Local forms for conservation laws, the divergence and the divergence theorem

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Overview

These notes cover the following:

- Converting an integral conservation law to a partial differential equation
- The divergence of a vector field
- The divergence theorem

Local form of a conservation law and the divergence of a vector field

In the notes on surface integrals, conservation of mass was written in the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho \,\mathrm{d}V = -\int_S \mathbf{q}\cdot\hat{\mathbf{n}}\,\mathrm{d}S. \tag{1}$$

In words, the rate at which the mass content of a volume V increases is rate at which mass flows into the volume, which is written here as minus the rate at which mass flows out of the volume. This equation must hold for *any* volume V. Instead of saying that, you will often hear that the equation must hold for an 'arbitrary' volume V. This means precisely that it must hold for any volume V, however that may be shaped, so long as V is finite.[1]

Equation (1) is clearly an equation that links changes in the density ρ tot he flux $\mathbf{q}$. The problem in making sense of this equation, or more precisely, in using

[1] Technically, V must also have a sufficiently smooth surface for the surface integral to be defined.

it to predict exactly how ρ changes in time, is that it involves an integral over an unspecified volume V and its surface S. What we would like to do is turn it into an equation that does not involve such integrals. Ideally, this alternative equation would only involve 'local' information about the function ρ and flux $\mathbf{q}$ — for instance, these functions themselves, or their derivatives. In other words, we would like a differential equation, in large part because we know more about how to deal with them than with equations of the form (1).

A strategy we can adopt is similar to how we dealt with describing pressure distribution with elevation in the atmosphere. Because V is arbitrary — meaning it can take any shape we choose — we can make it a cuboid with corners (x, y, z), $(x + \delta x, y, z)$, $x, y + \delta y, z)$, $(x, y, z + \delta z)$ etc. (see figure 1), and assume that δx, δy and δz are all small. This means the cuboid has a small volume of size $\delta x \delta y \delta z$, and we approximate the mass of the cuboid by

$$\int_V \rho \,\mathrm{d}V \approx \rho(x, y, z, t)\delta x \delta y \delta z$$

on the grounds that ρ should be approximately constant over the cuboid. Therefore

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V \rho \,\mathrm{d}V \approx \frac{\partial \rho}{\partial t} \delta x \delta y \delta z. \tag{2}$$

Note that the ordinary derivative on the left is an ordinary derivative because, for a given volume V, the integral depends on time t only, position having disappeared when computing the definite triple integrals. On the right, we need to turn the ordinary derivative back into a partial partial derivative because (x, y, z) have reappeared. The reason for this reappearance is that (x, y, z) actually tells us where the specific volume V we are looking at is located.

This still leaves the surface integral $-\int_S \hat{\mathbf{q}} \,\mathrm{d}S$. To deal with that, we will again want to use the fact that δx, δy and δz are small, so that we can treat the integrands as approximately constant. Before we can do so, we need to understand that the surface of the cuboid consists of six rectangles. Two of them are perpendicular to the x-axis, two are perpendicular to the y-axis, and two to the z-axis. We can look at these three pairs separately.

Look at the surfaces perpendicular to the x-axis first. These are at x and $x + \delta x$, on the left and right of the cuboid in figure 1. The figure is an attempt to show the three-dimensional geometry of the cuboid and may be somewhat confusing. Figure 2 shows a side-on view looking directly along the y-direction, so the left- and right-hand edges of the rectangle in figure 2 correspond to the surfaces in question, whereas the rectangle corresponds to the cuboid itself. Let the surfaces at x and $x+\delta x$ be labelled as S_1 and S_2, as indicated in figure 2.

The integrand in all cases is $\mathbf{q} \cdot \hat{\mathbf{n}}$. This is the normal component of $\mathbf{q}$, taken in the outward-pointing direction. Physically, think of it as the comonent of $\mathbf{q}$ that actually carries material *out* of the volume. To be definite, let

$$\mathbf{q}(x, y, z, t) = q_x(x, y, z, t)\mathbf{i} + q_y(x, y, z, t)\mathbf{j} + q_z(x, y, z, t)\mathbf{k}.$$

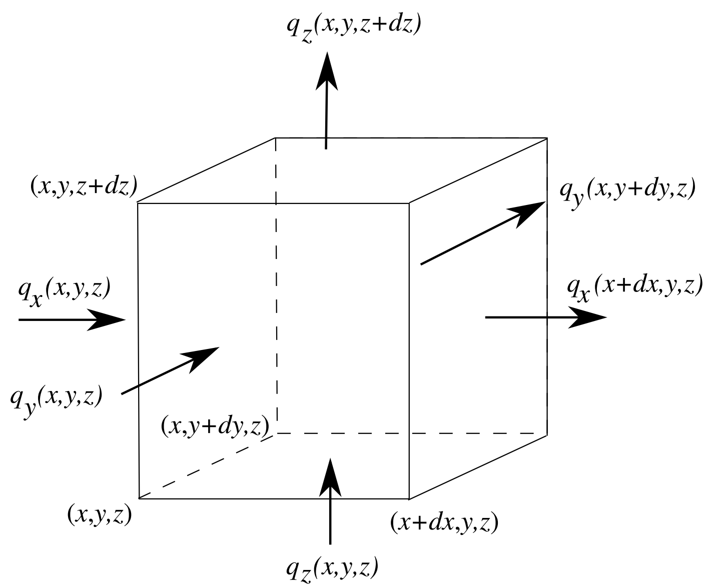

Figure 1: A small cuboid volume V, and the normal components of flux on its faces.

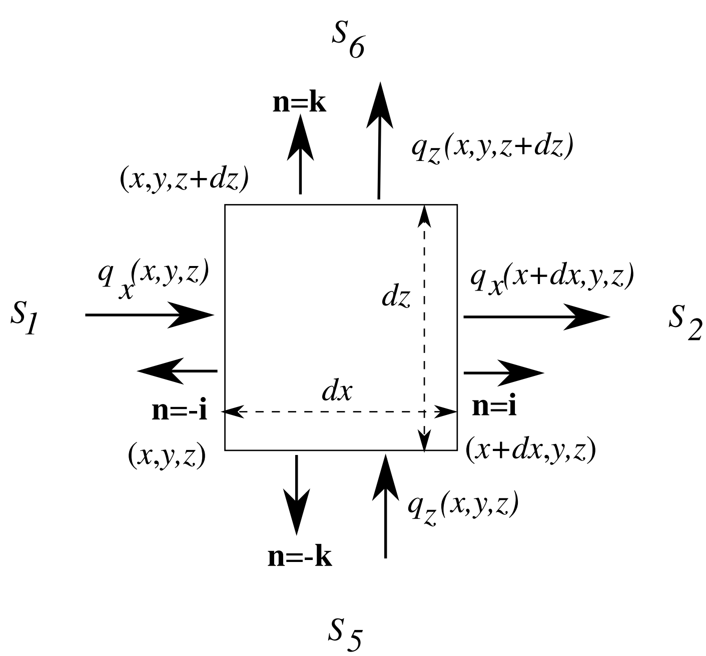

Figure 2: A side-on view of the small cuboid volume V.

Take the face S_1 located at x, meaning the left-hand edge of the rectangle in figure 2. The normal component of $\mathbf{q}$ is clearly parallel to the x-direction, as the face is perpendicular to that direction. However, in order for this to be the component that takes material *out* of the volume, we need to take the negative x-component $-q_x$ rather than q_x. We have $\mathbf{a} \cdot \hat{\mathbf{n}} = -q_x$. Taking the integral of $\mathbf{q} \cdot \hat{\mathbf{n}}$ over S_1, we can use the fact that S_1 is very small, and so approximate the integrand $\mathbf{q} \cdot \hat{\mathbf{n}} \approx -q_x(x, y, z)$ as approximately constant. Now the face S_1 has size $\delta y \delta z$. Treating the integrand as constant, we can calculate the integral as approximately

$$\int_{S_1} \mathbf{q} \cdot \hat{\mathbf{n}} \, \mathrm{d}S \approx -q_x(x, y, z) \delta y \delta z.$$

This is the rate at which mass flows out of the volume V through the face S_1; of course, this is the same as saying that $q_x(x, y, z)\delta y \delta z$ is the rate at which mass flows into the volume.

Now take the counterpart to S_1, the surface S_2 on the right of the rectangle in figure 2. We can follow the same steps as before. The only difference is that we now need to take the positive x-component of $\mathbf{q}$, namely q_x, as the integrand, and that this must be evaluated at $x + \delta x$ rather than at x. Therefore

$$\int_{S_2} \mathbf{q} \cdot \hat{\mathbf{n}} \, \mathrm{d}S \approx q_x(x + \delta x, y, z) \delta y \delta z.$$

Labelling the bottom and top of the cuboid in figure 1 as surfaces S_5 and S_6, respectively (see also figure 2), we can follow the same steps as above but switch the roles of x and z, so that

$$\int_{S_5} \mathbf{q} \cdot \hat{\mathbf{n}} \, \mathrm{d}S \approx -q_z(x, y, z) \delta x \delta y,$$

and

$$\int_{S_6} \mathbf{q} \cdot \hat{\mathbf{n}} \, \mathrm{d}S \approx q_z(x, y, z + \delta z) \delta x \delta y,$$

Similarly, if front and back of the cuboid are surfaces S_3 and S_4, we have

$$\int_{S_3} \mathbf{q} \cdot \hat{\mathbf{n}} \, \mathrm{d}S \approx -q_y(x, y, z) \delta x \delta z,$$

and

$$\int_{S_4} \mathbf{q} \cdot \hat{\mathbf{n}} \, \mathrm{d}S \approx q_z(x, y + \delta y, z) \delta x \delta z,$$

The integral over the whole surface is given by the sum of these six integrals, which we can write as

$$
\begin{aligned}
\int_S \mathbf{q}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = &\int_{S_1} \mathbf{q}\cdot\hat{\mathbf{n}}\,\mathrm{d}S + \int_{S_2} \mathbf{q}\cdot\hat{\mathbf{n}}\,\mathrm{d}S \\
&+ \int_{S_3} \mathbf{q}\cdot\hat{\mathbf{n}}\,\mathrm{d}S + \int_{S_4} \mathbf{q}\cdot\hat{\mathbf{n}}\,\mathrm{d}S \\
&+ \int_{S_5} \mathbf{q}\cdot\hat{\mathbf{n}}\,\mathrm{d}S + \int_{S_6} \mathbf{q}\cdot\hat{\mathbf{n}}\,\mathrm{d}S \\
= &- q_x(x,y,z)\delta y\delta z + q_x(x+\delta x,y,z)\delta y\delta z \\
&- q_y(x,y,z)\delta x\delta z + q_y(x,y+\delta y,z)\delta x\delta z \\
&- q_z(x,y,z)\delta x\delta y + q_z(x,y,z+\delta z)\delta x\delta y
\end{aligned} \tag{3}
$$

Note that the pairs of integrals have the same pattern: the first two have the same area term $\delta y\delta z$ and the same flux components q_x, except that hte flux components are evaluated a distance δx apart and have opposite signs. We can recongize this as being similar to taking a derivative: take for instance

$$-q_x(x,y,z)\delta y\delta z + q_x(x+\delta x,y,z)\delta y\delta z = [q_x(x+\delta x,y,z) - q_x(x,y,z)]\delta y\delta z$$

The right-hand side looks like a partial derivative of δx, except that it lacks the necessary division by δx. This is easy to remedy: we can divide by δx and multiply by δx again, so

$$[q_x(x+\delta x,y,z) - q_x(x,y,z)]\delta y\delta z = \frac{q_x(x+\delta x,y,z) - q_x(x,y,z)}{\delta x}\delta x\delta y\delta z.$$

With δx small, this allows us to write

$$-q_x(x,y,z)\delta y\delta z+q_x(x+\delta x,y,z)\delta y\delta z = \frac{q_x(x+\delta x,y,z) - q_x(x,y,z)}{\delta x}\delta x\delta y\delta z \approx \frac{\partial q_x}{\partial x}\delta x\delta y\delta z$$

as an approximation to the sum of the integrals over S_1 and S_2. Applying the same procedure to the remaining integrals, we have

$$-q_y(x,y,z)\delta x\delta z + q_y(x,y+\delta y,z)\delta x\delta z \approx \frac{\partial q_y}{\partial y}\delta y\delta x\delta z$$

and

$$-q_z(x,y,z)\delta x\delta y + q_z(x,y,z+\delta z)\delta x\delta y \approx \frac{\partial q_z}{\partial z}\delta z\delta x\delta y.$$

Substituting these back into (3) gives us

$$
\begin{aligned}
\int_S \mathbf{q}\cdot\hat{\mathbf{n}}\,\mathrm{d}S \approx &\frac{\partial q_x}{\partial x}\delta x\delta y\delta z + \frac{\partial q_y}{\partial y}\delta y\delta x\delta z + \frac{\partial q_z}{\partial z}\delta z\delta x\delta y \\
= &\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right)\delta x\delta y\delta z.
\end{aligned} \tag{4}
$$

Now we need to go back to our original equation (1). The left-hand side is given by (2), and the right-hand side by (4). Substituting gives

$$\frac{\partial \rho}{\partial t}\delta x \delta y \delta z = -\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right)\delta x \delta y \delta z.$$

We can cancel the volume of the cuboid $\delta x \delta y \delta z$ on both sides to give

$$\frac{\partial \rho}{\partial t} = -\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\right)$$

or, rearranging,

$$\frac{\partial \rho}{\partial t} + \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} = 0.$$

Note that there is a pattern to the partial derivatives of the components of $\mathbf{q} = q_x\mathbf{i} + q_y\mathbf{j} + q_z\mathbf{k}$ that appears: take the derivative of the x-component, q_x, with respect to the x-coordinate, and add the derivative of the y-component q_y with respect to the y-coordinate and the derivative of the z-component with respect to the z-coordinate.

This combination of partial derivatives occurs very frequently and has its own name: it is called the *divergence* of the vector field $\mathbf{q}$. It is usally written in an abbreviated form,

$$\nabla \cdot \mathbf{q} = \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}.$$

Take this to be the *definition* of what we mean by the notation $\nabla \cdot \mathbf{q}$ when $\mathbf{q}$ takes the form $\mathbf{q} = q_x\mathbf{i} + q_y\mathbf{j} + q_x\mathbf{k}$. The 'upside down triangle' ∇ is often called the 'nabla' or 'del' operator. An alternative notation, which we will not use here, writes

$$\operatorname{div}\mathbf{q} = \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}.$$

Again, this is a defintiion of what we mean by '$\operatorname{div}\mathbf{q}$'.

Note 1 *Think of ∇ as being a vector differential operator*

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}. \tag{5}$$

Then $\nabla \cdot \mathbf{q}$ as defined above is a natural interpretation of the 'scalar product' of ∇ and $\mathbf{q}$, as for a vector field $\mathbf{q} = q_x\mathbf{i} + q_y\mathbf{j} + q_z\mathbf{k}$, we have

$$\nabla \cdot \mathbf{q} = \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}.$$

However, this is no ordinary product, as we are differentiating, not multiplying. For an ordinary scalar product $\mathbf{a} \cdot \mathbf{b}$, we have commutativity, $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$. This is not *the case for the divergence, where we cannot write* $\nabla \cdot \mathbf{q} = \mathbf{q} \cdot \nabla$*. The latter would presumably have to mean*

$$\mathbf{q} \cdot \nabla = q_x \frac{\partial}{\partial x} + q_y \frac{\partial}{\partial y} + q_z \frac{\partial}{\partial z},$$

so the components of $\mathbf{q}$ *are not being differentiated. Exercise 2 will make more sense of this.*

The operator ∇ *also occurs in different forms. For instance, we will later come across the* gradient *of a scalar field like* ρ*, defined as*

$$\nabla \rho = \frac{\partial \rho}{\partial x}\mathbf{i} + \frac{\partial \rho}{\partial y}\mathbf{j} + \frac{\partial \rho}{\partial z}\mathbf{k}. \tag{6}$$

Again, this is the natural application of the vector differential operator ∇ *given by* (5) *to a scalar* ρ*, analogous to the multiplication of a vector (*∇*) by a scalar (*ρ*), except that we once again have a derivative rather than an ordinary product.*

With this notation, conservation of mass therefore leads to the partial differential equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{q} = 0. \tag{7}$$

This is known as the *local form* of the conservation law (1).

The divergence of a vector field

As stated above, the divergence of a vector field $\mathbf{q}(x, y, z) = q_x(x, y, z)\mathbf{i} + q_y(x, y, z)\mathbf{j} + q_z(x, y, z)\mathbf{k}$ is defined as

$$\nabla \cdot \mathbf{q} = \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}.$$

Obviously, $\nabla \cdot \mathbf{q}$ defined in this way is a scalar field.

To get some understanding of what the operator means, look at the following two observations.

Note 2 *Note that the right-hand side of* (4) *can be written as*

$$\left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} \right) V$$

where $V = \delta x \delta y \delta z$ *is the volume of the cuboid. The combination of partial derivatives*

$$\nabla \cdot \mathbf{q} = \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}$$

can therefore be though of as approximately equal to

$$\nabla \cdot \mathbf{q} = \frac{1}{V} \int_S \mathbf{q} \cdot \hat{\mathbf{n}} \, \mathrm{d}S \tag{8}$$

This combination of derivatives is therefore a measure of how much net flow the flux $\mathbf{q}$ *causes* out *of the surface* S *enclosing the small cuboid. This explains why this combination of derivatives is called the* divergence *of the vector field.*

Another way of understanding the same thing is to say that $\mathbf{q} \cdot \hat{\mathbf{n}}$ *is the outward-pointing normal component of flux* $\mathbf{q}$ *at any point on the surface* S*. Obviously* $\mathbf{q} \cdot \hat{\mathbf{n}}$ *is not, in general, constant over the surface of* S*. Taking the integral over* $\mathbf{q} \cdot \hat{\mathbf{n}}$ *and dividing by* V *effectively gives an 'average' over the normal component. If the divergence is positive, that average is positive: on average, flux points out of the surface rather than into it.*

Note 3 *Each of the terms making up the divergence can be understood in terms of whether the flow in a particular direction is speeding up or slowing down in the flow direction. Take*

$$\frac{\partial q_x}{\partial x}.$$

This is positive if q_x *increases with* x*: that is, if the flow in the* x*-direction gets faster as we move along the* x*-direction. Similarly,* $\partial q_y / \partial y$ *is positive if th flow in the* y*-direction gets faster as we move along the* y*-direction, and similarly for* $\partial q_z / \partial z$*.*

These terms show up because we are looking at the net rate at which mass flows out of the cuboid in figure 1, as described in the previous note. The net rate of flow out of the faces S_1 *and* S_2 *can be thought of as the rate at which mass flows out through faces* S_2 *minus the rate at which it flows in through face* S_1*,* $q_x(x+\delta x, y, z)\delta y \delta z$ *minus* $q_x(x, y, z)\delta y \delta z$*. It is the difference between flux* $q_x(x + \delta x, yz)$ *out on the right and flux in* $q_x(x, y, z)$ *on the left that gives the term* $\partial q_x / \partial x$*. Similar terms show up for flow out through faces* S_4 *minus flow in through face* S_3*, and flow out through faces* S_6 *minus flow in through face* S_5*.*

Example 1 *Let* $\mathbf{q} = x^2 y \mathbf{i} - y^2 x \mathbf{j}$*. This is of the form* $\mathbf{q} = q_x \mathbf{i} + q_y \mathbf{j} + q_z \mathbf{k}$ *if we identify* $q_x = x^2 y$*,* $q_y = -y^2 x$*,* $q_z = 0$*. Therefore*

$$\nabla \cdot \mathbf{q} = \frac{\partial (x^2 y)}{\partial x} + \frac{\partial (-y^2 x)}{\partial y} + \frac{\partial (0)}{\partial z} = 2xy - 2yx = 0.$$

Exercise 1 *Compute* $\nabla \cdot \mathbf{q}$ *for*

1.

$$\mathbf{q} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{[x^2 + y^2 + z^2]^{3/2}},$$

2.

$$\mathbf{q} = y\mathbf{i} - x\mathbf{j},$$

3.
$$\mathbf{q} = x^2y\mathbf{i} - y^2x\mathbf{j},$$

4.
$$\mathbf{q} = -x\mathbf{i} - y\mathbf{j}$$

5.
$$\mathbf{q} = x^2\mathbf{i} + y^2\mathbf{j}$$

6.
$$\mathbf{q} = 2x\mathbf{i} - y\mathbf{j}.$$

7.
$$\mathbf{q} = -x\mathbf{i} + y\mathbf{j}$$

8.
$$\mathbf{q} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

9. *Draw the vector fields (using arrows) defined under points 2–7 above. Consider the shape of the vector fields near the origin. For which ones is it obvious that divergence is definitely positive or negative at the origin from these plots alone? Why? (Hint. Think of how the components of the vector field change with x and y — can you tell when these rates of change are positive or negative? Then look at the definition of divergence in terms of the partial derivatives of the components of* $\mathbf{q}$*.)*

10. *Look at the plots in figure 3. Which panels show vector fields with positive divergences at the origin? Negative divergences? Give a reason for your answer — you should do this in much the same way as the previous problem.*

Exercise 2 *Again, we can think of the divergence operator as an extension of an ordinary derivative, and chain and product rules have extensions here too. The basic idea behind this follows an exercise in the notes on heat flux, which we illustrate with the following example: Let* $\mathbf{q}(x, y, z) = \rho(x, y, z)\mathbf{v}(x, y, z)$*. Then we can show that*

$$\nabla \cdot \mathbf{q} = \rho\nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla\rho$$

where $\nabla\rho$ *is the* gradient *of* ρ*, defined in* (6) *above. To do this, write write* $\mathbf{v}$ *in component form as* $\mathbf{v} = v_x\mathbf{i} + v_y\mathbf{j} + v_z\mathbf{k}$*, where* v_x*,* v_y *and* v_z *are functions of* x*,* y *and* z *whose form is determined by the vector field* $\mathbf{v}$*.*[2] *Then we have*

$$\mathbf{q} = \rho v_x\mathbf{i} + \rho v_y\mathbf{j} + \rho v_z\mathbf{k},$$

[2]To see this note that we can identify $v_x(x, y, z) = \mathbf{v}(x, y, z) \cdot \mathbf{i}$, and similarly for v_y and v_z.

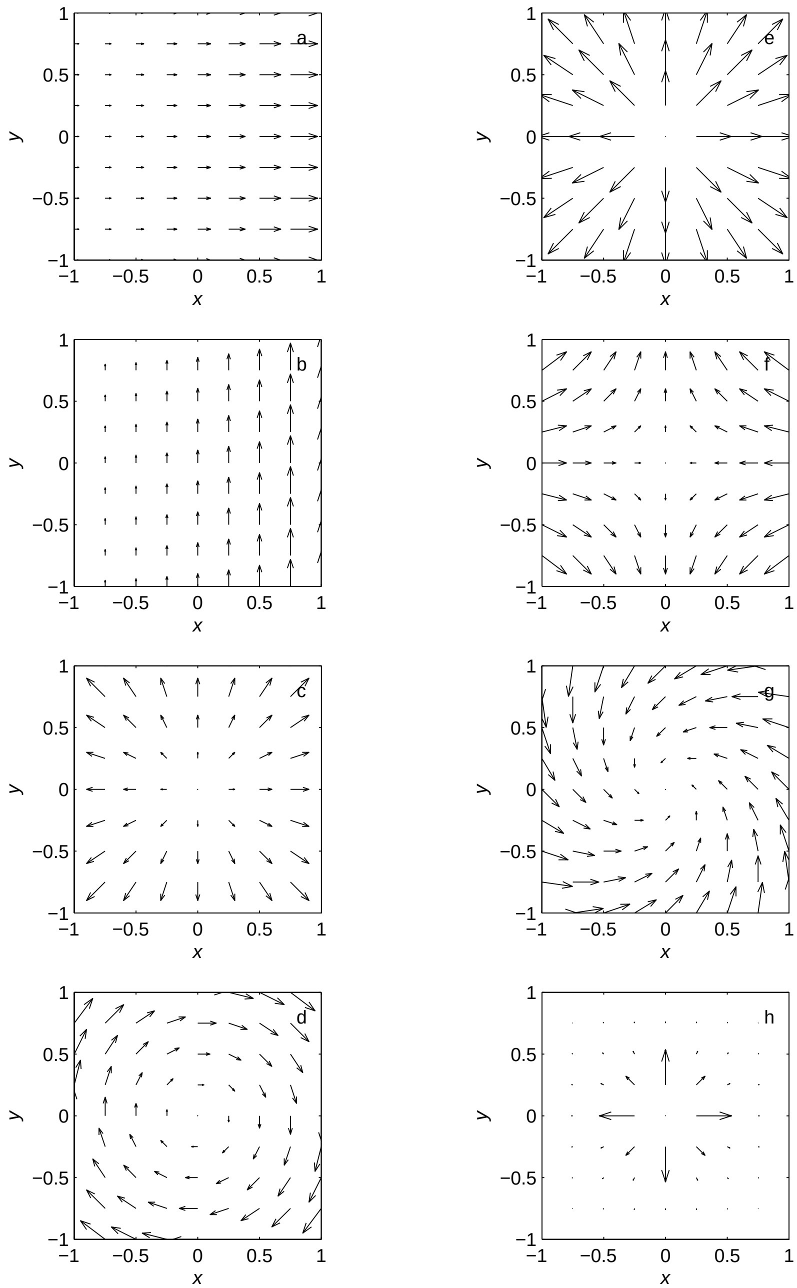

Figure 3: Various examples of vector fields, shownv at a number of grid points. Identify those examples for which the divergence $\nabla \cdot \mathbf{q}$ is definitely positive or negative at the origin. To do this, consider how the x- and y-components of the vector field shown change with x and y, and identify examples for which you can see that $\partial q_x/\partial x > 0$, $\partial q_y/\partial y > 0$ at the origin, or the equally, for which $\partial q_x/\partial x < 0$, $\partial q_y/\partial y < 0$ at the origin. To make this easier, you can try to sketch how $q_x(x, y)$ depends on x for $y = 0$ (i.e, along the x-axis), and similarly, you can sketch how $q_y(x, y)$ depends on y for $x = 0$ (i.e., along the y-axis).

and so

$$\nabla \cdot \mathbf{q} = \frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z}.$$

But now we can apply the product rule to each term on the right-hand side, e.g.,

$$\frac{\partial(\rho v_x)}{\partial x} = \frac{\partial \rho}{\partial x} v_x + \rho \frac{\partial v_x}{\partial x}$$

and similarly for $\partial(\rho v_y)/\partial y$ *and* $\partial(\rho v_z)/\partial z$. *Hence*

$$\begin{aligned}
\nabla \cdot \mathbf{q} &= \frac{\partial(\rho v_x)}{\partial x} + \frac{\partial(\rho v_y)}{\partial y} + \frac{\partial(\rho v_z)}{\partial z} \\
&= \left(\frac{\partial \rho}{\partial x} v_x + \rho \frac{\partial v_x}{\partial x}\right) + \left(\frac{\partial \rho}{\partial y} v_y + \rho \frac{\partial v_y}{\partial y}\right) + \left(\frac{\partial \rho}{\partial z} v_z + \rho \frac{\partial v_z}{\partial z}\right) \\
&= \left(\rho \frac{\partial v_x}{\partial x} + \rho \frac{\partial v_y}{\partial y} + \rho \frac{\partial v_z}{\partial z}\right) + \left(\frac{\partial \rho}{\partial x} v_x + \frac{\partial \rho}{\partial y} v_y + \frac{\partial \rho}{\partial z} v_z\right) \\
&= \rho \nabla \cdot \mathbf{v} + \mathbf{v} \cdot \nabla \rho.
\end{aligned}$$

The derivation above shows a that, when trying to derive results that involve the differential operator ∇, *it is usually necessary to write out the relevant expressions explicitly in terms of partial derivatives, and to manipulate these using the standard differentiation rules you already known about.*

Using a similar approach, as well as the specific result we have just derived, show the following:

1. *Let* $\mathbf{q}(x, y, z) = q(r)\hat{\mathbf{r}}$ *where* $r = \sqrt{x^2 + y^2 + z^2}$ *and* $\hat{\mathbf{r}} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})/r$ *and* $q(r)$ *is a function. Using the product rule result above, show that*

   $$\nabla \cdot \mathbf{q} = \frac{1}{r^2} \frac{\mathrm{d}(r^2 q)}{\mathrm{d}r}.$$

   Explain what this result is relevant for. What type of geometry / symmetry does the vector field $\mathbf{q}$ *have? Sketch an example of such a vector field.*

2. *Let* $\mathbf{q} = \mathbf{q}(\phi(x, y, z))$. *Show that*

   $$\nabla \cdot \mathbf{q} = \frac{\mathrm{d}\mathbf{q}}{\mathrm{d}\phi} \cdot \nabla \phi,$$

   where, if $\mathbf{q} = q_x\mathbf{i} + q_y\mathbf{j} + q_z\mathbf{k}$, *then the 'ordinary derivative' on the right-hand side is given by*

   $$\frac{\mathrm{d}\mathbf{q}}{\mathrm{d}\phi} = \frac{\mathrm{d}q_x}{\mathrm{d}\phi}\mathbf{i} + \frac{\mathrm{d}q_y}{\mathrm{d}\phi}\mathbf{j} + \frac{\mathrm{d}q_z}{\mathrm{d}\phi}\mathbf{k}.$$

3. *Let* $\mathbf{q}_1(x,y,z)$ *and* $\mathbf{q}_2(x,y,z)$ *be two different vector fields, and* c_1 *and* c_2 *be constants. Define* $\mathbf{q}(x,y,z) = c_1\mathbf{q}_1(x,y,z) + c_2\mathbf{q}_2(x,y,z)$*. Show that*

$$\nabla \cdot \mathbf{q} = c_1 \nabla \cdot \mathbf{q}_1 + c_2 \nabla \cdot \mathbf{q}_2.$$

Exercise 3 *Consider the divergence of a vector field in two dimensions,* $\mathbf{q} = q_x(x,y)\mathbf{i} + q_y(x,y)\mathbf{j}$*, so that*

$$\nabla \cdot \mathbf{q} = \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y}.$$

Now transform to a rotated coordinate system (x', y')*, in which*

$$x' = x\cos(\theta) + y\sin(\theta), \qquad y' = y\cos(\theta) - x\sin(\theta),$$

where θ *is a constant angle of rotation (see the notes on mathematical background). Define the unit vectors in the* $x'y'$*-coordinate system by*

$$\mathbf{i}' = \mathbf{i}\cos(\theta) + \mathbf{j}\sin\theta, \qquad \mathbf{j}' = -\mathbf{i}\sin(\theta) + \mathbf{j}\cos(\theta).$$

Use this to define components of $\mathbf{q}$ *in the* $x'y'$*-coordinate system so that*

$$\mathbf{q} = q'_{x'}\mathbf{i}' + q'_{y'}\mathbf{j}'.$$

1. *Express* $q'_{x'}$ *and* $q'_{y'}$ *in terms of* q_x*,* q_y *and* θ*.*

2. *Using the chain rule and your answer above, show that*

$$\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} = \frac{\partial q'_{x'}}{\partial x'} + \frac{\partial q'_{y'}}{\partial y'}.$$

   Why is this important?

The local form of a conservation revisited

In deriving the mass conservation equation (7) from (1), we skated over some technical detail. For instance, in writing down (3), we approximated the surface integral over the left-hand face S_1 by

$$\int_{S_1} \mathbf{q} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = -q_x(x,y,z)\delta y \delta z,$$

and the integral over S_2 by

$$\int_{S_2} \mathbf{q} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = -q_x(x+\delta z,y,z)\delta y \delta z$$

on the basis that we had moved a distance δx along the x-axis. This does give the right answer, but if you think about what we did carefully, you realize that we might not just want to worry about changes in going from x to $x + \delta x$. When integrating over S_1 and S_2, the other two coordinates also vary, from y to $y + \delta y$ and z to $z + \delta z$, and yet we are treating these coordinates as being constant when writing the flux components as $-q_x(x, y, z)$ and $-q_x(x + \delta z, y, z)$. Why not, for instance, write $-q_x(x, y + \delta y, z)$ and $q_x(x + \delta x, y, z + \delta z)$?

A better, more detailed way, of doing the same calculation would have been to write the integrals over S_1 and S_2 explicitly as

$$\int_{S_1} \mathbf{q} \cdot \hat{\mathbf{n}} \, \mathrm{d}S = \int_y^{y+\delta y} \int_z^{z+\delta z} -q_x(x, y', z') \, \mathrm{d}z' \, \mathrm{d}y'$$

and

$$\int_{S_2} \mathbf{q} \cdot \hat{\mathbf{n}} \, \mathrm{d}S = \int_y^{y+\delta y} \int_z^{z+\delta z} q_x(x + \delta x, y', z') \, \mathrm{d}z' \, \mathrm{d}y'$$

where y' and z' are the relevant dummy variables — we cannot use y and z simultaneously as limits and as integration variables.

Adding the two gives

$$\begin{aligned}
\int_{S_1} \mathbf{q} \cdot \hat{\mathbf{n}} \, \mathrm{d}S + \int_{S_2} \mathbf{q} \cdot \hat{\mathbf{n}} \, \mathrm{d}S &= \int_y^{y+\delta y} \int_z^{z+\delta z} q_x(x + \delta x, y', z') - q_x(x, y', z') \, \mathrm{d}z' \, \mathrm{d}y' \\
&= \int_y^{y+\delta y} \int_z^{z+\delta z} \frac{q_x(x + \delta x, y', z') - q_x(x, y', z')}{\delta x} \delta x \, \mathrm{d}z' \, \mathrm{d}y' \\
&\approx \int_y^{y+\delta y} \int_z^{z+\delta z} \frac{\partial q_x}{\partial x}(x, y', z') \delta x \, \mathrm{d}z' \, \mathrm{d}y',
\end{aligned}$$

and treating the partial derivative as approximately constant over the range $y < y' < y + \delta y$, $z < z' < z + \delta z$, we get back

$$\int_{S_1} \mathbf{q} \cdot \hat{\mathbf{n}} \, \mathrm{d}S + \int_{S_2} \mathbf{q} \cdot \hat{\mathbf{n}} \, \mathrm{d}S \approx \frac{\partial q_x}{\partial x} \delta x \delta y \delta z, \tag{9}$$

and we can do the same to the other faces of the cuboid in figure 1. Our original derivation therefore still works, but has become a lot more messy. There is a better approach that uses a very general and maybe surprising result that links surface and volume integrals. That result is the *divergence theorem*, also known as *Gauss's theorem*.

The divergence theorem

The divergence theorem states that, for a volume V with a sufficiently smooth surface S and a sufficiently smooth vector field $\mathbf{q}$,

$$\int_S \mathbf{q} \cdot \hat{\mathbf{n}} \, \mathrm{d}S = \int_V \nabla \cdot \mathbf{q} \, \mathrm{d}V,$$

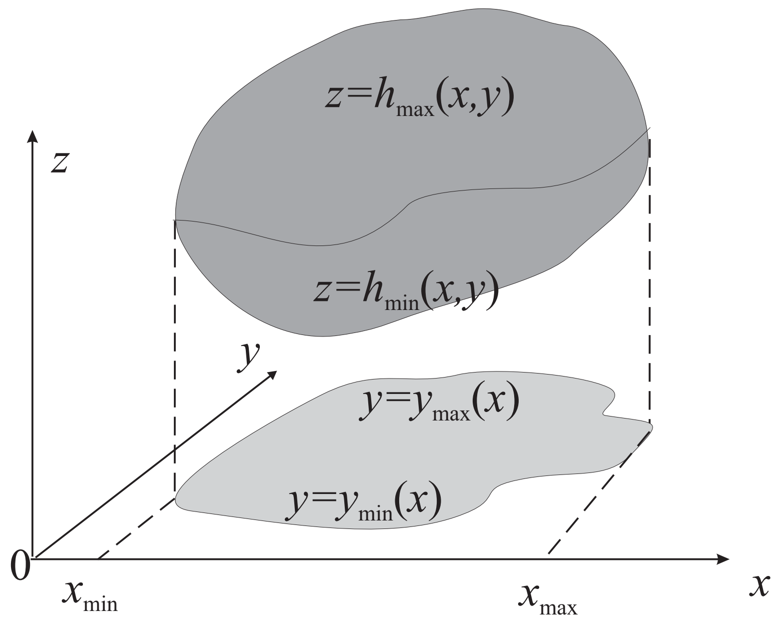

Figure 4: Geometry of the surface S of the volume V in the 'proof' of the divergence theorem.

where $\hat{\mathbf{n}}$ is the outward-pointing unit normal to S. As will become clearer below, we can think of the divergence theorem as a generalization of the fundamental theorem of calculus.

We will not present a general proof of the divergence theorem here. Instead, we simply demonstrate a brief calculation that shows that the divergence theorem is a plausible result for general geometries and vector fields, not just for the parallel-sided slab example above.

Let V be the volume between an upper surface S_{max} given by $z = h_{max}(x, y)$ and a lower surface S_{min} given by $z = h_{min}(x, y)$, which meet at a common boundary whose projection onto the xy-plane is a curve that can be divided into two parts, $y = y_{min}(x)$ and $y = y_{max}(x)$ (figure 4). Consider then the integral over V of the divergence of some vector field $\mathbf{q} = q_x\mathbf{i} + q_y\mathbf{j} + q_z\mathbf{k}$, which we can write as

$$\begin{aligned}\int_V \nabla \cdot \mathbf{q}\,\mathrm{d}V &= \int_V \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}\,\mathrm{d}V \\ &= \int_V \frac{\partial q_x}{\partial x}\,\mathrm{d}V + \int_V \frac{\partial q_y}{\partial y}\,\mathrm{d}V + \int_V \frac{\partial q_z}{\partial z}\,\mathrm{d}V\end{aligned}$$

Next, we focus only on the last integral, $\int_V \frac{\partial q_z}{\partial z}\,\mathrm{d}V$, in the expectation that there is nothing special about the x- or y-directions. But, in terms of multiple integrals, we can write the volume integral as

$$\int_V \frac{\partial q_z}{\partial z}\,\mathrm{d}V = \int_{x_{min}}^{x_{max}} \left[\int_{y_{min}(x)}^{y_{max}(x)} \left(\int_{h_{min}(x,y)}^{h_{max}(x,y)} \frac{\partial q_z}{\partial z}\,\mathrm{d}z \right) \mathrm{d}y \right] \mathrm{d}x.$$

Next, we can recognize that the innermost integral is the integral with respect to z of a partial derivative with respect to z. Using the fundamental theorem of calculus, we can evaluate this integral, and are left with

$$\begin{aligned}
\int_V \frac{\partial q_z}{\partial z}\,\mathrm{d}V &= \int_{x_{min}}^{x_{max}} \left[\int_{y_{min}(x)}^{y_{max}(x)} q_z(x,z,h_{max}(x,y)) - q_z(z,x,h_{min}(x,y))\,\mathrm{d}y \right] \mathrm{d}x \\
&= \int_{x_{min}}^{x_{max}} \left[\int_{y_{min}(x)}^{y_{max}(x)} q_z(x,z,h_{max}(x,y))\,\mathrm{d}y \right] \mathrm{d}x \\
&\quad - \int_{x_{min}}^{x_{max}} \left[\int_{y_{min}(x)}^{y_{max}(x)} q_z(x,z,h_{min}(x,y))\,\mathrm{d}y \right] \mathrm{d}x. \qquad (10)
\end{aligned}$$

At this point, we have to recall the formula for the integral of the normal component of a flux field $\mathbf{q}$ over a surface $z = h(x,y)$:

$$\int_S \mathbf{q}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \int_{x_{min}}^{x_{max}} \left[\int_{y_{min}(x)}^{y_{max}(x)} q_z(x,y,h(x,y)) - q_x(x,y,h(x,y))\frac{\partial h}{\partial x} - q_y(x,y,h(x,y))\frac{\partial h}{\partial y}\,\mathrm{d}y \right] \mathrm{d}x \qquad (11)$$

if $\hat{\mathbf{n}}$ is upward-pointing from S (i.e., upward-pointing relative to the z-axis) , and the same expression but with the signs on the right-hand side reversed if $\hat{\mathbf{n}}$ is downward-pointing. But we know that the outward-pointing unit normal $\hat{\mathbf{n}}$ is upward-pointing on S_{max} and downward-pointing on S_{min}, and so we can recognize

$$\int_{x_{min}}^{x_{max}} \left[\int_{y_{min}(x)}^{y_{max}(x)} q_z(x,z,h_{max}(x,y))\,\mathrm{d}y \right] \mathrm{d}x = \int_{S_{max}} q_z\mathbf{k}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$$

as well as

$$\int_{x_{min}}^{x_{max}} \left[\int_{y_{min}(x)}^{y_{max}(x)} q_z(x,z,h_{min}(x,y))\,\mathrm{d}y \right] \mathrm{d}x = \int_{S_{max}} q_z\mathbf{k}\cdot\hat{\mathbf{n}}\,\mathrm{d}S.$$

From (10), we therefore have

$$\begin{aligned}
\int_V \frac{\partial q}{\partial z}\,\mathrm{d}V &= \int_{S_{max}} q_z\mathbf{k}\cdot\hat{\mathbf{n}}\,\mathrm{d}S + \int_{S_{min}} q_z\mathbf{k}\cdot\hat{\mathbf{n}}\,\mathrm{d}S \\
&= \int_S q_z\mathbf{k}\cdot\hat{\mathbf{n}}\,\mathrm{d}S \qquad (12)
\end{aligned}$$

Now, if there is nothing special about the z-direction, then by symmetry, we should also have

$$\int_V \frac{\partial q_x}{\partial x}\,\mathrm{d}V = \int_S q_x\mathbf{i}\cdot\ \hat{\mathbf{n}}\,\mathrm{d}S \tag{13a}$$

$$\int_V \frac{\partial q_y}{\partial y}\,\mathrm{d}V = \int_S q_y\mathbf{j}\cdot\hat{\mathbf{n}}\,\mathrm{d}S. \tag{13b}$$

In fact, we should be able to reformulate volume and surface integrals by describing for instance the surface of S through its height above, say the xz-plane and integrating with respect to y first for the $\partial q_y/\partial y$ term, and subsequently integrating with respect to z and x, and similarly for the $\partial q_x/\partial x$ term. Each of these computations would then be perfectly analogous to how we showed that (12) holds, except with the roles of z and y or z and x reversed.[3]

Adding (13) to (12) then gives

$$\begin{aligned}\int_V \nabla\cdot\mathbf{q}\,\mathrm{d}V &= \int_V \frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}+\frac{\partial q_z}{\partial z}\,\mathrm{d}V\\ &= \int_S [q_x\mathbf{i}+q_y\mathbf{j}+q_z\mathbf{k}]\cdot\hat{\mathbf{n}}\,\mathrm{d}S\\ &= \int_S \mathbf{q}\cdot\hat{\mathbf{n}}\,\mathrm{d}S,\end{aligned}$$

which is the divergence theorem.

Exercise 4 *Verify the divergence theorem if V is the tetrahedron with vertices at $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$ and $\mathbf{q} = x^2\mathbf{i}+y^2\mathbf{j}+z^2\mathbf{k}$. That is, compute $\int_S \mathbf{q}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$ and $\int_V \nabla\cdot\mathbf{q}\,\mathrm{d}V$ and verify that they are the same.*

[3] To make this work, the surface of V would need to consist not only of a single upper and lower surface when view from the xy-plane but also from the xz-planes and yz-planes. Howver, as note 4 demonstrates, we can use the fact that the divergence theorem holds for simple volumes that do not fold back on themselves to show that it also holds for volumes that do fold back on themselves and do not have single upper and lower surfaces, so this is not the main issue.

Harder to see is that the formulation of the surface integral with z given on the surface as a function of x and y (so we are integrating over x and y) is indeed equivalent to formulating the surface as having y given as a function of x and z. This can be done by applying a change of variable to (11), but this is not trivial: what is required is a change of variables from y to z for the $q_y\partial h/\partial y$ term in the integrand in (11), and similarly a change of variables from x to z for the term $q_x\partial h\partial x$. This can be done by recognizing that we basically have $z = h(x,y)$ and so we expect that we can put

$$\int\int q_y\partial h/\partial y\,\mathrm{d}y\,\mathrm{d}x = \int\int q_y\,\mathrm{d}z\,\mathrm{d}x.$$

This however still requires careful handling of the change of variable, because the sign of $\partial h/\partial y$ and $\partial h/\partial x$ will generally change summer along the original upper and lower surfaces S_{max} and S_{min}. These are not complications worth discussing in detail here. As before, our aim is not mathematical rigour but physical understanding of why calculus is useful and what various operations represent physically.

Exercise 5 *Often you can use the divergence theorem to compute surface integrals much more easily. Let*

$$\mathbf{q} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}}.$$

If S is the triangle with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$, and $\hat{\mathbf{n}}$ is the upward-pointing unit normal, show that $\int_S \mathbf{q}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \pi/2$. Hint: *Apply the divergence theorem to the volume lying between the surface S and the part of the sphere with radius 1 that lies in the positive octant (i.e., for which $x > 0$, $y > 0$ and $z > 0$). Draw a diagram to understand this better.*

Exercise 6 *Use the divergence theorem to compute $\int_S \mathbf{q}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$ if S is the surface of the tetrahedron in example 4 above, and $\mathbf{q} = (x-y-z)\mathbf{i} + (x+y)\mathbf{j} + (z-x-y^2)\mathbf{k}$.*

Note 4 *The divergence theorem still holds if the volume V does not have a single upper and lower surface. Consider the volume in figure* **??**, *which folds back on itself. We can show that the divergence theorem holds for this volume by using the fact that it holds for volumes that do not fold back on themselves. Consider the split of V into the volumes V_1 and V_2 shown. Both of V_1 and V_2 have a single upper and lower surface, so we know the divergence theorem holds for each. When splitting V, we create a new surface S_{int}. S_{int} is both, part of the boundary S_1 of V_1 and part of the boundary S_2 of V_2, but the outward-pointing normals $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{n}}_2$ are equal and opposite in the two cases.*

The surface integral over the surface S of V is related to the surface integrals over S_1 and S_2 through

$$\int_S \mathbf{q}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \int_{S_1} \mathbf{q}\cdot\hat{\mathbf{n}}_1\,\mathrm{d}S - \int_{S_{int}} \mathbf{q}\cdot\hat{\mathbf{n}}_1\,\mathrm{d}S + \int_{S_2} \mathbf{q}\cdot\hat{\mathbf{n}}\,\mathrm{d}S - \int_{S_{int}} \mathbf{q}\cdot\hat{\mathbf{n}}_2\,\mathrm{d}S,$$

the subtraction being necessary to remove the contribution from the newly-created cuts. But, as $\hat{\mathbf{n}}_2 = -\hat{\mathbf{n}}_1$, we have

$$-\int_{S_{int}} \mathbf{q}\cdot\hat{\mathbf{n}}_1\,\mathrm{d}S - \int_{S_{int}} \mathbf{q}\cdot\hat{\mathbf{n}}_2\,\mathrm{d}S = -\int_{S_{int}} \mathbf{q}\cdot(\hat{\mathbf{n}}_1 - \hat{\mathbf{n}}_2)\,\mathrm{d}S = 0$$

and so

$$\int_S \mathbf{q}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \int_{S_1} \mathbf{q}\cdot\hat{\mathbf{n}}_1\,\mathrm{d}S + \int_{S_2} \mathbf{q}\cdot\hat{\mathbf{n}}\,\mathrm{d}S;$$

the contributions from the two sides of the cut cancel. But we can apply the divergence theorem to V_1 and V_2,

$$\int_{S_1} \mathbf{q}\cdot\hat{\mathbf{n}}_1\,\mathrm{d}S = \int_{V_1} \nabla\cdot\mathbf{q}\,\mathrm{d}V$$
$$\int_{S_2} \mathbf{q}\cdot\hat{\mathbf{n}}_2\,\mathrm{d}S = \int_{V_2} \nabla\cdot\mathbf{q}\,\mathrm{d}V$$

so

$$\int_S \mathbf{q}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \int_{V_1} \nabla\cdot\mathbf{q}\,\mathrm{d}V + \int_{V_2} \nabla\cdot\mathbf{q}\,\mathrm{d}V = \int_V \nabla\cdot\mathbf{q}\,\mathrm{d}V,$$

the last equality holding because V_1 and V_2 do not overlap and together make up the volume V. Therefore we have the divergence theorem for V,

$$\int_S \mathbf{q}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \int_V \nabla\cdot\mathbf{q}\,\mathrm{d}V,$$

Note 5 *The divergence theorem gives us a way of generalizing the interpretation we gave for a divergence in 2, which referred only to s mall cuboids $\delta x\delta y\delta z$. If we take the volume integral of a divergence over a very small volume V, we have approximately*

$$\int_V \nabla\cdot\mathbf{q}\,\mathrm{d}V \approx (\nabla\cdot\mathbf{q})V,$$

treating $\nabla\cdot\mathbf{q}$ as approximately constant over V. However, from the divergence theorem, we also have

$$\int_V \nabla\cdot\mathbf{q}\,\mathrm{d}V = \int_S \mathbf{q}\cdot\hat{\mathbf{n}}\,\mathrm{d}S,$$

where S is the surface of ΔV. Hence

$$\nabla\cdot\mathbf{q} \approx \frac{1}{V}\int_S \mathbf{q}\cdot\hat{\mathbf{n}}\,\mathrm{d}S,$$

or better,

$$\nabla\cdot\mathbf{q} = \lim_{V\to 0}\frac{1}{V}\int_S \mathbf{q}\cdot\hat{\mathbf{n}}\,\mathrm{d}S.$$

The divergence of a flux is the rate at which flux leads to mass flow out of a small volume, divided by the size of that volume, no matter the shape of the volume.

Exercise 7 *The result of the previous note often gives a quick way for determining the form of a divergence in a coordinate system that is not Cartesian. Take for instance a cylindrical polar coordinate system with coordinates (r,θ,z) linked to the Cartesian coordinates (x,y,z) through*

$$x = r\cos(\theta), \qquad y = r\sin(\theta).$$

In the polar coordinate system, a vector field would typically be represented in terms of unit vectors that are aligned with the coordinate axes. We will denote these by $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$ and $\hat{\mathbf{z}}$. Unlike in a Cartesian system, where the coordinate axes point in the same direction regardless of where a point (x,y,z) is located, the unit vectors $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ point in different directions at different points in space; they depend on r and θ. In fact, these unit vectors are related to the fixed unit vector $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ through

$$\hat{\mathbf{r}} = \mathbf{i}\cos(\theta) + \mathbf{j}\sin(\theta), \qquad \hat{\boldsymbol{\theta}} = -\mathbf{i}\sin(\theta) + \mathbf{j}\cos(\theta), \qquad \hat{\mathbf{z}} = \mathbf{k}.$$

If we write a vector field $\mathbf{q}$ *in the form*

$$\mathbf{q}(r,\theta,z) = q_r(r,\theta,z)\hat{\mathbf{r}} + q_\theta(r,\theta,z)\hat{\boldsymbol{\theta}} + q_z(r,\theta,z)\hat{\mathbf{z}}, \tag{14}$$

the divergence of $\mathbf{q}$ *will then* not *be given*

$$\nabla\cdot\mathbf{q} = \frac{\partial q_r}{\partial r} + \frac{\partial q_\theta}{\partial \theta} + \frac{\partial q_z}{\partial z}.$$

It is easy to see that this cannot be right on dimensional grounds alone. Imagine that $\mathbf{q}$ *has the units of a mass flux, so kg m*$^{-2}$ *s*$^{-1}$*. The divergence then has units of kg m*$^{-3}$ *s*$^{-1}$*, because differentiation with respect to a spatial coordinate like* x*,* y *or* z *leads to the units being divded by m. This means* $\partial q_r/\partial r$ *and* $\partial q_z/\partial z$ *would have the right units.* $\partial q_\theta/\partial\theta$ *on the other hand would not — angles essentially have no units (they are ratios of two distances) and so* $\partial q_\theta/\partial\theta$ *has units of kg m*$^{-2}$ *s*$^{-1}$*.*

Part 1 in exercise 2 gives you a flavour of how to be sure you get the right formula for divergence by starting from the definition

$$\nabla\cdot\mathbf{q} = \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}$$

in a Cartesian system, performing a change of variables and using the chain rule. In that exercise $\hat{\mathbf{r}}$*, is a radial unit vector in a spherical polar coordinate system, meaning a unit vector pointing radially away from the origin at a given point* (x,y,z)*, and* r *is the distance from the origin to that point. However, by assuming that the vector field takes the form* $\mathbf{q} = q(r)\hat{\mathbf{r}}$*, the exercise assumes a certain symmetry, so it does not give you a complete formula for the coordinate transformation.*

That said, we can pursue the same approach here by writing (14) *in Cartesian form. Substituting for* $\hat{\mathbf{r}}$*,* $\hat{\theta}$ *and* $\hat{\mathbf{z}}$ *in terms of* $\mathbf{i}$*,* $\mathbf{j}$ *and* $\mathbf{k}$ *gives*

$$\begin{aligned}\mathbf{q}(r,\theta,z) =& \left[q_r(r,\theta,z)\cos(\theta) - q_\theta(r,\theta,z)\sin(\theta)\right]\mathbf{i}\\ &+ \left[q_\theta(r,\theta,z)\cos(\theta) + q_r(r,\theta,z)\sin(\theta)\right]\mathbf{j} + q_z(r,\theta,z)\mathbf{k}\end{aligned}$$

so that

$$q_x = q_r(r,\theta,z)\cos(\theta) - q_\theta(r,\theta,z)\sin(\theta), \qquad q_y = q_\theta(r,\theta,z)\cos(\theta) + q_r(r,\theta,z)\sin(\theta),$$
$$q_z = q_z(r,\theta,z),$$

and the divergence becomes

$$\begin{aligned}\nabla\cdot\mathbf{q} =& \frac{\partial}{\partial x}\left[q_r(r,\theta,z)\cos(\theta) - q_\theta(r,\theta,z)\sin(\theta)\right]\\ r &+ \frac{\partial}{\partial y}\left[q_\theta(r,\theta,z)\cos(\theta) + q_r(r,\theta,z)\sin(\theta)\right] + \frac{\partial q_z}{\partial z}.\end{aligned} \tag{15}$$

There is a second, simpler way. If, as the divergence theorem tells us,

$$\nabla \cdot \mathbf{q} = \lim_{V \to 0} \frac{1}{V} \int_S \mathbf{q} \cdot \hat{\mathbf{n}} \, \mathrm{d}S$$

regardless of the shape of the small volume V (so long as it is 'small' in all directions), then we can pick a volume shape that makes the computation of the surface integral easy and leads to simple derivatives as we also did in (4) *(or better, as we did in* (4)*, though that is harder). The obvious thing to do is to pick a small volume, shaped so that any point (r, θ, z) inside that volume satisfies $r_0 < r < r_0 + \delta r$, $\theta_0 < \theta < \theta_0 + \delta\theta$, $z_0 < z < z_0 + \delta z$ for some fixed r_0, θ_0, z_0 and δr, $\delta\theta$, δz.*

1. *Sketch the r, θ coordinate plane with lines of constant r and θ indicated by dashed lines. This is like looking at the (r, θ, z) coordinate system down the z-axis. Plot what the volume V looks like in this perspective.*

2. *By looking the size faces of the the volume just described, show that we have*

$$\begin{aligned}\int_S \mathbf{q} \cdot \hat{\mathbf{n}} \, \mathrm{d}S \approx & (r_0 + \delta r)\delta\theta\delta z q_(r_0 + \delta r, \theta_0, z_0) - (r_0 + \delta r)\delta\theta\delta z q_(r_0 + \delta r, \theta_0, z_0) \\ & + \delta r \delta z q_\theta(r_0, \theta_0 + \delta\theta, z_0) - \delta r \delta z q_\theta(r_0, \theta_0, z_0) \\ & + r\delta\theta\delta r q_z(r_0, \theta_0, z_0 + \delta z) - r\delta\theta\delta r q_z(r_0, \theta_0, z_0)\end{aligned}$$

   Be careful to show why the factors multiplying the flux components should take the form they do — remember these factors describe the surface areas these flux components pass through. Show also that

$$V \approx r\delta\theta\delta r\delta z.$$

   Therefore show that

$$\nabla \cdot \mathbf{q} = \frac{1}{r}\frac{\partial (r q_r)}{\partial r} + \frac{1}{r}\frac{\partial q_\theta}{\partial \theta} + \frac{\partial q_z}{\partial z}. \tag{16}$$

3. *Apply the chain rule to* (15) *to show directly that* (16) *holds.*

Exercise 8 *Adapt the method used in part 2 above to a spherical polar coordinate system (r, θ, ϕ), with r the radial coordinate, θ the longitude and ϕ the colatitude, to show that*

$$\nabla \cdot \mathbf{q} = \frac{1}{r^2}\frac{\partial (r^2 q_r)}{\partial r} + \frac{1}{r\sin\phi}\frac{\partial q_\theta}{\partial \theta} + \frac{1}{r\sin(\phi)}\frac{\partial(\sin(\phi) q_\phi)}{\partial \phi}.$$

Note that some texts define θ as co-latitude, meaning π radians or 90 degrees minus the ordinary latitude, and ϕ as longitude, in which case the roles of θ and ϕ are switched.

Deriving local forms of conservation laws using the divergence theorem

One of the main applications of, and our main motivation for introducing, the divergence is to turn integral conservation laws of the form (1) into partial differential equations. We already have such a differential equation in (7). Here, we will see an alternative derivation of the same equation using a more general method.

Based on what we have done above, this is now quite easy. Applying the divergence theorem to (1), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\,\mathrm{d}V = -\int_V \nabla\cdot\mathbf{q}\,\mathrm{d}V. \tag{17}$$

The last thing that is needed is to recognize that we can turn the derivative of an integral on the left-hand side into the integral of a derivative. To do so, remember how a derivative is defined: in general

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \lim_{\delta t\to 0}\frac{f(t+\delta t)-f(t)}{\delta t}.$$

Here, f is given by an integral,

$$f(t) = \int_V \rho(x,y,z,t)\,\mathrm{d}V$$

where the volume V does not change with time, and the defintie integration over the fixed volume V makes sure that f does not depend on x, y or z. Using the definition of the derivative, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\,\mathrm{d}V = \lim_{\delta t\to 0}\frac{\int_V \rho(x,y,z,t+\delta t)\,\mathrm{d}V - \int_V \rho(x,y,z,t)\,\mathrm{d}V}{\delta t}.$$

Rearranging the right-hand side, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\,\mathrm{d}V = \lim_{\delta t\to 0}\int_V \frac{\rho(x,y,z,t+\delta t)-\rho(x,y,z,t)}{\delta t}\,\mathrm{d}V.$$

Now, in theory there are technical issues associated with taking the limit *inside* the integral — basically, the integral is already a limit itself (through the Riemann sum), and one could worry about whether the order in which limits are teken matters. However, in keeping with the rest of the course, we will not address this here. For sufficiently smooth functions, the order of taking limits can be interchanged, and so

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\,\mathrm{d}V = \int_V \lim_{\delta t\to 0}\frac{\rho(x,y,z,t+\delta t)-\rho(x,y,z,t)}{\delta t}\,\mathrm{d}V = \int_V \frac{\partial\rho}{\partial t}\,\mathrm{d}V.$$

Armed with this, we can then finally turn (17) into the form

$$\int_V \frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{q}\,\mathrm{d}V = 0. \tag{18}$$

Now all we have to do is recognize that V can again be any volume we like. So if there was a region in which the integrand was positive,[4] we could make V be that region and the integral would be positive, in contradiction to (18). This means there cannot be a region in which the integrand is positive. The same argument can be made to show that there cannot be a region in which the integrand is negative. If there are no regions in which the integrand is positive or negative, it must be zero everywhere. [5]. As a result,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{q} = 0. \tag{19}$$

This is once again the local form of the conservation law (1).

Note 6 *A simpler way of saying the same thing is: we need (18) to hold for any volume V, so we can pick a very small volume $V = \delta V$. For a very small volume, we can approximate the integrand as constant* — provided the integrand is continuous, i.e., has no abrupt jumps — *and so*

$$\int_{\Delta V} \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{q} \, \mathrm{d}V \approx \left[\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{q} \right] \delta V.$$

However, by (18), this must equal zero. Dividing by δV, we get

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{q} = 0.$$

The steps in this derivation can be summed up as

1. Start with the integral form
$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V \rho \, \mathrm{d}V = -\int_S \mathbf{q} \cdot \hat{\mathbf{n}} \, \mathrm{d}S$$

2. Turn the left-hand side into the integral over the derivative,
$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V \rho \, \mathrm{d}V = \int_V \frac{\partial \rho}{\partial t} \, \mathrm{d}V$$

3. Use the divergence theorem on the right to turn the surface integral into a volume integral,
$$-\int_S \mathbf{q} \cdot \hat{\mathbf{n}} \, \mathrm{d}S = -\int_V \nabla \cdot \mathbf{q} \, \mathrm{d}V$$

[4] f is positive if $f > 0$, not $f \geq 0$.

[5] A more technical statement would be: we assume that the integrand is continuous. If it were positive at some point, we could make V a sufficiently small volume around that point such that the integrand remains positive in V, so (18) would not hold for that V. Similarly, if the integrand were negative at some point, we could find an analogous small volume so the integrand remains negative in V, and (18) would not hold for that volume. Consequently, there cannot be any points at which the integrand is postive or negative, and it must therefore be zero.

4. Combine the last two steps to give

$$\int_V \frac{\partial \rho}{\partial t}\,\mathrm{d}V = -\int_V \nabla \cdot \mathbf{q}\,\mathrm{d}V,$$

or, as a single integral

$$\int_V \frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{q}\,\mathrm{d}V = 0.$$

5. This is the conceptually hard step: if the integral is zero for any shape and size of volume V, the *integrand* cannot be positive or negative anywhere (because we could make the *integral* positive or negative as a result). If the integrand is not positive or negative, it must be zero:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{q} = 0.$$

Of course, once we have the local form (19), it is possile to work backwards through these steps to get the original integral form (1), that is, the local form of the conservation law ensures that the integral form (1) holds for any volume V.

Exercise 9 *Remember that the mass flux* $\mathbf{q}$ *was given by* $\mathbf{q} = \rho\mathbf{v}$, *so* (19) *is the same as*

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{v}) = 0$$

Assume the material is incompressible, so ρ *is a constant. Find a differential equation satisfied by* $\mathbf{v}$ *alone (meaning, a differential equation that does not contain* ρ*). Write that equation out explicitly in terms partial derivatives of the components* v_x, v_y *and* v_z *of* $\mathbf{v}$. *Suppose you have a flow that slows in one direction (for instance,* v_x *decreases in the* x*-direction). What can you say about the other components? Describe a practical, everyday manifestation of this.*