Heat conduction

February 27, 2019

Overview

These notes cover the following:

- Gradient of a scalar field
- Conductive heat flux and the gradient of temperature: Fourier's law
- Heat capacity
- Derivation of the heat equation

Conservation of energy revisited

We have previously seen how to formulate conservation of energy in integral form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \varepsilon \,\mathrm{d}V = -\int_S \varepsilon \mathbf{v}\cdot\hat{\mathbf{n}}\,\mathrm{d}S - \int_S \mathbf{q}_c\cdot\hat{\mathbf{n}}\,\mathrm{d}S + \int_V a\,\mathrm{d}V. \tag{1}$$

and as a differential equation that is equivalent to the integral form (meaning the integral form requires the partial differential equation to hold, and if the partial differential equation holds, then so too must the integral form for any volume V):

$$\frac{\partial \varepsilon}{\partial t} + \nabla\cdot(\varepsilon\mathbf{v}) + \nabla\cdot\mathbf{q}_c - a = 0. \tag{2}$$

We have *two* fluxes here: the *advective* flux $\varepsilon\mathbf{v}$ and the *conductive* flux $\mathbf{q}_c$.

*except figure 4

We have also seen that this equation as stated contains too many unknows to be solved. Our stated aim is to relate both, thermal energy density ε and conductive heat flux $\mathbf{q}_c$ to the temperature field $T(x, y, z, t)$. We begin by looking at conduction. Conduction of heat is the exchange of thermal energy between different bits of material. This flow of heat is driven by temperature *differences*: heat flows from hot to cold. We need to quantify this. By 'quantify', we mean that we need to come up with a *plausible* mathematical model. Whether the model works or not can in practice only be determined by experiment.

Intuitively, conduction of heat requires the bits of material exchanging thermal energy to be in contact with each other. We would like to say that a larger temperature difference will lead to faster conduction of heat between the bits of material. However, in a continuum, there are no readily-defined 'bits of material' such that we can compute the temperature difference between them. In fact, temperature $T(x, y, z, t)$ is a smoothly-varying field without abrupt jumps that would define a 'difference', and we need to move a finite distance in order to experience a finite temperature change.

The point is not just that larger temperature differences will drive faster conduction, but that the distance over which those temperature differences occur matters. Going from a point (x, y, z, t) in any given direction, we expect that the change δT in temperature is proportional to the distance travelled δs, so long as δs is small. Heat should then preferentially flow in the direction in which the temperature change $|\delta T|$ corresponding to a given distance δs is as large as possible. As heat flows from hot to cold, the direction also needs to be such that δT is negative. The rate of heat flow should depend on $\delta T/\delta s$: the larger this ratio, the bigger the temperature difference δT between two different bits of material spaced a distance δs apart. To compute $\delta T/\delta s$ easily, we need a derivative of T that takes into account the direction associated with the distance δs. This derivative is the gradient of T.

The gradient of a scalar field

The *gradient* of the temperature field T is the quantity that allows us to determine both, the rate at which temperature changes with distance and the dependence of that rate on direction. The gradient of $T(x, y, z, t)$ is defined as

$$\nabla T = \frac{\partial T}{\partial x}\mathbf{i} + \frac{\partial T}{\partial y}\mathbf{j} + \frac{\partial T}{\partial z}\mathbf{k}. \qquad (3)$$

Note that while T is a scalar vector field, ∇T is a vector field.

Note 1 *The gradient ∇T defined in this way is the natural application of the vector differential operator*

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

to a scalar field $T(x, y, z, t)$, Note that gradient ∇T is a vector field. We can contrast this with the divergence of a vector field. The divergence is $\nabla \cdot \mathbf{q}$ is one natural way of applying ∇ to a vector field $\mathbf{q}(x, y, z, t)$, and turns that vector field into a scalar field.

The gradient of T has a number of useful attributes. The most important is this: consider the temperature change in going from a point with a general position vector

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

to another nearby point with position vector

$$\mathbf{r} + \delta\mathbf{r} = (x + \delta x)\mathbf{i} + (y + \delta y)\mathbf{j} + (z + \delta z)\mathbf{k}.$$

so $\delta\mathbf{r} = \delta x\mathbf{i} + \delta y\mathbf{j} + \delta z\mathbf{k}$. Let δT be the temperature difference between the two points,

$$\delta T = T(x + \delta x, y + \delta y, z + \delta z, t) - T(x, y, z, t).$$

Then δT is related to $\delta\mathbf{r}$ through the gradient as

$$\delta T \approx \nabla T \cdot \delta\mathbf{r}, \tag{4}$$

provided $\delta\mathbf{r}$ is very short.

This is easy to demonstrate if we write

$$\begin{aligned}
\delta T =& T(x + \delta x, y + \delta y, z + \delta z, t) - T(x, y, z, t) \\
=& T(x + \delta x, y + \delta y, z + \delta z) - T(x + \delta x, y + \delta y, z) \\
&+ T(x + \delta x, y + \delta y, z) - T(x + \delta x, y, z) \\
&+ T(x + \delta x, y, z) - T(z, y, z) \\
=& \frac{T(x + \delta x, y + \delta y, z + \delta z) - T(x + \delta x, y + \delta y, z)}{\delta z}\delta z \\
&+ \frac{T(x + \delta x, y + \delta y, z) - T(x + \delta x, y, z)}{\delta y}\delta y \\
&+ \frac{T(x + \delta x, y, z) - T(z, y, z)}{\delta x}\delta x
\end{aligned}$$

But the terms after the last equality can be recongized as being approximately equal to partial derivatives, so

$$\delta T \approx \left.\frac{\partial T}{\partial z}\right|_{(x+\delta x, y+\delta y, z)} \delta z + \left.\frac{\partial T}{\partial y}\right|_{(x+\delta x, y, z)} \delta y + \left.\frac{\partial T}{\partial x}\right|_{(x, y, z)} \delta y,$$

where the subscripts $_{(x,y,z)}$ indicate where the derivative is evaluated. But, if the partial derivatives are continuous, then for small δx, δy,

$$\left.\frac{\partial T}{\partial z}\right|_{(x+\delta x, y+\delta y, z)} \approx \left.\frac{\partial T}{\partial z}\right|_{(x, y, z)}, \qquad \left.\frac{\partial T}{\partial y}\right|_{(x+\delta x, y, z)} \approx \left.\frac{\partial T}{\partial y}\right|_{(x, y, z)}$$

and we can evaluate all three derivatives at (x, y, z), so the subscripts become obsolete. We have

$$\delta T = \frac{\partial T}{\partial x}\delta x + \frac{\partial T}{\partial y}\delta y + \frac{\partial T}{\partial z}\delta z.$$

With ∇T given by (3) and $\delta\mathbf{r} = \delta x\mathbf{i} + \delta y\mathbf{j} + \delta z\mathbf{k}$, we have

$$\delta T = \frac{\partial T}{\partial x}\delta x + \frac{\partial T}{\partial y}\delta y + \frac{\partial T}{\partial z}\delta z = \nabla T \cdot \delta\mathbf{r}$$

as promised.

Note 2 *If you are familiar with Taylor expansions of functions of three variables, you will know that*

$$T(x,y,z) = T(x_0,y_0,z_0) + \left.\frac{\partial T}{\partial x}\right|_{(x_0,y_0,z_0)}(x-x_0) + \left.\frac{\partial T}{\partial y}\right|_{(x_0,y_0,z_0)}(y-y_0) + \left.\frac{\partial T}{\partial z}\right|_{(x_0,y_0,z_0)}(z-z_0) + \ldots$$

where '...' stands for terms that are quadratic and higher order polynomials in $(x-x_0)$, $(y-y_0)$, $(z-z_0)$. *A bit of relabelling, putting* $\delta x = x - x_0$, $\delta y = y - y_0$, $\delta z = z - z_0$ *so that* $(x, y, z) = (x_0 + \delta x, y_0 + \delta y, z_0 + \delta z)$, *gives*

$$T(x_0 + \delta x, y_0 + \delta y, z_0 + \delta z) - T(x_0, y_0, z_0) = \frac{\partial T}{\partial x}\delta x + \frac{\partial T}{\partial y}\delta y + \frac{\partial T}{\partial z}\delta z$$

where the partial derivatives are evaluated at (x_0, y_0, z_0). (4) *is therefore nothing more than retaining only the linear terms in a Taylor expansion of* T *about the point* (x, y, z).

We can also express δT in terms of the distance between the two points $\mathbf{r}$ and $\mathbf{r} + \delta\mathbf{r}$. Define that distance

$$\delta s = |\delta\mathbf{r}| = \sqrt{\delta x^2 + \delta y^2 + \delta z^2}.$$

Also define a *unit vector* in the direction of the displacement $\delta\mathbf{r}$. We denote this unit vector by $\hat{\mathbf{t}}$,

$$\hat{\mathbf{t}} = \frac{\delta\mathbf{r}}{|\delta\mathbf{r}|}$$

so that $\delta\mathbf{r} = \hat{\mathbf{t}}\delta s$. Then (4) becomes

$$\delta T = \left(\nabla T \cdot \hat{\mathbf{t}}\right)\delta s. \qquad (5)$$

With (5), we can now show the following attributes of ∇T

1. ∇T points in the direction in which T increases most rapidly (in the direction of 'steepest ascent')

2. The magnitude $|\nabla T|$ is the derivative of T with distance in that direction

3. ∇T at (x, y, z) is perpendicular to the temperature contour that passes through that point

To show the first of these, note that for a distance δs travelled, the temperature change δT depends on the direction of travel $\hat{\mathbf{t}}$ through

$$\delta T = \nabla T \cdot \hat{\mathbf{t}}\delta s = |\nabla T||\hat{\mathbf{t}}|\delta s \cos(\theta) = |\nabla T|\delta s \cos(\theta) \leq |\nabla T|\delta s$$

where θ is the angle between the gradient ∇T and the unit vector $\hat{\mathbf{t}}$ in the direction of the displacement $\delta\mathbf{r}$. The last inequality becomes an equality if the angle θ is zero: that is, when the direction of travel is the same as the direction of thegradient ∇T. This maximizes the temperature difference δT.

The second follows immedaitely. With $\theta = 0$, we have

$$\delta T = |\nabla T|\delta s,$$

and dividing by δs,

$$|\nabla T| = \frac{\delta T}{\delta s}$$

To see that the third is true, consider moving from (x, y, z) to another point $(x+\delta x, y+\delta y, z+\delta z)$ that lies on the same contour. The displacement $\delta\mathbf{r} = \delta x\mathbf{i}+\delta y\mathbf{j}+\delta z\mathbf{k}$ is then parallel to the contour, and the direction vector $\hat{\mathbf{n}} = \delta\mathbf{r}/|\delta\mathbf{r}|$ is a unit tangent to the contour. Also, if the two points lie on the same contour, they are at the same temperature, so $T(x + \delta x, y + \delta y, z + \delta z, t) = T(x, y, z, t)$ and $\delta T = 0$, But $\delta T = 0$ means

$$\nabla T \cdot \hat{\mathbf{t}}\delta s = 0,$$

so the gradient ∇T is perpendicular to the tangent of the contour $\hat{\mathbf{t}}$.

Note 3 *Items 1 and 2 can be interpreted in two ways: We can either say that we move a certain distance δs in a direction $\hat{\mathbf{t}}$ that can be varied. Choosing the direction in which the temperature difference δT is largest, we find that the magnitude of the gradient ∇T is the derivative as $\delta T/\delta s$.*

Alternatively, we could fix a temperature difference δT: we could say that the point (x, y, z) lies on the contour $T = T(x, y, z, t)$, and we will look at points that lie on a nearby contour with contour level $T + \delta T$. The distance between a point on that contour and the original point (x, y, z) is then given by

$$\delta s = \frac{\delta T}{\nabla T \cdot \hat{\mathbf{t}}}.$$

We minimize that distance by making the denominator as large as possible, so when we move in a direction $\hat{\mathbf{t}}$ that is parallel to ∇T (and therefore, as point 3 shows, perpendicular to the contours). In that case, the minimal distance is

$$\delta s = \frac{\delta T}{|\nabla T|}.$$

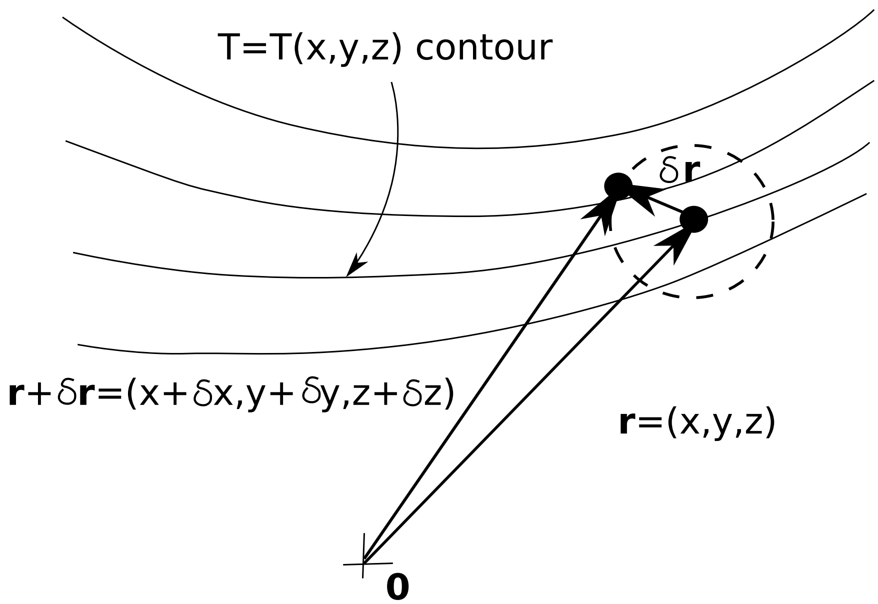

Figure 1: Illustration of the difference in temperature between position $\mathbf{r}$ and $\mathbf{r} + \delta\mathbf{r}$. The circle indicates what happens if the length of $\delta\mathbf{r}$ is kept constant but the direction $\hat{\mathbf{t}}$ is changed. The maximum temperature difference between $\mathbf{r}$ and $\mathbf{r} + \delta\mathbf{r}$is achieved when $\hat{\mathbf{t}}$ is perpendicular to the $T(x, y, z)$ contour. No temperature difference occurs when $|\hat{\mathbf{t}}|$ is parallel to the contour.

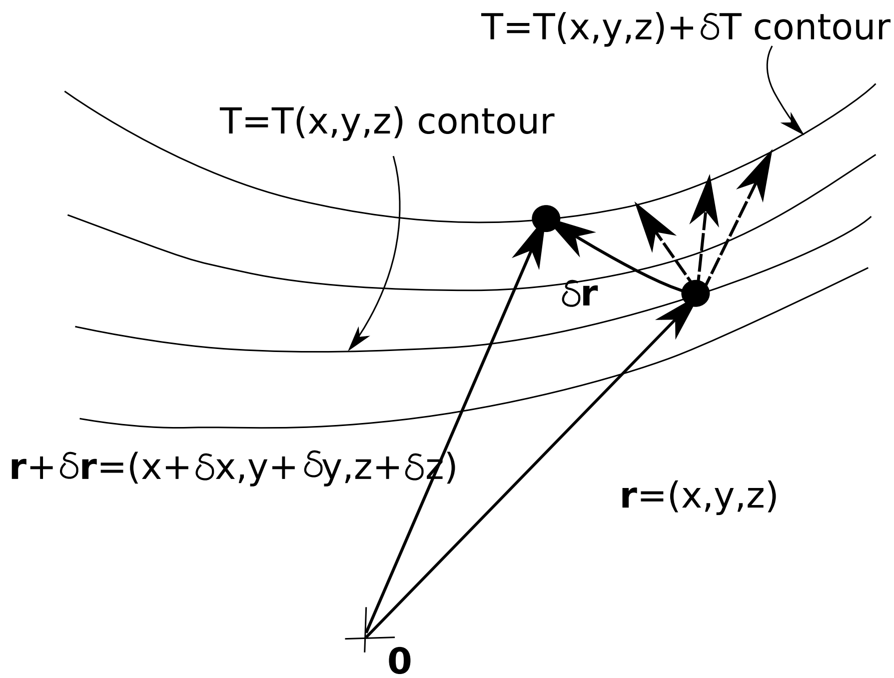

Figure 2: If we keep the temperature difference δT between $\mathbf{r}$ and $\mathbf{r}+\delta\mathbf{r}$ constant, we force the point $\mathbf{r}+\delta\mathbf{r}$ to lie on a given contour different from the $T(x,y,z)$. Changing the direction $\hat{\mathbf{t}}$ then changes the length $\delta s = |\delta\mathbf{r}|$ as is indicated by the dashed arrows. The minimum distance (which we would probably call the 'distance between the two contours') is attained when $\hat{\mathbf{t}}$ is perpendicular to the contour.

In other words, for a small contour interval δT, that distance is proportional to the contour interval, and $1/|\nabla T|$ is the constant of porportionality

$$\frac{1}{|\nabla T|} = \frac{\delta s}{\delta T}$$

Where ∇T is large, contours are closely spaced.

Example 1 *Let $T(x, y) = x^2 - y^2$, so*

$$\nabla T = 2x\mathbf{i} - 2y\mathbf{j}.$$

The contours to this vector field are the hyperbolae

$$y = \pm\sqrt{x^2 + c}$$

where c is the contour level, see the example on sketching contour lines in the notes on volume integrals. We can also compute streamlines for the problem, if we put

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \nabla T,$$

so that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\partial T}{\partial x} = 2x, \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\partial T}{\partial y} = -2y.$$

We can combine these into a single equation by dividing

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t}\Big/\frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{y}{x}.$$

Separating variables and integrating gives $\log(y) = -\log(x) + C = \log(C'/x)$, where $C = log(C')$. Therefore streamlines are given by

$$y = \frac{C'}{x}$$

Figure 3 shows contours of T along with ∇T as vectors on the left, and contours of T along with the streamlines of ∇T on the right. The properties of ∇T derived above should be evident: ∇T goes from high to low T, and the magnitude of ∇T is largest where the contours of T are highly bunched. The direction of ∇T as indicated by the streamlines is also perpendicular to the contours.

Exercise 1 *Compute the gradients for the following functions*

1. $T(x, y) = xy$
2. $T(x, y) = x^2 + y^2$

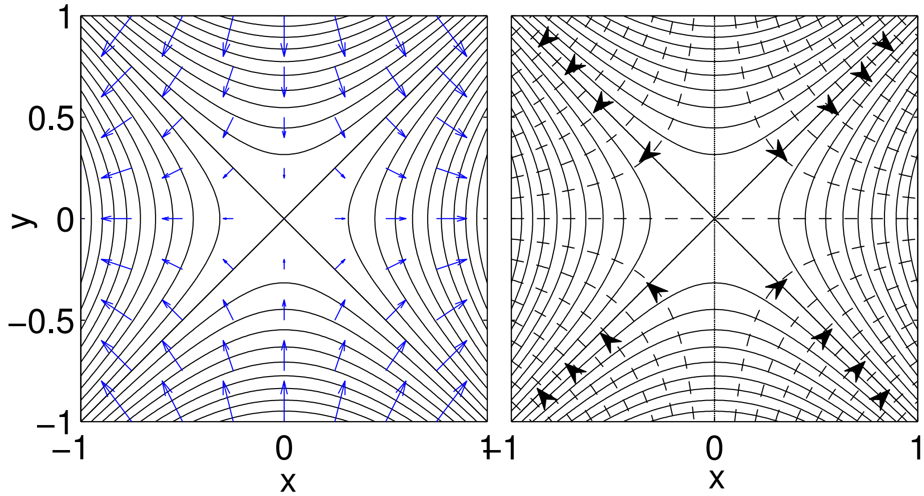

Figure 3: Contour lines of $T = x^2 - y^2$ are shown as solid lines in both plots. High values of T lie along the x-axis, both in the positive and negative x-direction. Low values lie along the y-axis. The panel on the left shows ∇T as blue vectors, the panel on the right shows streamlines.

3. $T(x, y) = \cos(x) - \sin(y)$

Exercise 2 *Sketch the contours of the functions in exercise 1, and sketch the vector field ∇T as arrows.*

Exercise 3 *The gradient of a scalar field is an extension of the idea of a derivative of a function of a single variable. Perhaps unsurprisingly, there are analogues of the usual differentiation rules, like product and chain rule. As an example, take the case where $T = f(g(x, y, z))$. We can then show that*

$$\nabla T = \frac{\mathrm{d}f}{\mathrm{d}g} \nabla g.$$

To see this, recall that

$$\nabla T = \frac{\partial T}{\partial x}\mathbf{i} + \frac{\partial T}{\partial y}\mathbf{j} + \frac{\partial T}{\partial z}\mathbf{k}.$$

But now we can apply the chain rule for partial derivatives[1] *to each term in this sum,*

[1]If this is unfamiliar, recall that a partial derivative is simply an 'ordinary' derivative with all the independent variables but one held constant, and use the chain rule as it applies to that 'ordinary' derivative. In other words, treating y and z as constants and defining

$$\delta g = g(x + \delta x, y, z) - g(x, y, z),$$

i.e.

$$\frac{\partial T}{\partial x} = \frac{\partial f(g)}{\partial x} = \frac{\mathrm{d}f}{\mathrm{d}g}\frac{\partial g}{\partial x}.$$

Similar results obviously apply to $\partial T/\partial y$ and $\partial T/\partial z$, so that

$$\begin{aligned}
\nabla T &= \frac{\partial T}{\partial x}\mathbf{i} + \frac{\partial T}{\partial y}\mathbf{j} + \frac{\partial T}{\partial z}\mathbf{k} \\
&= \frac{\mathrm{d}f}{\mathrm{d}g}\frac{\partial g}{\partial x}\mathbf{i} + \frac{\mathrm{d}f}{\mathrm{d}g}\frac{\partial g}{\partial y}\mathbf{j} + \frac{\mathrm{d}f}{\mathrm{d}g}\frac{\partial g}{\partial z}\mathbf{k} \\
&= \frac{\mathrm{d}f}{\mathrm{d}g}\left[\frac{\partial g}{\partial x}\mathbf{i} + \frac{\partial g}{\partial y}\mathbf{j} + \frac{\partial g}{\partial z}\mathbf{k}\right] \\
&= \frac{\mathrm{d}f}{\mathrm{d}g}\nabla g,
\end{aligned}$$

as required. In other words, to derive general results (so-called 'identities') involving the differential operator ∇, it can be useful to write out each expression in terms of partial derivatives, and to apply the usual calculus rules to these partial derivatives. Using a similar approach, as well as the chain rule result above, show the following:

1. *If $r = \sqrt{x^2 + y^2 + z^2}$ and $T = T(r)$, show that*

$$\nabla T = \frac{\mathrm{d}T}{\mathrm{d}r}\hat{\mathbf{r}}$$

   where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\hat{\mathbf{r}} = \mathbf{r}/r$.

2. *Let $T = f(x, y, z)g(x, y, z)$. Show that*

$$\nabla T = g\nabla f + f\nabla g.$$

3. *Let $f_1(x, y, z)$ and $f_2(x, y, z)$ be two different scalar fields, and let c_1 and c_2 be constants. Define $f(x, y, z) = c_1 f_1(x, y, z) + c_2 f_2(x, y, z)$. Show that*

$$\nabla f = c_1\nabla f_1 + c_2\nabla f_2.$$

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we have

$$\begin{aligned}
\frac{\partial f(g(x,y,z))}{\partial x} &= \lim_{\delta x\to 0}\frac{f(g(x+\delta x,y,z)) - f(g(x,y,z))}{\delta x} \\
&= \lim_{\delta x\to 0}\frac{f(g(x+\delta x,y,z)) - f(g(x,y,z))}{g(x+\delta x,y,z) - g(x,y,z)}\frac{g(x+\delta x,y,z) - g(x,y,z)}{\delta x} \\
&= \lim_{\delta x\to 0}\frac{f(g+\delta g) - f(g)}{\delta g}\frac{g(x+\delta x,y,z) - g(x,y,z)}{\delta x} \\
&= \frac{\mathrm{d}f}{\mathrm{d}g}\frac{\partial g}{\partial x}.
\end{aligned}$$

Exercise 4 *The gradient also appears in the product rule for divergences. Let* $\mathbf{v} = v_x(x, y, z, t)\mathbf{i} + v_y(x, y, z, t)\mathbf{j} + v_z(x, y, z, t)\mathbf{k}$ *and* $\phi = \phi(x, y, z, t)$*. Use the definition of*

$$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

and the definition of the gradient to show that the chain rule holds in the following form

$$\nabla \cdot (\phi \mathbf{v}) = \phi(\nabla \cdot \mathbf{v}) + \mathbf{v} \cdot (\nabla \phi).$$

Exercise 5 *Consider the gradient in two dimensions,*

$$\nabla T = \frac{\partial T}{\partial x}\mathbf{i} + \frac{\partial T}{\partial y}\mathbf{j}.$$

Now transform to a rotated coordinate system (x', y')*, in which*

$$x' = x\cos(\theta) - y\sin(\theta), \qquad y' = y\cos(\theta) + x\sin(\theta),$$

where θ *is a constant angle of rotation (see the notes on mathematical background). Define the unit vectors in the* $x'y'$*-coordinate system by*

$$\mathbf{i}' = \mathbf{i}\cos(\theta) + \mathbf{j}\sin\theta, \qquad \mathbf{j}' = -\mathbf{i}\sin(\theta) + \mathbf{j}\cos(\theta).$$

Using the chain rule, show that

$$\frac{\partial T}{\partial x}\mathbf{i} + \frac{\partial T}{\partial y}\mathbf{j} = \frac{\partial T}{\partial x'}\mathbf{i}' + \frac{\partial T}{\partial y'}\mathbf{j}'.$$

Why is this important?

Gradients and normals to surfaces

The fact that ∇T is perpendicular to contours of T can be used to construct normal vectors to known surfaces. Recall that in the notes on surface integrals, we constructed the upward-pointing normal to the surface $z = h(x, y)$ as

$$\hat{\mathbf{n}} = \frac{\mathbf{k} - \frac{\partial h}{\partial x}\mathbf{i} - \frac{\partial h}{\partial y}\mathbf{j}}{\sqrt{1 + \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2}}. \tag{6}$$

We did so by tiling the surface with small parallelograms, and using the edges of the parallelograms to compute the normal direction.

We could equally use the gradient to compute the same normal. A contour of a function $T(x, y, z)$ is nothing more than an *implicitly* defined surface, given by

$$T(x, y, z) = C,$$

where C is a constant, which then has normal $\mathbf{n} = \nabla T$, or unit normal

$$\hat{\mathbf{n}} = \frac{\nabla T}{|\nabla T|}. \tag{7}$$

If we already have an explicit formula like $z = h(x, y)$, we can re-write it in the form $T(x, y, z) = C$ by defining

$$T(x, y, z) = z - h(x, y)$$

and putting $C = 0$. Taking T defined in this way, we get

$$\nabla T = -\frac{\partial h}{\partial x}\mathbf{i} - \frac{\partial h}{\partial y}\mathbf{j} + \mathbf{k}, \qquad |\nabla T| = \sqrt{1 + \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2},$$

and hence $\hat{\mathbf{n}} = \nabla T/|\nabla T|$ takes the form of (6).

Writing the normal in the form (7) allows us to compute normals more easily in cases where the explicit formula $z = h(x, y)$ is complicated but an implicit formula is not. Take for instance the normal to a circle of radius R about the origin. This surface is most easily written implicitly as

$$x^2 + y^2 + z^2 = R^2$$

The left hand side is the square of the distance of a point (x, y, z) from the origin. This is set equal to the square of the radius of the circule. The equation above defines a contour of the function $T(x, y, z) = x^2 + y^2 + z^2$ with $C = R^2$. We have

$$\nabla T = 2(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}), \qquad |\nabla T| = 2\sqrt{x^2 + y^2 + z^2}$$

Hence

$$\hat{\mathbf{n}} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}}. \tag{8}$$

If we define the position vector of a point as $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, with $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$, we get

$$\hat{\mathbf{n}} = \frac{\mathbf{r}}{r} = \hat{\mathbf{r}}$$

where $\hat{\mathbf{r}}$ is the unit vector in the direction of $\mathbf{r}$. Of course, to be on the $x^2+y^2+z^2 = R^2$ sphere, we cannot treat x, y and z as independent above. We could go back and write z as a function of x and y,

$$z = \sqrt{R^2 - x^2 - y^2},$$

which would give us

$$\hat{\mathbf{n}} = \frac{x\mathbf{i} + y\mathbf{j} + \sqrt{R^2 - x^2 - y^2}\mathbf{k}}{R}.$$

or we can keep the symmetric form (8) on the understanding that x, y and z are also linked through

$$x^2 + y^2 + z^2 = R^2.$$

Exercise 6 *Consider the surface $x^2 + y^2 - z^2 = R^2$. Compute the normals to this surface.*

Heat conduction and the temperature gradient: Fourier's law

We have seen that ∇T gives the direction in which temperature increases most rapidly, and its magnitude $|\nabla T|$ is the rate of temperature increase with distance. It follows that $-\nabla T$ gives the direction in which temperature *decreases* most rapidly as well as the rate of temperature decrease with distance. At any given point, we expect heat to flow in that direction — from hot to cold, in the direction in which T decreases most rapidly — and the rate to heat flow to be greater when T decreases more rapidly with distance. A plausible model assumes that the conductive heat flux $\mathbf{q}_c$ is simply proportional to $-\nabla T$, so

$$\mathbf{q}_c = -k\nabla T$$

This is *Fourier's law.* k is called the *thermal conductivity*, with units of W m^{-1} K^{-1}, and its value depends on the material present. An insulating material will have a low thermal conductivity, while a good thermal conductor will have a high thermal conductivity. We will usually treat k as constant. In some materials, k actually varies slightly as temperature changes, in which case $k = k(T)$ is a function of temperature.

Note that, despite the name containing the word 'law', Fourier's law is no Newton's law — meaning there is no fundamental reason why it *must* hold other than that it conforms to our intuition about heat flow.

Exercise 7 *Design an experiment that would allow you to test whether Fourier's law holds for a particular material, and to determine k. The only pieces of equipment you are allowed are polystyrene, cut to any shape you like (this is a good insulator), a constant voltage power supply, a resistor of known strength, a thermometer and a plentiful supply of the material you are meant to test, which can be cut to any shape you like (and you can have as many pieces of it as you like).*

Exercise 8 *This exercise is about heat flow through a window. Glass has a thermal conductivity of k = 1.05 W m^{-1} K^{-1}. For a single glass window pane of thickness 1 cm with a temperature of 20 C on one side and 5 C on the other, what is the heat flux through the glass? If the area of the window is 1.5 m^2, what is the rate of heat loss through the window?*

Exercise 9 *Let $T = xy + z^2$, and let S be the surface of the prism with vertices at $(0, 0, 0)$, $(1, 0, 0)$, $(0, 0, 1)$ and at $(0, 1, 0)$, $(1, 1, 0)$ and $(0, 1, 1)$. If $k = 1$ and $\mathbf{q}_c$ is given by Fourier's law, calculate $\int_S \mathbf{q}_c \cdot \hat{\mathbf{n}}\, \mathrm{d}S$, where $\hat{\mathbf{n}}$ is the outward-pointing unit normal.*

Energy density

In general, we expect thermal energy density ε to increase with temperature. The precise form of that relationship depends on the material and can, in general, only be established by measurement. A simple and frequently made assumption is that the material in question has a constant *specific heat capacity.*

You probably know specific heat capacity c of a given material as 'the amount of energy it takes to raise 1 kg of the material by 1 K'. This is fine for our purposes if we can treat c as a constant, and all the energy goes into what we have termed 'thermal energy'.[2] Consider the mass δm contained in a small volume δV. If c is constant, then we know that the energy required to raise a mass δm to a temperature T is $cT\delta m$, and therefore

$$\varepsilon(T) = \frac{\delta e}{\delta V} = \frac{cT\delta m}{\delta V} = \frac{\delta m}{\delta V} cT = \rho c T$$

Exercise 10 *Let V be the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(1,0,0)$, and let $T(x,y,z) = x^2 + y^2$. With $\rho = c = 1$, compute the total thermal energy content of V.*

The thermal energy density ε appears in (2) in the first two terms only. If we substitute for ε, we get get

$$\frac{\partial \varepsilon}{\partial t} + \nabla \cdot (\varepsilon \mathbf{v}) = \frac{\partial(\rho c T)}{\partial t} + \nabla \cdot (\rho c T \mathbf{v}). \tag{9}$$

There is a trick that allows us to simplify this using the equation for conservation of mass,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0. \tag{10}$$

We can recognize that derivatives of ρ and $\rho\mathbf{v}$ occur in both, and apply the product rule to get the same derivatives from (9) as we have in (10):

$$\begin{aligned}
\frac{\partial \varepsilon}{\partial t} + \nabla \cdot (\varepsilon \mathbf{v}) &= \frac{\partial(\rho c T)}{\partial t} + \nabla \cdot (\rho c T \mathbf{v}) \\
&= \frac{\partial \rho}{\partial t} cT + \rho c \frac{\partial T}{\partial t} + \nabla \cdot (\rho \mathbf{v}) cT + \rho c \mathbf{v} \cdot \nabla T \\
&= cT \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) \right) + \rho c \frac{\partial T}{\partial t} + \rho c \mathbf{v} \cdot \nabla T \\
&= \rho c \frac{\partial T}{\partial t} + \rho c \mathbf{v} \cdot \nabla T, \qquad (11)
\end{aligned}$$

using (10) to derive the last equality.

[2]We have discussed the slightly artifical distinction between 'chemical potential' and 'thermal energy' before; in thermodynamics, you would simply talk about changes in internal energy rather than separating it into a chemical and a thermal part. The point is that none of the energy should go into mechanical work or kinetic energy.

Exercise 11 *If specific heat capacity is not constant, c is defined through*

$$c = \frac{\mathrm{d}}{\mathrm{d}T}\left(\frac{\varepsilon}{\rho}\right), \tag{12}$$

assuming that ε/ρ depens only on T.[3]

The definition above allows us to make sense of 'specific heat capacity' when c is not necessarily a constant. Take c to be defined as above. Write $\varepsilon = \rho \times (\varepsilon/\rho)$ and substitute into the first two terms of (2)*. Apply the product and chain rules to show that*

$$\frac{\partial \varepsilon}{\partial t} + \nabla \cdot (\varepsilon \mathbf{v}) = \frac{\varepsilon}{\rho}\left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v})\right) + \rho c \frac{\partial T}{\partial t} + \rho c \mathbf{v} \cdot \nabla T$$

and then use (10) *to show that*

$$\frac{\partial \varepsilon}{\partial t} + \nabla \cdot (\varepsilon \mathbf{v}) = \rho c \frac{\partial T}{\partial t} + \rho c \mathbf{v} \cdot \nabla T$$

This generalizes (11) *to the case of non-constant specific heat capacity.*

Note 4 *There are simple situations in which $\varepsilon = \rho c T$ cannot hold, even in an approximate sense. When a material goes through a phase change, its internal energy content changes by a finite amount without any temperature change at all. For instance, it takes 3.35×10^5 J to turn 1 kg of ice at 0 C into 1 kg of water at 0 C. In that case, the heat capacity as defined by* (12) *is actually infinite.*

Next, we put together our models for conductive flux $\mathbf{q}_c$ and thermal energy density ε to derive a single equation for temperture T, the *heat equation.*

The heat equation

With $\varepsilon = \rho c T$ and $\mathbf{q}_c = -k\nabla T$, the energy conservation equation (2) becomes

$$\frac{\partial(\rho c T)}{\partial t} + \nabla \cdot (\rho c T \mathbf{v}) - \nabla \cdot (k \nabla T) = a \tag{13}$$

and we can identify the advective heat flux as $\rho c T \mathbf{v}$ and the conductive heat flux is $-k\nabla T$.

[3]If you are familiar with with thermodynamics, you will know that in general it is important to specify which other variables are held constant when T is varied. Given our formulation of the energy conservation law in (2), it turns out the derivative is taken at constant ρ — or at 'constant volume' in the language of thermodynamics — so

$$c = \frac{1}{\rho}\left(\frac{\partial \varepsilon}{\partial T}\right)_\rho.$$

We can apply (11) to the first two terms and get a slightly simpler form

$$\rho c \frac{\partial T}{\partial t} + \rho c \mathbf{v} \cdot \nabla T - \nabla \cdot (k \nabla T) = a. \tag{14}$$

In many ways, we should call this equation the 'heat equation', because it tells us about the flow of heat in a fairly general way. In mathematics, the name *heat equation* is however usually used for a more restrictive version of this equation.

If we assume that there is no motion of material (for instance, if we have a solid that does not deform) *and* the thermal conductivity is constant, then we can simplify (14) to

$$\rho c \frac{\partial T}{\partial t} - k \nabla \cdot \nabla T = a.$$

The combination $\nabla \cdot \nabla T$ is the divergence of the gradient of T. This occurs frequently enough that it gets is own symbol,

$$\nabla \cdot \nabla T = \nabla^2 T,$$

called the *Laplacian* of T. With this notation, we get

$$\rho c \frac{\partial T}{\partial t} - k \nabla^2 T = a.$$

This is the heat equation.

Exercise 12 *Write out the gradient ∇T of T in terms of partial derivatives of T. Then take the divergence of the gradient, again writing this out in terms of partial derivatives. This will give you a formula for $\nabla^2 T$. Then do the same to write $\nabla \cdot (k \nabla T)$ in terms of partial derivatives with respect to x, y and z.*

Exercise 13 *Figure 4 shows a sketch of temperature contours in a subducting oceanic plate. The lines with arrows in the left-hand figure are streamlines of $\mathbf{v}$. Assume the the slab moves at constant speed as shown, with the direction of motion parallel to the surface of the slab. Copy out the figure twice, showing temperture contours and the outline of the slab. On one copy sketch the* advective *heat flux $\rho c T \mathbf{v}$, on the other sketch the conductive heat flux $-k \nabla T$. Obviously you do not have the parameter values ρ, c or k, but all they do is scale the arrows you draw.*

1. *Assume there is no conversion of other forms of energy into thermal energy. Suppose you had only advective heat flux but no conductive heat flux. On the plot where you have sketched advective heat flux, indicate where temperature would be going up or down as a result of the advective heat flux field you have sketched.*

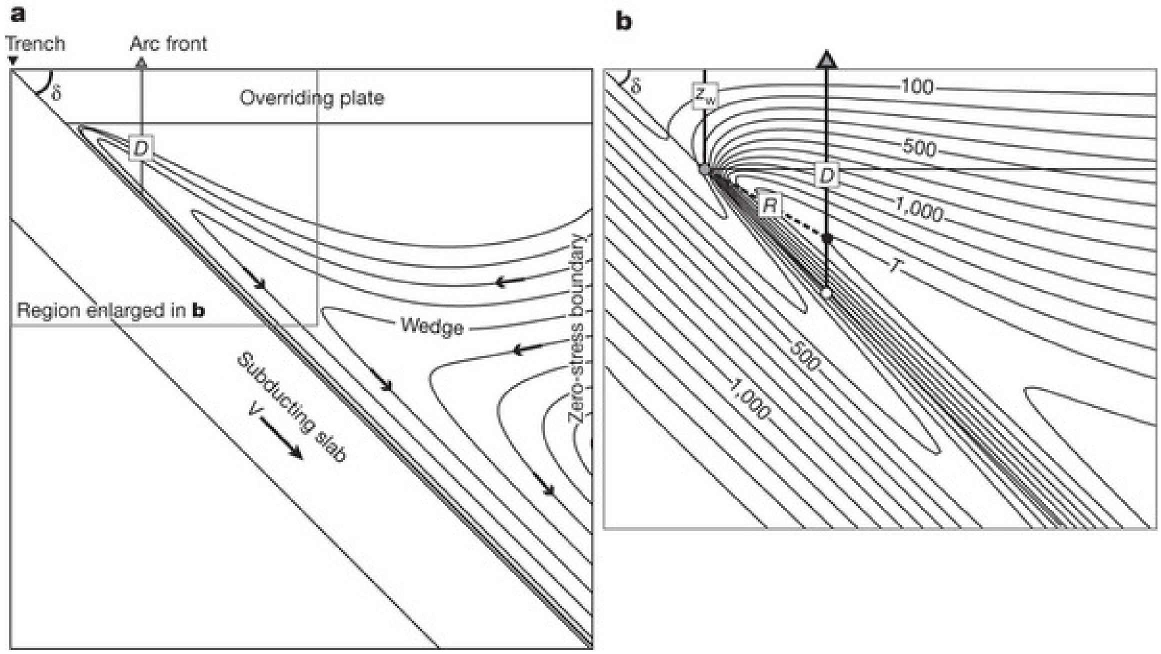

Figure 4: The temperature field in a subducting oceanic plate. Figure from England and Katz, *Nature*, 467, 700703, 2010. Note that panel b does not show the same region as panel a, but enlarges the box at top left in panel a marked as 'Region enlarged in b' in the figure. Note also that there is no motion $\mathbf{v} = \mathbf{0}$ in the 'overriding plate'.

2. *Next, suppose you had only conductive heat flux but no advective heat flux. On the plot where you have sketched advective heat flux, indicate where temperature would be going up or down as a result of the conductive heat flux field you have sketched.*

3. *Explain how, when you combine advective and conductive heat flux, the temperature field can be steady in time (i.e., how advective and conductive heat flow together can ensure that the temperature field remains as shown in the figure).*

4. *If there really was no conduction but only advection, what would a steady temperature distribution look like? That is, if there was no conduction but only advection and you waited for a long time, what would temperature distribution would eventually evolve?*

5. *If there really was no advection but only convection, what would a steady temperature distribution look like? That is, if there was no advection but only conduction and you waited for a long time, what would temperature distribution would eventually evolve?*

6. *For* $\rho = 3300$ *kg* m^{-3}, $c = 1000$ *J* kg^{-1} K^{-1}, $k = 2$ *J* m^{-1} K^{-1} *and a subduction velocity of 5 cm per year, estimate the magnitudes of maximum advective and conductive heat fluxes for the situation shown in the figure (in standard SI units).*