometimes you can determine from intercepts, asymptotes or singularities whether a stationary point is a minimum or maximum

7. Before you plot, you should also look for symmetries. If $f(-x) = f(x)$, the graph is symmetric about the y-axis. $f(-x) = -f(x)$, the graph has rotational symmetry: a 180° rotation about the origin does not change the graph. There are also less obvious symmetries, for instance if $f(a - x) = f(a + x)$, the graph is symmetric under reflections about the vertical line $x = a$.

In the case of our solution (6), we can work through these points in order

1. The intercept on the vertical axis is easy to find. We know that $v = v_0$ at $t = 0$; this is the initial condition.

2. Intercepts on the horizontal axis correspond to $v = 0$. This implies

$$\left(v_0 + \frac{mg}{C}\right)\exp\left(-\frac{Ct}{m}\right) - \frac{mg}{C} = 0$$

which has a solution

$$\begin{aligned} t &= -\frac{m}{C}\log\left(\frac{\frac{mg}{C}}{v_0 + \frac{mg}{C}}\right) \\ &= -\frac{m}{C}\log\left(\frac{1}{1+\frac{Cv_0}{mg}}\right) \end{aligned}$$

This solution is only well-defined if the argument of the logarithm is not negative, so we need

$$1 + \frac{Cv_0}{mg} > 0$$

or

$$v_0 > -\frac{mg}{C}.$$

The logarithm is positive if its argument is greater than one, in which case the value of t at the intercept is negative and lies to the left of the vertical (v-) axis. The intercept lies to the right of the v-axis if the argument of the lograithm is less than one, which happens if $1 + cv_0/mg > 1$, or $v_0 > 0$.

3. There are no singularities: there is no finite value of t for which v becomes infinite; the exponential function is well-defined for all values of t

4. As $t \to +\infty$, the exponential function $\exp\left(-\frac{Ct}{m}\right)$ goes to zero. The solution therefore asymptotes to $v \to -mg/C$

5. To see if there are local maxima or minima, we can use the fact that, from the original differential equation we have

$$\frac{\mathrm{d}v}{\mathrm{d}t} = -g - \frac{C}{m}v$$

Substituting for $v(t)$ on the right gives

$$\begin{aligned} \frac{\mathrm{d}v}{\mathrm{d}t} &= -g - \left[\left(\frac{cv_0}{m} + g\right)\exp\left(-\frac{Ct}{m}\right) - g\right] \\ &= -\left(\frac{cv_0}{m} + g\right)\exp\left(-\frac{Ct}{m}\right) \end{aligned}$$

This can never be zero unless $cv_0/m = g$, or $v_0 = -mg/C$, in which case $\mathrm{d}v/\,\mathrm{d}t$ is always zero. There are therefore no local minima or maxima.

6. There are also no obvious symmetries to exploit

As you can see, it is not trivial to work through the list of tasks needed to sketch a solution carefully, but there is also no magic to it. Now that we can sketch the solution

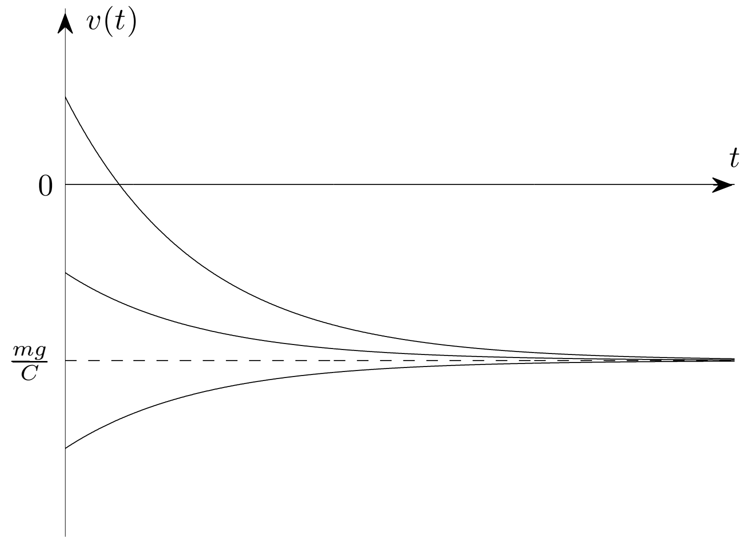

Figure 1: The solution $v(t)$ as a function of t for $v_0 > 0$, $-mg/C < v_0 < 0$ and $v_0 < -mg/C$.

There are three cases to distinguish, and they go back to point 2 above — whether we have a t-intercept. There is no t-intercept if $v_0 < -mg/C$. Looking at point 4, this corresponds to the graph starting below its asymptote. We can explain this physically: the asymptote is the *terminal velocity* of the object, which it approaches as t goes to infinity (which of course ignores the possiblity of eventually hitting the ground, somthing our model is not set up to describe). If the objects starts with a downward velocity greater than its terminal velocity, its downward motion will simply slow to spproach terminal velocity. There will not be a point in time for which velocity is zero, which would be a point at which the object is not moving.

There is a t-intercept if $0 > -v_0 > -mg/C$, but it lies on the negative half of the t-axis, that is, it comes before the point in time $t = 0$ at which the object was launched with its initial velocity. This is the case if the initial velocity v_0 is negative: we launch the object downward, and there is no reason why it should stop moving at any point after that. A t-intercept on the positive t-axis occurs if $v_0 > 0$: this is the case where the object is launched upwards; naturally we expect it to slow down, stop, and then begin falling downwards, eventually approaching terminal velocity. In all three cases, the velocity either continually decreases (becomes more negative) or continually increases (becomes less negative); there is no local velocity minimum or maximum.

Exercise 1 *From* (5)*, derive a solution for* $x(t)$ *if* $x(0) = 0$*. Identify the different* *parameter ranges ('parameter regimes') that will give different-looking graphs, and sketch the solution for each of those.*

Note 4 *The solution for $v(t)$ above takes the form*

$$v(t) = a \exp(-bt) + c$$

for choices of a, b and c that depend on the parameters m, g, C and v_0. Stripping the solution down to a simpler-looking form like this can sometimes help you plot it more easily. Immediately, c is evident as the asymptote, and $a \exp(-bt)$ is the height of the graph above the asymptote.

Defining that 'height above the asymptote' as $V = v - c$, we have

$$V(t) = a \exp(-bt)$$

The height above the asymptote 'decays exponentially' to zero. The key characteristic of exponential decay you should know about that is that V decays by a constant fraction each time a fixed time interval elapses. If that time interval is Δt, we have

$$V(t + \Delta t) = a \exp[-b(t + \Delta t)] = a \exp(-bt) \exp(-\Delta t) = \frac{V(t)}{\exp(b\Delta t)}$$

and

$$V(t + n\Delta t) = \frac{V(t)}{[\exp(b\Delta t)]^n}$$

This gives rise to the idea of a 'half-life'. If we put

$$\Delta t = \log(2)/b$$

we have $\exp(b\Delta t) = 2$, and

$$V(t + n\Delta t) = \frac{V(t)}{2^n}.$$

Every time a half-life Δt passes, V is halved. This reduction is very rapid: after four half-lives, only one sixteenth of the original value of V is left, and after five half-lives, only 1/32 of the original value.

Closely related and perhaps more widely used is the idea of an e-folding time. If

$$\Delta t = 1/b$$

then

$$V(t + n\Delta t) = \frac{V(t)}{e^n}.$$

where e is the base of the exponential function, $e = 2.71\ldots$. Every time an e-fodling time elapses, the solution is reduced by a factor of $1/e$.

Whether we think of exponential decay in terms of half-lives or e-folding times does not matter much. The important observation is that both, half-life and e-folding time are inversely proportional to b (that is, they are given by a constant divided by b; the e-folding time is simply one divided by b). $1/b$ therefore gives a characteristic time scale for decay. Even though V never goes to zero, we can *identify a meaningful time scale over which it 'approaches' zero.*

Exercise 2 *Sketch the following functions. This means you have to identify intercepts, minima/maxima, singularities and asymptotes before you start drawing.*

1. $f(x) = \frac{1}{1+(x-1)^2}$
2. $f(x) = \frac{1}{1-(x-1)^2}$
3. $f(x) = \frac{1}{x} + x$
4. $f(x) = -\frac{1}{x} + x$
5. $f(x) = \frac{1}{x} + x^2$
6. $f(x) = -\frac{1}{x} + x^2$

Note 5 *It is worth knowing the shapes of a few basic functions, like $f(x) = \exp(x)$, $f(x) = \sin(x)$, $f(x) = \log(x)$, $f(x) = 1/x$, $f(x) = x$, $f(x) = x^2$, $f(x) = x^2$ etc. Often, your solution will be a sum or product of such functions, possibly stretched or shifted. Stretching or shifting happens if you do any of the following*

1. *The plot of $y = f(-x)$ looks like the mirror image of the plot of $y = f(x)$ about the y-axis. Note that if $f(x) = f(-x)$, the plot is symmetric about the y-axis (as is the case for instance for $y = x^2$).*
2. *The plot of $y = -f(x)$ looks like the mirror image of the plot of $y = f(x)$ about the x-axis*
3. *The plot of $y = f(ax)$ looks like the plot of $y = f(x)$ but is compressed horizontally by a factor of $1/a$*
4. *The plot of $y = af(x)$ looks like the plot of $y = f(x)$ but is stretched vertically by a factor of a*
5. *The plot of $y = f(x + a)$ looks like the plot of $y = f(x)$ but is shifted to the left by an amount a*
6. *The plot of $y = f(x) + a$ looks like the plot of $y = f(x)$ but is shifted upwards by an amount a*

To plot the sum of two functions $y = f(x) + g(x)$, you need to add the height of the plots of the two functions. Plotting the product of two functions $y = f(x)g(x)$ is harder. y then has x-intercepts whereever the graph of either $y = f(x)$ or $y = g(x)$ has an intercept, provided the other function does not have a singularity there. The graph of $y = f(x)g(x)$ lies above the x-axis wherever either both f and g are positive, or where both are negative. $y = f(x)g(x)$ has singularities where f has a singularity and g is non-zero, and vice versa.

Exercise 3 *Use the information in note 5 and knowledge of the plots of the functions* $\exp(x)$, $1/x$, x, x^2 *to sketch the following sets of functions*

1. $y = \exp(x)$, $y = \exp(-x)$, $y = \exp(-ax)$ *with* $a > 1$, $y = -b\exp(-ax) - c$ *with* $b > 0$, $c > 0$, *all on the same plot.*
2. $y = x$, $y = 1/x$, $y = x + 1/x$, *all on the same plot.*
3. $y = x^2$, $y = 1/x$, $y = x^2 + 1/x$, *all on the same plot.*
4. $y = x^2$, $y = 1/x$, $y = 1/(x-1)$, $y = x^2/(x-1)$, *all on the same plot.*
5. $y = x^2$, $y = 1/x$, $y = x^2/x$, *all on the same plot.*
6. $y = 1/x$, $y = 1/(x+1)$, $y = \exp(x)/(x+1)$, *all on the same plot.*

Separation of variables and integrating factors

Unless stated otherwise, all differential equations you are asked to solve in this course can be solved by separation of variables. It is therefore worth formalizing the method. It can be stated generally as follows:

1. Let y be the dependent variable in the differential equation and x be the independent variable
2. Write the differential equation in the form
$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y).$$
If this factorization (or *separation of variables*) is not possible, then the method cannot be used.
3. Divide both sides by $g(y)$:
$$\frac{1}{g(y)}\frac{\mathrm{d}y}{\mathrm{d}x} = f(x).$$

4. Integrate both sides with respect to x:

$$\int \frac{1}{g(y)} \frac{\mathrm{d}y}{\mathrm{d}x} \, \mathrm{d}x = \int f(x) \, \mathrm{d}x.$$

5. Change variables on the left, so

$$\int \frac{1}{g(y)} \, \mathrm{d}y = \int f(x) \, \mathrm{d}x.$$

6. Evaluate the integrals on both sides, not forgetting a constant of integration.

7. The left-hand side is then a function of y, and the right-hand side a constant of x. Solve the resulting equation for y.

8. The solution for y contains a constant of integration. This must be found be applying a boundary condition: a piece of physical information not contained in the differential equation.

Exercise 4 *Decide whether the following differential equations can be solved by separation of variables, and if they can, find a general solution and sketch it. Comment on any unusual attributes the solutions may have.*

1.
$$\frac{\mathrm{d}v}{\mathrm{d}t} = t + v$$

2.
$$\frac{\mathrm{d}v}{\mathrm{d}t} = -(t + vt + v + 1)$$

3.
$$\frac{\mathrm{d}v}{\mathrm{d}t} = -2tv$$

4.
$$\frac{\mathrm{d}v}{\mathrm{d}t} = -2tv + 1$$

5.
$$\frac{\mathrm{d}v}{\mathrm{d}t} = tv^2$$

6.
$$\frac{\mathrm{d}v}{\mathrm{d}t} = t \exp(v) + t$$

(You may need the result that $1/(\exp(x) + 1) = \exp(-x)/(1 + \exp(-x))$*)*

7.
$$\frac{\mathrm{d}v}{\mathrm{d}t} = \exp(v+t).$$

Exercise 5 *For each of the differential equations in exercise 4, determine the constant of integration and write down the actual solution corresponding to $v(0) = 1$.*

Related to separation of variables but applicable to somewhat different differential equations is the method of integrating factors. While separation of variables can be used for equations in the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)g(y), \tag{7}$$

integrating factors are used to solve equations of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)y + h(x). \tag{8}$$

The method works as follows.

1. Re-write the equation as
$$\frac{\mathrm{d}y}{\mathrm{d}x} - f(x)y = h(x).$$

2. The trick is to multiply both sides by a specially-chosen function $I(x)$,
$$I(x)\frac{\mathrm{d}y}{\mathrm{d}x} - I(x)f(x)y = I(x)h(x) \tag{9}$$
so that the left-hand side can be turned into a single derivative by running the product rule in reverse.

3. To do that, remember that the product rule says
$$\frac{\mathrm{d}}{\mathrm{d}x}[I(x)y(x)] = I(x)\frac{\mathrm{d}y}{\mathrm{d}x} + y(x)\frac{\mathrm{d}I}{\mathrm{d}x}.$$
We can make that equal to the left-hand side of (9) by requiring the function $I(x)$ to satisfy
$$y(x)\frac{\mathrm{d}I}{\mathrm{d}x} = -I(x)f(x)y(x), \tag{10}$$
or, cancelling y,
$$\frac{\mathrm{d}I}{\mathrm{d}x} = -I(x)f(x).$$

4. This can be solved by separation of variables:

$$\frac{1}{I}\frac{\mathrm{d}I}{\mathrm{d}x} = -f.$$

   Integrating both sides gives

$$\log(I) = -\int f(x)\,\mathrm{d}x.$$

   We could add a constant of integration but do not need to — all we need is *a* function I that satisfies (10). Exponentiating gives

$$I(x) = \exp\left(-\int f(x)\,\mathrm{d}x\right).$$

5. With (10) satisfied, (9) becomes

$$\frac{\mathrm{d}}{\mathrm{d}x}[I(x)y(x)] = I(x)h(x).$$

   This can now be solved by simply integrating

$$I(x)y(x) = \int I(x)h(x)\,\mathrm{d}x$$

   and

$$y(x) = \frac{1}{I(x)}\int I(x)h(x)\,\mathrm{d}x.$$

6. The right-hand side will generate a constant of integration through the integral $\int I(x)h(x)\,\mathrm{d}x$. This needs to be determined using an initial condition.

Exercise 6 *Some problems can be solved by either separation of variables or by integrating factors. These take the form*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = f(x)y + cf(x), \tag{11}$$

This is of the form (7) *with* $g(y) = y + c$, *and of the form* (8) *with* $h(x) = cf(x)$. *Equation* (3) *can be written in the form of* (11). *Use the method of integrating factors to re-derive the solution* (6).

Exercise 7 *Re-consider the differential equations in exercise 4. Determine which ones can be solved by integrating factors, and solve them. All can be solved by either separation of variables or by integrating factors. At least two can be solved by both methods.*

Exercise 8 *Consider the falling object model again, but now assume that the air it is falling through is moving up and down (for instance on a windy day, where there are eddies in the air). The drag is really proportional to the velocity of the object relative to that of air, so the drag force is* $-C(v - v_{air})$*, where we assume that* v_{air} *is a known function of time. In that case*

$$m\frac{\mathrm{d}v}{\mathrm{d}t} = -mg - C(v - v_{air}(t))$$

Solve this for the case where you are given that $v_{air}(t) = w_0 \cos(\omega t)$*, with* w_0 *and* ω *being constants. Use* $v(0) = v_0$ *as your initial condition again. Sketch the solution (don't worry about finding all the zeros, which you cannot do analytically — but figure out in what parameter range you expect to have infinitely many zeros).*

Deriving a differential equation: pressure in the atmosphere

Our previous differential equation example was relatively simple to construct because we had Newton's laws to rely on. In reality, Newton's law turned out to *be* a differential equation in which we just needed to supply the right F. The real work was then in solving the equation.

Often, the hard part is in *constructing* the right differential equation; once you have formulated an equation in a standard form like (7) or (8) above, solving that equation can turn into exercise in 'cranking the handle': you simply have to follow a standard set of steps and not make a mistake. That may be hard work — there may be a lot of steps to take, and you can't let errors slip in — but it is not necessarily very inventive.

To illustrate the process of formulating a differential equation whose form isn't handed to you in the shape of Newton's laws, consider the question of how deonsity and pressure vary with elevation in the atmosphere. Experience tells us that both must go down as you go up in elevation, as you will know if you have ever climbed a tall enough mountain: the air gets thinner. But how fast?

You could argue that it is pointless to try to write down equations for this, and that you should just make measurements. That partly misses the point, however. Measurements and models have different purposes. A model is useful if you want to make a prediction for something that will happen in the future, or in an inaccessible place. It is also useful if you want to see if you understand *why* something works the way it does: you can represent mathematically the physics you believe to be behind an observable phenomenon (like pressure and density drop with elevation in the atmosphere), and then see if the solution of that mathematical model agrees with actual measurements. Seen from this perspective, the role of measurements is then to test the validty of a model, or help determine the parameter values in the model.

In order to construct a model for how pressure varies with elevation in the atmosphere, we need to know in what kind of physics the variable 'pressure' might appear. You will know pressure as 'force over area', so pressures are associated with forces.

Exercise 9 *Why do we define pressure as 'force over area'? Why not 'force over square root of area' or something else?*

The obvious starting point is to consider the balance of forces in the atmosphere. If we take a 'blob' of atmosphere, we expect forces to result from gravity acting on the blob, and from pressure acting on the surface of the blob, where the rest of the atmosphere presses inward on it. When we add these forces, Newton's second law tells us that we should get mass times acceleration for the blob.

Accounting for that acceleration would make life quite difficult, so we start by making some simplifying assumptions. We assume that any acceleration that occurs is small, so that the forces on a blob of atmosphere approximately balance. We also want to keep the geometry simple, so we assume that pressure$p$ and density ρ (this is the Greek letter 'rho', not a funny p) depend only on elevation above the ground, z. We also assume that the atmosphere has a limited extent compared with the radius of the planet. This means we neglect the curvature of the planet, and treat gravitational acceleration g as constant.

With these five simplifications, life becomes much easier, but we still need to compute gravitational forces and pressure forces on the 'blob' to make progress. This is difficult for two reasons. First, if the density varies wth elevation, how can we compute the mass and hence the gravitational force on the blob? If pressure varies with elevation, how can we compute the pressure force on the possibly uneven surface of the blob?

These questions actually go to the heart of what we mean by 'density' and 'pressure'. You will get a lot more detail on the idea of 'density' when we talk about volume integrals in the next part of the course; for a similar treatment for 'pressure', you will need a continuum of fluid mechanics course. However, here, it suffices if you realize that we can compute the mass of the blob if the extent of the blob is small enough for $\rho(z)$ to be treated as constant.

Note 6 *Saying that the blob is small enough to be treated as having constant density is analogous to what we did in saying that displacement δx could be computed as $v(t)\delta t$ over time intervals δt that are short enough for v to be treated as constant.*

To compute forces, it also makes sense to choose the blob to have surfaces that make pressure forces easy to compute. As pressure is a function of z, this is easiest if the main surfaces are flat and have constant elevations. This means it is easiest to look at a thin slice of the atmosphere, lying between z and $z + \delta z$ and with base area A (see figure 2).

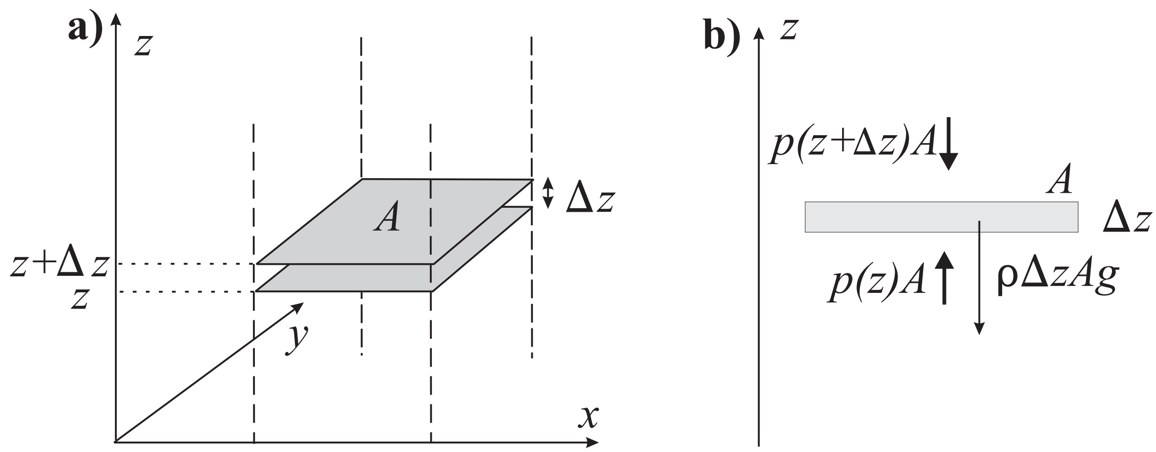

Figure 2: A thin slice of the atmosphere: panel a) shows a three-dimensional view, panel b) a side-on view with the forces acting on the slice.

Pressure forces then act on the surfaces of the slice (at z and $z + \delta z$) due to interactions with the atmosphere above and below the slice, and gravity acts on the interior of the slice. The forces due to pressure are given by pressure times surface area, and taking upward forces as positive, we have a positive force $p(z)A$ due to the atmosphere below the slice pushing up, and a negative force $p(z + \delta z)A$ due to the atmosphere above pushing down. In addition, the gravitational force on the slice is mass times acceleration due to gravity. Mass can be approximated as density of the slice times its volume, so the force due to gravity is $\rho(z) \times A\delta z \times (-g)$. The minus sign in front of g reflects that gravity acts downwards.

Summing these forces to zero, we have

$$p(z)A - p(z + \delta z)A - \rho(z) \times A\delta z g = 0. \tag{12}$$

The next step is key: it requires recognition that we have a difference between p evaluated at z and at $z+\delta z$ on the left of (12), and that this is the pattern we expect to see in a derivative. We cancel the A's on both sides immediately to give

$$-\rho(z)\delta z g = p(z + \delta z) - p(z).$$

To turn the right into a derivative, we simply need to divide by δz,

$$\rho(z)g = \frac{p(z + \delta z) - p(z)}{\delta z}.$$

so that

$$\frac{\mathrm{d}p}{\mathrm{d}z} = -\rho g, \tag{13}$$

which must hold at any height z in the atmosphere.

Exercise 10 *Why is it essential that δz is small? How would you compute the mass of a slice of the atmosphere between z and $z + \Delta z$ if Δz was not small?*

The point about (13) is that it is a *differential equation*: it contains the unknown p as a derivative. This differential equation must be solved to find pressure as a function of z. For an 'atmosphere' with constant density ρ this is simple, as we can just integrate,

$$\frac{\mathrm{d}p}{\mathrm{d}z} = -\rho g \qquad \Rightarrow \qquad p(z) = \int -\rho g \, \mathrm{d}z = -\rho g z + C.$$

The constant of integration must be determined from some extra piece of information that is not supplied by the differential equation (13). For instance, suppose we know that the atmosphere had constant density ρ and an upper surface at $z = h$ where pressure is zero. In that case, pressure is given at any elevation z by $p(z) = -\rho g z + C$ as shown above. This must therefore also be true at $z = h$, so $p(h) = C - \rho g h$. But we also know that pressure at $z = h$ is zero, so $p(h) = 0$. Equating the two expressions for $p(h)$ then gives us an equation that must be satisfied by C, which we can solve

$$0 = -\rho g h + C, \qquad C = \rho g h.$$

Once we have found C, we have to put this back into the expression for general $p(z)$,

$$p(z) = \rho g(h - z).$$

Note 7 *what we have just derived is in fact the high school formula '$p = \rho g h$'. If we sit at the bottom of the 'atmosphere', we are at $z = 0$, and the depth of fluid above us is h. The pressure is $\rho g h$. The fact that the solution formula says $p = \rho g(h - z)$ reflects the fact that, as we go up and z increases above zero, the depth of fluid above us decreases to $h - z$.*

Variable density: the need for a constitutive relation

In general, differential equations are not as simple as the example with a constant density. For instance, in the atmosphere we know that density is not constant, but decreases with height. This decrease is actually the result of pressure also decreasing with height — so density and pressure are in fact coupled.

The differential equation (13) is based on one simple principe — that forces on any slice of the atmosphere balance. This comes from the principle of conervation of momentum, if we add tha additional assumption that there are no acceleration terms. The equation is not solveable by itself, as we have two unknowns (p and ρ) but only one equation. We therefore need another equation, a constitutive relation, in order to be able to solve the problem

Note 8 *Our previous assumption, that ρ is constant and known, is in fact a constitutive relation, although a very simple one.*

One model for the relationship between density and pressure is the *ideal gas law*,

$$\rho = \frac{mp}{RT} \tag{14}$$

where m is the mass of one mole of gas, T is absolute temperature (in Kelvins) and R is a constant. Another expression for *constitutive relation* in this case would be *equation of state*, which means the same thing here for practical purposes.

One complication here is that density depends not only on pressure, but also on a further variable, temperature, for which we also need a model. To start with, let us make our job easier and pretend that temperature is constant at $T = T_0$ in the atmosphere (this is reasonable provided temperature in the atmosphere does not vary by much, when measured in kelvins). Then (13) becomes

$$\frac{\mathrm{d}p}{\mathrm{d}z} = -\frac{mg}{RT_0}p. \tag{15}$$

To solve this, we can use either separation of variables or integrating factors If we know atmospheric pressure p_0 at sea level, $z = 0$, then the solution becomes

$$p(z) = p_0 \exp\left(-\frac{mg}{RT_0}z\right). \tag{16}$$

Exercise 11 *Derive* (16) *using separation of variables and by using integrating factors.*

Exercise 12 *As we mentioned previously, models are always mathematical idealizations of reality. It is important to know the 'weak points' of models: what assumptions made in constructing a model are most questionable? The first step is of course to be aware of all the assumptions that have gone into a model.* (15) *is a model for pressure distribution in the atmospher, and anumber of assumptions have already gone into constructing* (13). *List the most important assumptions in words, and identify the ones that you think are most likely to be violated in reality.*

Exercise 13 *Take the molar weight of air to be 29 g mol^{-1} and $R = 8.314$ J K^{-1} mol^{-1}, g = 9.8 m s^{-2}, and assume an atmospheric temperature of $T_0 = 273$ K. Over what height does atmospheric pressure decrease by 50 %?*

Exercise 14 *Instead of putting $T = T_0$ in (14), assume that temperature decreases linearly above sea level at a constant lapse rate c, so that*

$$T = T_0 - cz. \tag{17}$$

Rewrite (13) using (14) with this description of temperature. Use separation of variables to solve this differential equation. If atmospheric pressure at $z = 0$ is p_0, show that

$$p(z) = p_0 \left(1 - \frac{cz}{T_0}\right)^{\frac{mg}{Rc}}.$$

Does pressure decrease slower or faster than in the model with constant temperature? What is the physical reason for this? What is an obvious problem with the temperature model (17)?

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Exercise 15 *Assuming a fixed lapse rate is not necessarily ideal; after all, why should the lapse rate do what it does. Below, we consider a model that assumes that the relationship between temperature, pressure and density in a planetary atmosphere is that the atmosphere behaves* adiabatically.[3] *For an ideal gas that behaves adiabatically, pressure is proportional to density to a fixed power γ (which in turn is determined by the heat capacity, and hence the molecular structure, of the gas):*

$$p \propto \rho^\gamma. \tag{18}$$

Here γ is a constant that is greater than one.

1. *Turn the relation (18) into an* equation *involvin a constant of proportionality.*
2. *Let z be height in the atmosphere. Denoting atmospheric pressure at sea level ($z = 0$) by p_0 and the corresponding atmospheric density at sea level by ρ_0, compute the constant of proportionality in the relationship (18) in terms of p_0, ρ_0 and γ. Write down a differential equation for pressure $p(z)$ in the atmosphere. What does this differential equation assume about forces on any slice of the atmosphere?*
3. *Use the method of separation of variables to solve the differential equation, and find the constant of integration that you obtain.*
4. *Given p_0, ρ_0, γ and acceleration due to gravity g, is there again a height z at which the atmosphere ends?*

[3]This means that temperature in the atmosphere is neither constant, nor decreases in a prescribed way with height in the atmosphere. Instead, temperature is related to density and pressure because changes in the volume of a sample of gas in the atmosphere (and hence a change in its density) correspond to mechanical work being done on the gas. This changes the heat content of the sample, and therefore its temperature. For an ideal gas, in which density is related to temperature and pressure through

$$\rho(p, T) = \frac{mp}{RT},$$

this means that density can be written as a function of pressure alone.

5. *For an atmosphere that is made of a diatomic gas (like nitrogen or oxygen), statistical physics gives* $\gamma = 7/5$. *Take* $p_0 = 10^5$ *Pa,* $\rho_0 = 1.2$ *kg* m^{-3}, $g = 9.8$ *m* s^{-2} *for the atmosphere on Earth. At what height does* $p(z)$ *drop to one-third its value at the surface* $z = 0$*?*

6. *From part 2 above, you should have an equation relating* ρ *and* p*, involving constants* γ, p_0 *and* ρ_0. *Suppose you also have the ideal gas law*

$$\rho = \frac{mp}{RT}.$$

Eliminate ρ *between these equations to find a relationship between* p *and* T *that involves (apart from* $p(z)$ *and* $T(z)$*) only the constants* γ, ρ_0, p_0, R *and* m*). Using your solution in part 3 above, use this relationship to solve for* $T(z)$ *as a function of* z *(where this function again involves only the constants* γ, ρ_0, p_0, R *and* m*). Show that you actually end up predicting a constant lapse rate. If* $m = 29$ *g* mol^{-1}*, what is the temperature at sea level (*$z = 0$*)? If the lapse rate is defined as* $-\frac{dT}{dz}$*, what is that lapse rate?*

Exercise 16 *Consider again an adiabatic atmosphere as in exercise 15, but assume now that the thickness of the atmosphere is not small compared with the radius* r_0 *of the planet which it surrounds. In that case, acceleration due to gravity can no longer be treated as constant. Now we need to solve the following:*

$$\frac{dp}{dr} = -\rho g$$
$$p(r) \propto \rho(r)^\gamma$$
$$g(r) = g_0 r_0^2 / r^2,$$

where r *measures distance from the centre of the planet and* r_0 *is the radius of the solid part of the planet (and therefore a constant). The atmosphere extends upward from* $r = r_0$ *(i.e.,* $r > r_0$ *in the atmosphere).* g_0 *is acceleration due to gravity at* r_0. *Assume again that, at the surface of the planet (*$r = r_0$*),* $p = p_0$ *and* $\rho = \rho_0$ *to compute the constant of proportionality.*

1. *Draw a diagram of the geometry in question.*

2. *Derive an equation relating* p *to* ρ *that involves* p_0 *and* ρ_0. *Then derive a differential equation containing* $p(r)$ *as the only dependent variable. This will involve the constants* g_0, r_0, p_0, ρ_0 *and* γ.

3. *Solve this differential equation using separation of variables. What is the initial condition you need to apply to determine your constant of integration?*

4. *Find the radius r_s at which pressure drops to zero. The thickness of the atmosphere is now $r_s - r_0$. Take the thickness you computed in part (4) of exercise 15. Put $g = g_0$ into that expression. Is the thickness computed in exercise 15 greater or smaller than the thickness you have just computed here? Does this make sense physically?*

Exercise 17 *So far, we have assumed that pressure and density are both known at the base of the atmosphere $z = 0$. In exercise 15, assume you know the* mass m_0 *of the atmosphere per unit surface area of the planet, and you know the surface temperature T_0 of the planet. Find ρ_0 and p_0.* Hint. Given a solution $\rho(z)$, you need to figure out how to compute the mass of atmosphere that sits above a unit area of planet surface from that function ρ. Think about how we computed the mass of a slice of atmosphere between elevations z and $z + \delta z$ and work from there.

Exercise 18 *Consider a liquid which is slightly compressible, so*

$$\rho = \rho_0 \times (1 + cp)$$

where ρ_0 is a constant density, c is a constant expansion coefficient and p is pressure. Let z measure height above the bottom of a tank filled with the liquid, and let the surface of the liquid be at $z = h$, where pressure $p = 0$.

1. *Using the method of separation of variables, find pressure in the liquid as a function of z.*

2. *If the tank is a cuboid with base area A, what is the mass of fluid in the tank?*