The heat equation for slabs, cylinders and spheres

March 20, 2016

Overview

These notes cover the following:

- The steady state heat equation in one dimension
- Solving by separation of variables and applying boundary conditions
- Other geometries with simple symmetries: cylinders and spheres
- A note on point sources

The heat equation in one dimension

In the previous set of notes, we derived a conservation law for energy in the form

$$\rho c \frac{\partial T}{\partial t} + \rho c \mathbf{v} \cdot \nabla T - \nabla \cdot (k \nabla T) = a. \tag{1}$$

In the case where there is no motion of material, $\mathbf{v} = \mathbf{0}$, and therefore no advective heat flux $\rho c T \mathbf{v}$, this can be reduced to

$$\rho c \frac{\partial T}{\partial t} - \nabla \cdot (k \nabla T) = a. \tag{2}$$

Technically, we also have to put k equal to a constant in order to arrive at the 'heat equation' that you will see in mathematics courses,

$$\rho c \frac{\partial T}{\partial t} - k \nabla^2 T = a, \tag{3}$$

*except figure 3

where

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

is called the *Laplacian*.

Writing down differential equations without ever solving them is a reasonably pointless exercise, so we will now try to solve some heat flow problems. In their most general form, none of the differential equations we have written down above can be solved exactly using pencil and paper, and so-called numerical methods become necessary. Numerical methods are not the point of this course: they are a large and non-trivial subject in their own right.

In order to allow us to find exact solutions, we will look at a number of special cases that allow us to turn the *partial differential equations* above into *ordinary differential equations*.

Note 1 *Partial differential equations are equations that involve the partial rather than ordinary derivatives of the unknown, in this case the temperature* $T(x, y, z, t)$

This will generally involve two steps: first, we assume that the temperature field is in *steady state*, meaning it does not change over time. In other words $T = T(x, y, z)$ does not depend on t. This ensures that the time derivative $\partial T/\partial t = 0$. Second, we assume a spatial symmetry that allows us to turn the partial derivatives in x, y and z into a derivative with respect to a single variable.

The first step turns (2) into

$$-\nabla \cdot (k\nabla T) = a, \tag{4}$$

or with k constant, we would get

$$-k\nabla^2 T = a. \tag{5}$$

(5) in particular occurs often enough that, like the heat equation (3), it has a name: *Poisson's equation*.

Wether the assumptions we are making are valid of course depends on the particular heat flow problem we are interested in. They may well not be, and comparison of results with actual observations can be a useful guide as to whether the assumptions we have made are valid.

Note 2 *We will generally start with* (4)*, and then assume that we have some spatial symmetry. In starting with* (4)*, we are already assuming that temperature is in steady state, and that velocity* $\mathbf{v} = \mathbf{0}$*. These two assumptions are part of the list of possibly questionable assumptions we are making.*

The easiest symmetry to work with is to assume that T, as well as k (if we do not assume it to be constant to begin with) and a depend only on one spatial variable, say x. In that case

$$\mathbf{q}_c = -k\nabla T = -k\frac{\partial T}{\partial x}\mathbf{i}.$$

In fact, since T depends only on x but not on y or z, and we have assumed that T is also in a steady state, so T does not depend on t either, we have

$$\mathbf{q}_c = -k\frac{\mathrm{d}T}{\mathrm{d}x}\mathbf{i}$$

and

$$-\nabla \cdot (k\nabla T) = \nabla \cdot \mathbf{q}_c = -\frac{\mathrm{d}}{\mathrm{d}x}\left(k\frac{\mathrm{d}T}{\mathrm{d}x}\right).$$

Equation (4) then becomes

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(k\frac{\mathrm{d}T}{\mathrm{d}x}\right) = a, \tag{6}$$

where T, k and a can only be functions of x but not y, z or t.

Note 3 *We have now moved on from original derivation of the heat conservation problem* (1)*, so the connection to the original conservation law may seem remote. It is actually easy to derive* (6) *from first principles. Consider a thin slab of material lying between x and $x + \delta x$, as show in figure 1. We can compute the net amount of heat that flows out of the slab in time δt as the amount of heat that flows out at the top (at $x + \delta x$) minus the amount that flows out at the bottom (at x). If the base area of the slab is A, we get*

$$q_c(x + \delta x)A\delta t - q_c(x)A\delta t, \tag{7}$$

where q_c is the conductive heat flux. Note that the quantity we have just calculated is in fact equal to $\int_S \mathbf{q}_c \cdot \hat{\mathbf{n}}\,\mathrm{d}S$; this follows from the normal direction $\hat{\mathbf{n}}$ on the two surfaces and the fact that the heat flux $\mathbf{q}_c$ is uniform on each surface.

In steady state, the amount of heat that flows out must be the amount of heat that is produced. If the rate of production per unit volume is a, then with a volume $A\delta x$, the amount of heat produced in time δt is

$$aA\delta x\delta t.$$

Equating this to the amount of heat that flows out of the slab, we get

$$q_c(x + \delta x)A\delta t - q_c(x)A\delta t = aA\delta x\delta t,$$

so

$$\frac{q_c(x + \delta x) - q_c(x)}{\delta x} = a,$$

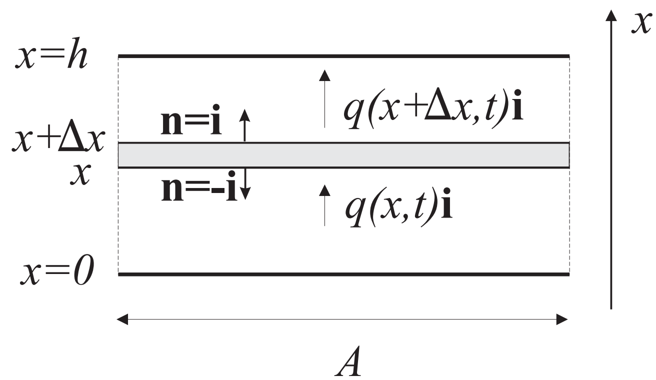

Figure 1: Geometry for deriving the heat equation. Note that A represents an area.

or, recognizing the left-hand side as a derivative,

$$\frac{\mathrm{d}q_c}{\mathrm{d}x} = a. \tag{8}$$

In one dimension, Fourier's law is simply that heat flux q_c depends on the temperature gradient as

$$q_c = -k\frac{\mathrm{d}T}{\mathrm{d}x}$$

and so

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(k\frac{\mathrm{d}T}{\mathrm{d}x}\right) = a$$

Exercise 1 *The derivation of* (8) *is analogous to how we derived the pressure equation*

$$\frac{\mathrm{d}p}{\mathrm{d}z} = -\rho g.$$

Show which quantities are analogous to each other in the derivations.

Solving the heat equation: examples

Equation (6) is a *second-order* ordinary differential equation, meaning it contains second derivatives of the unknown, T. This is slighlty more complicated than the *first-order* problems we have met so far, but we will generally only deal with problems where the same methods we are already familiar with can be applied.

We can always think of (6) as a set of two first-order problems,

$$\frac{\mathrm{d}q_c}{\mathrm{d}x} = a, \tag{9a}$$

$$-k\frac{\mathrm{d}T}{\mathrm{d}x} = q_c. \tag{9b}$$

Often (but not always) we can solve them in sequence, finding q_c first and then T.

The easisest case is that of a and k constant. We can then simply integrate (9a) once to find

$$q_c(x) = ax + C_1,$$

where C_1 is a constant of integration. Then substitute into (9b),

$$-k\frac{\mathrm{d}T}{\mathrm{d}x} = ax + C_1,$$

divide by $-k$ and integrate again

$$T = -\frac{ax^2}{2k} - \frac{C_1 x}{k} + C_2, \tag{10}$$

where C_2 is another constant of integration.

Note 4 *There is no reason why the constants of integration should be the same — they have different units to begin with — so do not use the same symbol for both.*

The procedure we have just followed is straightforward to adapt to cases where a depends on x, k depends on x or k depends on T, see the exercises below. In all cases, we have to integrate twice because the heat flow problem (6) contains a second derivative of T. This means we get two different constants of integration that we still need to determine. This is done using *boundary conditions*. Boundary conditions are extra equations that tell us about the physics of the surfaces of the slab of material, and play a similar role to the initial conditions we encountered in the simpler first-order differential equations we have previously studied. There are two very common types of boundary conditions, called Dirichlet and Neumann conditions:

1. Dirichlet conditions prescribe temperature at a surface

2. Neumann conditions prescribe heat flux at a surface

More complicated boundary conditions, for instance coupling flux at a surface to temperature, also exist. The heat equation requires *one* boundary condition on *each* surface of the 'domain' of the problem.

Note 5 *By* domain *we simply mean the region in space in which temperature T is to be computed.*

Example 1 gives an example of how to implement boundary conditions. You need to identify where your boundaries are, and what conditions you will apply. Then it is a relatively straightforward case of substituting both, the position of the boundary for x, and the known value of the quantity given on the boundary — here T or q_c — into the general solution.

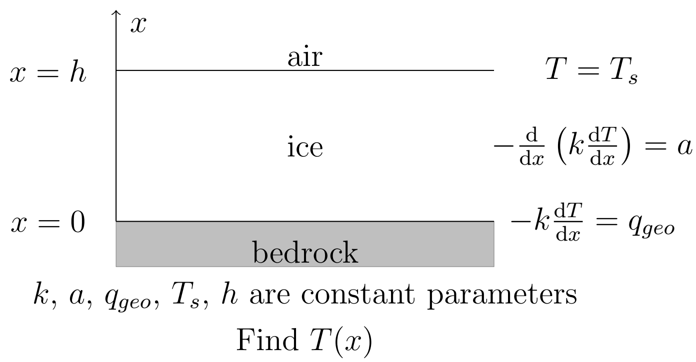

Figure 2: The set-up for the problem to be solved in example 1. It is worth drawing a sketch like this before you start calculating things, showing the shape of the region in which you are going to compute the solution T, and showing which equations hold where.

Example 1 *The following example shows how boundary conditions can fix the constants of integration C_1 and C_2. Consider the following question: an ice sheet of thickness h = 2000 m with a constant rate of heat production per unit volume of a = 2.6×10^{-6} W m^{-3} and a surface temperature T_s = -40 C. Ice has a thermal conductivity of k = 2.2 W m^{-1} K^{-1}. If geothermal heat flux going into the base of the ice sheet is q_geo = 0.04 W m^{-2}, what is the temperature at the base of the ice?*

The basic steps in the solution are:

1. *We assume that* (6) *describes the temperature profile in the ice sheet. We need to define a coordinate system, and ideally we should make a quick sketch of the domain in which we are trying find T, of what the boundary conditions are, and where they hold. Here, the obvious coordinate system to choose has the x-axis pointing across the ice (so, uncharacteristically, upwards), with $x = 0$ at the base of the ice and $x = h$ at the surface. See figure 2.*

2. *We find a general solution to the heat flow problem. Because a and k are constant, we can follow the steps that lead to* (10) *exactly, so we get*

$$q = -\,k\frac{\mathrm{d}T}{\mathrm{d}x} = ax + C_1. \tag{11a}$$

$$T(x) = -\frac{ax^2}{2k} - \frac{C_1 x}{k} + C_2, \tag{11b}$$

   You should not try to memorize these formulae but understand how to derive them. In general, if a or k are not constant, the formulae above will not hold, but the same method of solution is likely to give you the correct, alternative formulae.

3. Always *use letters to denote parameter values.* Do not *substitute their numerical values. You may later want to change those values, and not have to go through all the computations again.*

4. *One boundary condition tells us that this flux at the base is* q_{geo}, *so*

$$q_c(0) = q_{\text{geo}}.$$

From the formula (11a) *for flux,* q *at the base of the ice* ($x = 0$) *is*

$$q_c(0) = C_1.$$

5. *Substitute this value of* C_1 *into the temperature solution* (11b)*:*

$$T(x) = -\frac{a}{2k}x^2 - \frac{q_{\text{geo}}}{x}x + C_2. \tag{12}$$

6. *At the surface we have* $x = h$ *and* $T = T_{\text{s}}$. *Making sure that the function* $T(x)$ *given by* (12) *satisfies this,*

$$T(h) = -\frac{ah^2}{2k} - q_{\text{geo}}h + C_2 = T_{\text{s}}.$$

Hence C_2 *must be*

$$C_2 = T_{\text{s}} + \frac{q_{\text{geo}}h}{k} + \frac{h^2}{2}.$$

7. *Substituting* C_2 *back into equation* (12) *gives the solution for the temperature field* $T(x)$, without *any unknown constants of integration:*

$$T(x) = T_{\text{s}} + \frac{q_{\text{geo}}(h - x)}{k} + \frac{a(h^2 - x^2)}{2k}.$$

8. *At the base of the ice,* $x = 0$, *and*

$$T(0) = T_{\text{s}} + \frac{q_{\text{geo}}h}{k} + \frac{ah^2}{2k}. \tag{13}$$

With the values of T_{s}, q_{geo}, h *and* k *given, this predicts*

$$T(0) = -1.3\,\text{C}.$$

Note that this is actually above the melting point of ice at a depth of 2000 m (the melting point of solids is pressure-dependent. For ice, it decreases *with pressure). This means that the boundary conditions of fixed heat flux at the base are actually unrealistic. In reality, we should instead apply a temperature equal to the melting point at depth, and use the difference between the flux into the ice and the geothermal heat flux arriving at the base of the ice to compute a rate at which the base of the ice sheet melts.*

Exercise 2 *Suppose you were* not *given numerical values for any of the parameters in example 1. Work through the steps required to sketch the solution: intercepts, maxima / minima, singularities etc. Bear in mind that we* only *need the solution in the interval* $0 < x < h$ *that is actually occupied by ice. Identify conditions under which features you identify, like maxima / minima etc., actually lie in that interval.*

Note 6 *It may seem like it would also be possible to apply two boundary conditions on one boundary. It turns out that this is an artifcat of having reduced our problem to one spatial dimension. If you tried to have prescribed temperature* and *heat flux on one boundary and no boundary condition on the other in the more general heat equation* (4)*, you would generally get unphysical solutions in which temperature becomes infinite inside your domain. This actually tells you something about physics: you generally cannot simultaneously impose temperature* and *heat flux at a single surface.*

Exercise 3 *This question is about dealing with heat production rates* $a(x)$ *that are not constant but depend on* x*. Ice sheets produce heat as they flow, but unlike what was assumed above, the rate of heat production is not usually distributed evenly with depth. Forces are greatest near the ice sheet bed, and so is heat production. For an ice sheet with thickness* h*, the rate of heat heat production above the ice sheet base* $x = 0$ *can be modelled as*

$$a(x) = \alpha(h - x)^{n+1}, \tag{14}$$

where α *and* n *are positive constants that describe the flow of the ice. Assume that temperature at the ice sheet bed is fixed at some value* T_{b}*, while surface temperature is fixed at a different value* T_{s}*. Treat all of these parameters as known constants. Their numerical values will be given later.*

If the temperature field in the ice sheet is in steady state and k *denotes thermal conductivity, answer the following questions:*

1. *Write down the steady state heat equation for this problem.*
2. *Find the general solution analogous to* (10) *for the temperature field in the ice in terms of* h*,* k*,* α *and* n *and two constant of integration, by the same method by which we found* (10)*. Do not yet apply boundary conditions.*
3. *Apply boundary conditions to find the constants of integration in terms of* T_{b} *and* T_{s} *as well as* h*,* k*,* α *and* n*.*
4. *Write down formulae for* $T(x)$ *and* $q_{\mathrm{c}}(x)$ *in terms of* T*,* T_{s}*,* h*,* k*,* α *and* n*.*
5. *Sketch the solution. Make sure to find intercepts, maxima etc.*
6. *If* $\alpha = 1.2 \times 10^{-17}\,\mathrm{W\,m^{-7}}$*,* $n = 3$*,* $h = 2000\,\mathrm{m}$*,* $k = 2.2\,\mathrm{W\,m^{-1}\,K^{-1}}$*,* $T_{\mathrm{b}} = 0\,\mathrm{C}$*,* $T_{\mathrm{s}} = -40\,\mathrm{C}$*, calculate the heat flux* q *at the base of the ice sheet and at the top surface.*

7. *Take a column of ice between $x = 0$ and $x = h$ with a base area of A. What is the rate at which thermal energy is produced in this column (i.e., this volume) in terms of h, α and n, and what is its numerical value?*

8. *Compute the difference between the rate of heat flow out at the top of the ice and heat flow in at the base, $q_c(h)A - q_c(0)A$. Compare with your answer in part 7. How does this relate to equation* (9a)*?*

Fourier's law, $\mathbf{q}_c = -k\nabla T$ with constant thermal conductivity k, is not completely general. The microscopic processes that lead to heat conduction can be affected for instance by temperature, so thermal conductivity may not be a constant but can for some materials depend on temperature. This isn't too problematic, however, as our derivation of (4) and therefore of (6) did not rely on constant k. We can therefore treat $k = k(T)$ as a function of temperature. The following exercise shows that steady state solutions to the heat equation with non-constant thermal conductivity can sometimes be compute following the same method as we followed above.

Exercise 4 *The thermal conductivity of ice depends on temperature, and a good model for k in ice is $k = k_0 \exp(-\beta T)$ where $k_0 = 9.8$ W m^{-1} K^{-1} and $\beta = 5.7 \times 10^{-3}$ K^{-1}. T here* must *be expressed in Kelvins. Consider an ice sheet of thickness $h = 2000$ m with surface temperature $T_{\text{surface}} = -50$ C, and with a geothermal heat flux $q_{\text{geo}} = 0.04$ W m^{-2} flowing into its base. Assuming that the rate of heat production in the ice is given by* (14) *with $\alpha = 1.2 \times 10^{-17}\,\mathrm{W\,m^{-7}}$ and $n = 3$, and that temperature is in steady state, answer the following questions:*

1. *Write down* (9)*, but with k a function of T.*

2. *Follow the same procedure as we did to find* (10)*. On the second integration step, make sure to remember how and why separation of variables works for first-order ordinary differential equations.*

3. *Apply boundary conditions and find the constants of integration.*

4. *What is the temperature at the base of the ice sheet?*

5. *By what percentage does thermal conductivity at the base of the ice sheet differ from conductivity at the surface? How necessary was it to allow k to depend on T in our model?*

Exercise 5 *Remember that models always leave out some physics, and should be checked against data. Example 1 and exercises 3 and 4 try to compute the temperature field $T(x)$ as a function of position in the ice. Temperatures in ice sheets are however known from borehole measurements. Some examples are given in figures 3. They are plotted in an unusual way, with position plotted on the veritcal axis, and temperature*

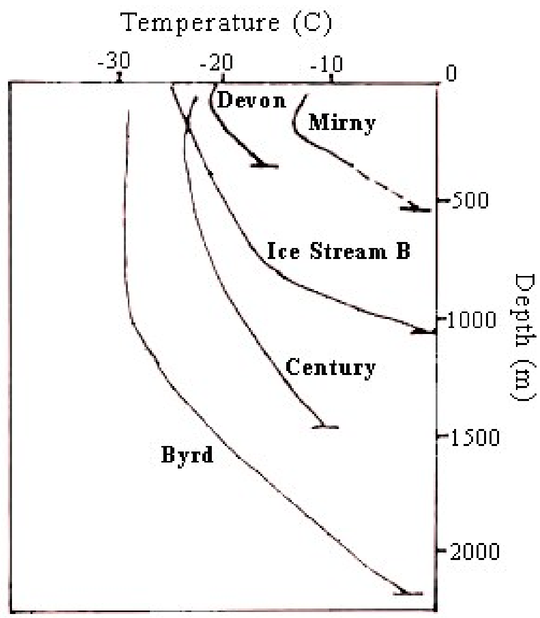

Figure 3: Temperature profiles measured in real ice sheets. *Temperature is plotted horizontally, depth below the surface is plotted vertically.* When plotting your results for $T(x)$ vs. x, it is likely that you will intuitively plot T on the vertical and x on the horizontal, so remember that you have to flip this graph on its side to compare with your own results.

on the horizontal. Sketch your solutions to to exercises 3 and 4 in the same way. Do they look similar to observed temepratures, even approximately (or qualitatitvely)? If not, what is the main qualitative difference? If we treat k as constant, equation (6) *says*

$$-k\frac{\mathrm{d}^2T}{\mathrm{d}x^2} = a$$

where the heat production rate a is positive. *Even without knowing anything about the distribution of a, or the value of k or the boundary conditions, what does this equation tell you about the shape of the solution curve? Can it ever look like any of the solution curves in figure 3?*

Exercise 6 *There are some boundary conditions that do not make sense physically with equation* (6)*, and this is reflected in the mathematics. Consider the following problem: You have a parallel-sided slab between $x = 0$ and $x = h$ with constant thermal conductivity k, heat production $a = a_0$ also constant, and temperature $T = T(x)$ dependent on x only. Let flux $q_c(0) = q_b$ is prescribed at the base of the slab, and flux $q_c(h) = q_s$ is prescribed at the top of the slab.*

1. *Assume* (6) *holds. Find a general solution for $T(x)$ including two constants of integration.*

2. *Apply boundary conditions. What problem do you find mathematically?*

3. *Interpret this problem physically. What conditions must be satisfied in order for a solution to exist? Is this solution unique?*

4. *Equation* (6) *was based on a number of assumptions that simplified an earlier, more general equation. If we now have a problem that has no solution, which of these assumptions might have been wrong? How might 'undoing' the assumptions fix the problem of having no solution? (Think physically.)*

Other geometries with symmetry: spheres

Most of our work has been in Cartesian coordinate systems. For some geometries, other types of coordinate systems are more useful because they make better use of possible symmetries. Obvious cases are spheres and cylinders, in which spherical and cylindrical polar coordinate systems are the obvious choice. For instance, for a sphere of radius R, we would have a surface defined by $r = R$ instead of $\sqrt{x^2 + y^2 + z^2} = R$, where r is the distance from the origin to a given point. In some of the exercises in the notes on divergences and gradients, we have touched on polar coordinate systems to a limited extent. We will build on these here.

Consider a temperature field with rotational symmetry: temperature T at a point (x, y, z), depends only on distance $r = \sqrt{x^2 + y^2 + z^2}$ of the point from the origin, but not on its 'latitude' or 'longitude'. This situation is likely to arise when conduction occurs in a sphere, provided the production rate density a and boundary conditions preserve the symmetry of the sphere.

Note 7 *Note that, even though we are talking about a sphere, r is* not *in general the radius of the sphere. Instead, it is the distance from the origin to a general point* inside *the sphere, so r is typically* less *than the radius of the sphere. We will use a different symbol, for instance R, to denote the radius of the sphere.*

If $T = T(r)$, one of the exercises in the notes on gradients is to use the chain rule to show that

$$\nabla T = \frac{\mathrm{d}T}{\mathrm{d}r}\nabla r = \frac{\mathrm{d}T}{\mathrm{d}r}\hat{\mathbf{r}}. \tag{15}$$

where $\hat{\mathbf{r}}$ is the unit vector pointing in the direction from the origin to the point (x, y, z),

$$\hat{\mathbf{r}} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}},$$

so that conductive heat flux can be written in the form

$$\mathbf{q}_c = -k\nabla T = -k\frac{\mathrm{d}T}{\mathrm{d}r}\hat{\mathbf{r}}.$$

Assuming that T depends only on r only makes sense if thermal conductivity k is either constant or depends on r only. With $T = T(r)$, $\mathrm{d}T/\,\mathrm{d}r$ is also a function of r only, so we can write the flux as

$$\mathbf{q}_c = q_c(r)\hat{\mathbf{r}}, \qquad q_c(r) = -k\frac{\mathrm{d}T}{\mathrm{d}r}.$$

One of the exercises in the notes on the divergence theorem is to demonstrate using the product and chain rules that

$$\nabla \cdot (q_c(r)\hat{\mathbf{r}}) = \frac{1}{r^2}\frac{\mathrm{d}(r^2 q_c)}{\mathrm{d}r} \tag{16}$$

Exercise 7 *Derive* (15) *and* (16) *if you have not done so yet.*

Equation (4) states that

$$\nabla \cdot \mathbf{q}_c = -\nabla \cdot (k\nabla T) = a.$$

With T and k dependent only on r, we can substitute from above to get

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 q_c(r)\right) = a, \qquad q_c(r) = -k\frac{\mathrm{d}T}{\mathrm{d}r}, \tag{17}$$

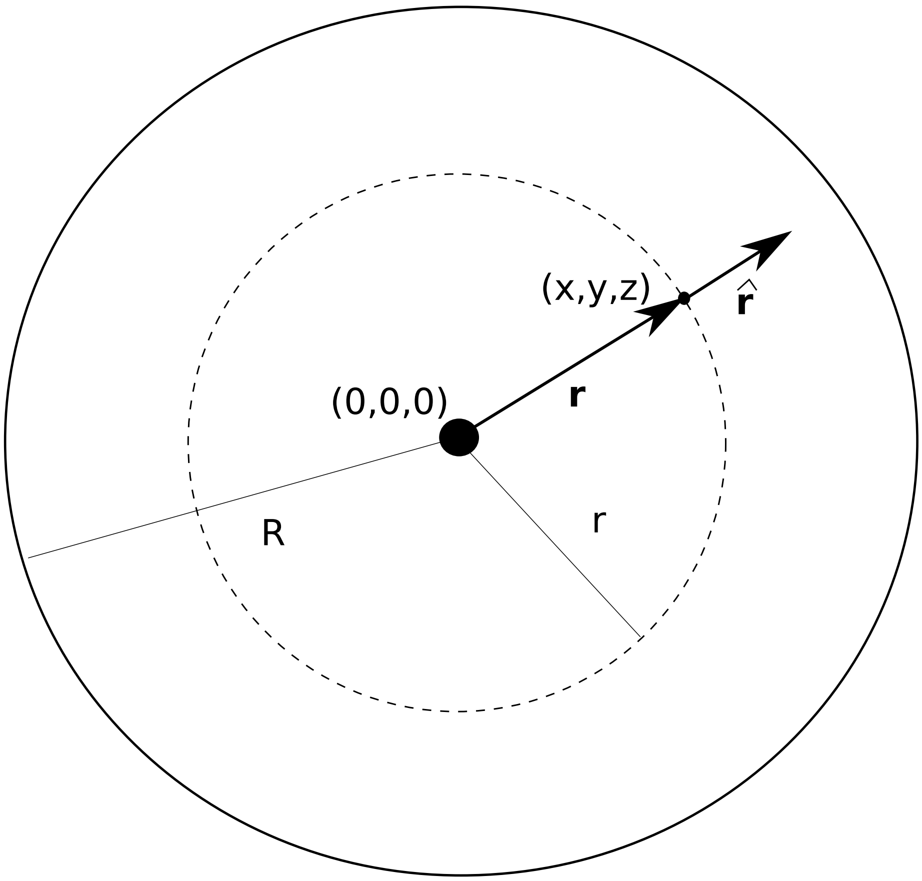

Figure 4: The position vector $\mathbf{r}$ of a point and the corresponding unit vector $\hat{\mathbf{r}}$, with the tail of the vector located at the point (x, y, z). Note that $\hat{\mathbf{r}}$ is normal to the spherical surface of radius $r = |\mathbf{r}|$ about the origin, so $\hat{\mathbf{n}} = \hat{\mathbf{r}}$ for this surface. If we have heat conduction in a sphere, this surface of radius r does not have to be the outer surface of the sphere: r is the distance from the origin to any given point (x, y, z) in the sphere, and the dashed spherical surface is the set of points that are at the same distance from the oriigin as (x, y, z). The outer surface of the sphere is at some fixed radius R. In deriving the heat equation for a sphere, we assume that each temperature contour is a spherical surface like the dashed surface shown, with every point on the surface at the same distance r from the origin.

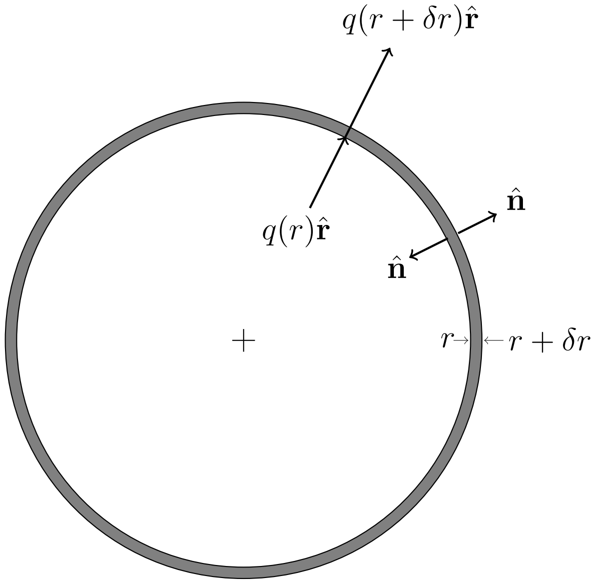

Figure 5: A spherical shell

or as a single second-order equation,

$$-\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 k \frac{\mathrm{d}T}{\mathrm{d}r}\right) = a. \tag{18}$$

This is a differential equation for $T(r)$. In order to obtain a solution that genuinely depends only on r, we have to assume that a is also a function of r only, if it depends on position at all.

Note 8 *We have derived* (17) *and* (18) *by using a number of non-trivial manipulations involving gradients and divergences. This does not make the physical origin of these equations very clear. It is actually not that difficult to derive the same equations from first principles, by adapting with what we did in note 3.*

First, if T depends only on r, then temperature contours are concentrical spherical surfaces centered on the origin. From the properties of the gradient, we know that ∇T and therefore the heat flux $\mathbf{q}_c = -k\nabla T$ is oriented perpendicularly to the contours. This means that heat flows in a radial direction. We also know, again from the properties of the gradient, that the magnitude of the temperature gradient ∇T is the derivative of temperature with distance in the direction perpendicular to the contours: the magnitude of ∇T is the derivative of T with respect to r. So the result

$$\nabla T = \frac{\mathrm{d}T}{\mathrm{d}r}\hat{\mathbf{r}}, \qquad \mathbf{q}_c = -k\frac{\mathrm{d}T}{\mathrm{d}r}\hat{\mathbf{r}}$$

follows directly from the properties of the gradient.

Knowing that heat flux is oriented radially with magnitude

$$q_c(r) = -k\frac{\mathrm{d}T}{\mathrm{d}r}$$

in the outward-pointing direction, we can then enforce conservation of energy. The easiest thing to do in a spherical geometry is to look at a spherical shell *between radii r and $r + \delta r$, meaning the volume that lies between the spherical surfaces of radii r and $r + \delta r$. This is the equivalent of looking at a slab between elevations x and $x + \delta x$ as we did previously when T depended only on x.*

The flux field is prependicular to the surfaces of the shell, and, because q_c depends only on r, the magnitude of flux is also uniform over each of the two surfaces. This means we can compute the amount of heat that flows into the shell from within in time δt as flux times area times time elapsed, or

$$q_c(r)4\pi r^2 \delta t,$$

and similarly, the amount of heat that flows out through the outer surface is

$$q_c(r + \delta r)4\pi(r + \delta r)^2 \delta t.$$

The net amount of heat that flows out of the shell is therefore

$$q_c(r + \delta r)4\pi(r + \delta r)^2 \delta t - q_c(r)4\pi r^2 \delta t.$$

This formula is equivalent to (7) *for a thin slab in note 3, and is equal to $\int_S \mathbf{q} \cdot \hat{\mathbf{n}}\, \mathrm{d}S \delta t$ for the surface S of the thin shell.*

In steady state, we need to equate the amount of heat that flows out of the shell to the amount of thermal energy produced within it. This is given by a times the volume of the shell. The volume of the shell is the difference between the volumes of a sphere of radius $r + \delta r$ and a sphere of radius r, or

$$\frac{4}{3}\pi(r + \delta r)^3 - \frac{4}{3}\pi r^3.$$

There are various ways of simpifying this. Here we use basic ideas of differentiation, writing

$$\frac{4}{3}\pi(r + \delta r)^3 - \frac{4}{3}\pi r^3 = \frac{4}{3}\pi\frac{(r + \delta r)^3 - r^3}{\delta r}\delta r \approx \frac{4}{3}\pi\frac{\mathrm{d}r^3}{\mathrm{d}r}\delta r = 4\pi r^2 \delta r.$$

This should not be surprising: the volume of the shell is the surface area of the shell times its thickness. This we could have intuited if we had split the surface of the shell into lots of pieces δS, each of which corresponds to a piece of the shell that has base area δS and thickness δr, giving a volume $\sum_{\delta S} \delta S \delta r = S \delta r$.

Multiplying roduction rate density by volume and time elapsed, we get the amount of thermal energy generated as

$$a4\pi r^2\delta r.$$

Equating this to the net amount of heat that flows out of the shell, we get

$$q_c(r+\delta r)4\pi(r+\delta r)^2\delta t - q_c(r)4\pi r^2\delta t = a4\pi r^2\delta r.$$

Rearranging,

$$a = \frac{1}{r^2}\frac{q_c(r+\delta r)(r+\delta r)^2 - q_c(r)r^2}{\delta r}.$$

Recalling the definition of a derivative,

$$\frac{\mathrm{d}f}{\mathrm{d}r} = \lim_{\delta r\to 0}\frac{f(r+\delta r)-f(r)}{\delta r},$$

we see that the fraction on the right-hand side corresponds to the derivative of $f(r) = r^2 q_c(r)$*, so*

$$a = \frac{1}{r^2}\frac{\mathrm{d}(r^2 q_c)}{\mathrm{d}r}.$$

Solving the steady state heat equation (18) for a sphere is a bit harder to do than for its equivalent for a slab (6). The key is to use separation of variables. How to do this is easiest to recognize if we again write it as a system of equations, this time in the slightly altered form

$$\frac{1}{r^2}\frac{\mathrm{d}Q}{\mathrm{d}r} = a \tag{19a}$$

$$Q = r^2 q = -r^2 k\frac{\mathrm{d}T}{\mathrm{d}r}. \tag{19b}$$

Here Q is simply a variable we have invented to make the separation of variables clearer. Physically, Q has units of watts. In fact, $4\pi Q$ would be the rate at which heat flows out through a spherical surface of radius r, as we have $4\pi Q = 4\pi r^2 q(r)$.

We will work through this for the simplest case of constant k and a. Start with (19a). Separating variables gives

$$\frac{\mathrm{d}Q}{\mathrm{d}r} = r^2 a,$$

and integrating gives

$$Q = \frac{ar^3}{3} + C_1. \tag{20}$$

Next, substitute for (19b) to find

$$r^2 q = -r^2 k\frac{\mathrm{d}T}{\mathrm{d}r} = \frac{ar^3}{3} + C_1. \tag{21}$$

Once again, separate variables,

$$\frac{\mathrm{d}T}{\mathrm{d}r} = -\frac{ar}{3k} - \frac{C_1}{kr^2}.$$

Now we can integrate again to give

$$T = -\frac{ar^2}{6k} + \frac{C_1}{kr} + C_2 \tag{22}$$

Equation (22) is the general solution of (18) for constant k and a. The procedure we have followed can also be adapted to cases where k and a are not constant. As in the case of equation(10) for slabs, we have two unknown constants, C_1 and C_2, which must be fixed through boundary conditions.

Again, we can consider Dirichlet or Neumann conditions (known flux or temperature), or more complicated physics. If our domain is a solid sphere with radius R, then we generally have a problem, however. We have only one boundary, the surface of the sphere at $r = R$. Suppose we have a known surface temperature $T(R) = T_s$. This will give us one equation, but we need to determine two constants.

The key to this is to realize that the centre of the sphere is not a surface, but that we still need a sensible solution to hold there. Our general solution (22) in general predicts that temperature T will become infinite as $r \to 0$ because it contains the term $C_1/(kr)$, and $1/r$ becomes infinite at $r = 0$.

Exercise 8 *Assume that $C_1 > 0$, and $a > 0$, $k > 0$ (which is physically necessary). Sketch the solution* (22)*.*

The only way to avoid this is to put $C_1 = 0$, in wich case (22) becomes

$$T = -\frac{ar^2}{6k} + C_2$$

and we have only one constant left to determine. Putting $T(R) = T_s$ gives

$$T_s = -\frac{aR^2}{6k} + C_2$$

or

$$C_2 = T_s + \frac{aR^2}{6k}.$$

The solution then becomes

$$T(r) = \frac{a(R^2 - r^2)}{6k} + T_s. \tag{23}$$

Exercise 9 *Sketch the solution* (23)*, first by plotting the graph of $T(r)$ against r, and then by plotting the contours of T as a function of x and y (putting $z = 0$).*

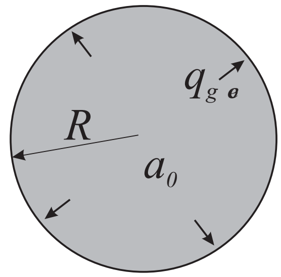

Figure 6: Heat production in the Earth and geothermal heat flux.

Note that there is a much deeper explanation of why in general we must have $C_1 = 0$ above, and what the physical meaning of C_1 is when we do not set it equal to zero. This is discussed at the end of these notes.

Exercise 10 *This question is about solving the steady-state heat equation in a solid sphere, and about heat transport in the interior of the Earth. Conductive heat flux reaching the surface of the Earth from below is calle geothermal heat flux.*

Assume geothermal heat flux at the Earth's surface is a uniform $q_{\mathrm{geo}} = 0.04\ W\,m^{-2}$*, and that the Earth has a surface temperature of* $T_{\mathrm{s}} = 280\ K$*. Assume also that the Earth is a perfect sphere of radius* $R = 6380\ km$*, and that it has uniform thermal conductivity* $k = 2\ W\ m^{-1}\ K^{-1}$*. Let heat be produced at a uniform rate* a_0 *per unit volume inside the Earth, and suppose the Earth is in a thermal steady state. The aim of the question is to find* a_0 *and the temperature profile* $T(r)$ *inside the Earth.*

1. *Keep all your calculations in terms of symbols (*k*,* a_0*,* R*,* q_{geo}*,* T_{surf} *etc.) until told to substitute numerical values.*
2. *Write down the steady-state heat equation inside the Earth.*
3. *By analogy with the case of the cylindrical steady-state heat equation, find a general solution for* $T(r)$*, involving two constants of integration* C_1 *and* C_2*.*
4. *Find a condition involving the limit* $r \to 0$ *that allows you to compute one of the constants of integration* C_1 *and* C_2*.*
5. *What boundary conditions can you apply at* $r = R$*, again using symbols not numbers?*
6. *Use one of these conditions to find* a_0 *in terms of* q_{geo} *and* R*. Evaluate a numerical answer.*
7. *How could you have found* a_0 *directly from conservation of energy for the whole sphere (meaning, not the differential equation* (1) *but its integral form*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \varepsilon\,\mathrm{d}V = -\int_S \varepsilon\mathbf{v}\cdot\hat{\mathbf{n}}\,\mathrm{d}S - \int_S \mathbf{q}_c\cdot\hat{\mathbf{n}}\,\mathrm{d}S + \int_V a\,\mathrm{d}V,$$

when we take into account the assumption that we have made in our model (no motion of material, steady state, a_0 spatially uniform, $\mathbf{q}_c$ points in a radial direction).

8. *Use the other condition at $r = R$ to find the remaining constant of integration in terms of a_0, k and T_s*

9. *Write down the temperature profile $T(r)$ inside the Earth in terms of r, k, a_0, R and T_s.*

10. *Find a numerical value for temperature at the centre of the Earth, and at a distance of 3000 km from the centre of the Earth.*

11. *The outer core of the Earth has a radius of approximately 2900 km. Assume that no rock can remain solid above 5000 K at pressures found inside the Earth's mantle. Comment. What is wrong with the model? Suggest something the model may be missing.*

Exercise 11 *This exercise is a harder variation of exercise 10, allowing for variations in thermal conductivity due to either a dependence on temperature or an explicit dependence on radial position (for instance because of chemical variations). The problem below is to find the degree of variation in k required to achieve a certain temperature at the centre of the Earth.*

Assume as before that the rate of heat production $a = a_0$ is constant in the Earth, which has radius R and has a geothermal heat flux q_{geo} at its surface. Assume also that the Earth is thermally in a steady state, and that its temperature field as well its thermal conductivity depend only on r radial distance from the centre of the earth but not on angular position (latitude, longitude). Let the surface temperature of the Earth be a uniform value T_surf.

1. *Show that the rate of heat production a_0 required to account for the geothermal heat flux q_{geo} is the same as in the previous question, regardless of whether k is constant or not.*

2. *Let k be of the form*
$$k = k_0 \exp\left[(R - r)/R_0\right],$$
where k_0 and R_0 are constants (and R is the radius of the Earth). For given values of k_0, a_0, T_surf and R, find an equation that must be satisfied by R_0 to ensure a temperature at the centre of the Earth of T_c.[1] *This equation cannot be solved analytically, but for given parameter values can be solved numerically or graphically. If k_0 = 2 W m^{-1} K^{-1}, T_surf = 280 K and T_c = 3000 K, and a_0 and R as in exercise 10, what is R_0 to three significant figures? Sketch the corresponding temperature distribution in the Earth.*

[1] Recall that the integral $\int x \exp(x)\,\mathrm{d}x$ can be done through integration by parts.

3. Let k be of the form

$$k = k_0 \exp\left[(T - T_{\mathrm{surf}})/T_0\right]$$

where T_0 is a constant. For given values of k_0, a_0, T_{surf} and R, find an equation that must be satisfied by T_0 in order to produce a a temperature at the centre of the Earth of T_c? If $k_0 = 2$ W m^{-1} K^{-1}, $T_{\mathrm{surf}} = 280$ K and $T_c = 3000$ K, and a_0 and R as in exercise 10, find T_0 to three significant figures. Sketch the corresponding temperature distribution in the Earth.

More geometries with symmetry: cylinders

Cylindrical geometries work much the same way as spherical geometries. If we have heat conduction in a cylinder, we may expect that temperature depends only on distance from the centreline of the cylinder. If we align that centreline with the z-axis, that distance is equal to

$$r = \sqrt{x^2 + y^2}.$$

Unfortunately, this *is* the same symbol as we used for distance from the origin for spherical geometries; which version of r is meant therefore depends on context.[2]

We therefore assume that $T = T(r)$, and similarly that k and a can depend at most on r. With $r = \sqrt{x^2 + y^2}$ and a radial unit vector defined now as

$$\hat{\mathbf{r}} = \frac{x\mathbf{i} + y\mathbf{j}}{\sqrt{x^2 + y^2}},$$

we can now show that

$$\nabla T = \frac{\mathrm{d}T}{\mathrm{d}r}\hat{\mathbf{r}}, \qquad \nabla \cdot (q(r)\hat{\mathbf{r}}) = \frac{1}{r}\frac{\mathrm{d}(rq(r))}{\mathrm{d}r}. \tag{24}$$

Substituting into the steady state heat equation (4) gives

$$-\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(rk\frac{\mathrm{d}T}{\mathrm{d}r}\right) = a. \tag{25}$$

Exercise 12 *Use the product and chain rules to demonstrate that* (24) *holds.*

Exercise 13 *We can also derive* (25) *from first principles, following notes 3 and 8. Assume $T = T(r)$ as above, with r distance from the cylinder centreline, and $\hat{\mathbf{r}}$ the radial unit vector pointing away from the centreline.*

By analogy with note 8, show that the properties of the gradient ensure that $\mathbf{q}_c = -k\frac{\mathrm{d}T}{\mathrm{d}r}\hat{\mathbf{r}}$. Let

$$q_c(r) = -k\frac{\mathrm{d}T}{\mathrm{d}r}.$$

[2] This is not a deliberate attempt to be confusing: we are following standard notation here.

Show for a cylinrical shell with inner radius r, outer radius $r + \delta r$ and height h that the net amount of heat conducted out of the shell in time δt is

$$q_c(r + \delta r)2\pi(r + \delta r)h\delta t - q_c(r)2\pi rh\delta t,$$

and that the amount of thermal energy generated in the shell is

$$a2\pi rh\delta t,$$

and derive (25).

Solving the heat equation in a cylindrical geometry follows the procedure for spherical geometries almost exactly. The following exercise will give you practice doing this.

Exercise 14 *Assume k and a are constant in* (25). *Show that a general solution takes the form*

$$T(r) = -\frac{a_r^2}{4k} - \frac{C_1}{k}\log(r) + C_2.$$

Consider the following two cases:

1. *Assume the cylinder is solid, from $r = 0$ to an outer boundary $r = R$. Assume the temperature on the outer boundary is prescribed as $T(R) = T_s$, Find C_1 and C_2, and write down the full solution for $T(r)$ in terms of a, k, R.*

2. *Assume that you have a pipe with inner radius R_1 and outer radius R_2. Assume that the inner temperature is T_1 and the outer temperature is T_2, and that no heat is produced in the pipe. Find $T(r)$ in terms of T_1, T_2, R_1, R_2 and k. Compute the rate at which heat is lost from a length h of the pipe.*

Point sources

Recall that in the general solution for temperature in a sphere, equation (22), we had a term $-C_1/(kr)$ that we decided needed to be excluded by setting C_1 because temeprature would otherwise become infinite. There is actually a real reason why this term appears, and we can attach physical meaning to it. To understand this, we need to back up a little from the general solution.

Remember that we are solving the steady state heat equation with constant thermal conductivity k, which can be written as (5),

$$-k\nabla^2 T = a.$$

Under the assumption that T and a depend on position only through distance r to the origin, this can be written as (18), or in turn as a system of equations

$$\frac{1}{r^2}\frac{\mathrm{d}(r^2 q(r))}{\mathrm{d}r} = a$$
$$q(r) = -k\frac{\mathrm{d}T}{\mathrm{d}r},$$

where q is heat flux. Separating variables in the first of these equations gives

$$\frac{\mathrm{d}(r^2 q(r))}{\mathrm{d}r} = ar^2,$$

and integrating gives

$$r^2 q(r) = \frac{ar^3}{3} + C_1$$

assuming that a is constant; variable a would lead to a similar result.

It follows that

$$q(r) = \frac{ar}{3} + \frac{C_1}{r^2}.$$

The problem is therefore not just that T becomes infintie if we do not set C_1 to zero, but $q(r)$ does too. In fact, $q(r)$ goes to infinity as $1/r^2$. This is often called an inverse square law.

What does this mean physically? Remember that $q(r)$ is a flux, a rate of heat transfer per unit area. More precisely, $q(r)$ is the flux at a distance r from the origin, and it passes at right angles through a spherical surface around the origin that has area $4\pi r^2$. We can therefore calculate that rate at which heat passes through that spherical surface as flux times area,

$$Q = 4\pi r^2 q(r),$$

with units of watts.[3] Substituting for $q(r)$, we get

$$Q = \frac{4}{3}\pi r^3 + 4\pi C_1.$$

The first term on the right is easy to interpret. $4pir^3/3$ is the volume enclosed by the spherical surface, and

$$\frac{4}{3}\pi r^3 a$$

is the rate at which thermal energy is generated due to the production rate density a in that volume. So what is $4\pi C_1$ doing on the right-hand side?

[3] This Q is slightly different from the Q we defined previously in (19); the two differ by a factor of 4π.

The answer to this lies in shrinking the volume to zero radius, that is taking the limit $r \to 0$. The term $4\pi r^3 a/3$ obviously goes away, so we are left with

$$\lim_{r\to 0} Q = 4\pi C_1.$$

This the rate at which heat passes through a spherical surface of infinitely small radius around the origin. The only possible interpretation is that $4\pi C_1$ is the strength of a point source of heat located at the origin. Setting $C_1 = 0$ is equivalent to saying there is no such point source.

Exercise 15 *Consider the heat conduction problem in a cylinder, in the form*

$$\frac{1}{r}\frac{\mathrm{d}(rq(r))}{\mathrm{d}r} = a,$$
$$q(r) = -k\frac{\mathrm{d}T}{\mathrm{d}r}.$$

Show that

$$q(r) = \frac{ar}{2} + \frac{C_1}{r}, \tag{26}$$

and consider the rate at which heat is conducted through a cylinder of radius r and length h. Interpret the two terms on the right-hand side. Show that $2\pi C_1$ is the strenght of a line source *at the centreline of the cylinder.*

Now we know that $q(r) = C_1/r^2$ corresponds to a heat source at the origin. As $Q_0 = 4\pi C_1$ is the strength of the heat source, we can write the flux alternatively as $Q_0/(4\pi r^2)$. From here, we can generalize to multiple heat sources at different locations. We focus on the steady state heat equation with constant thermal conductivity, for which Poisson's equation (5) holds:

$$-k\nabla^2 T = a \tag{27}$$

Recall that the temperature field corresponding to $q(r) = C_1/r^2$ is

$$T(r) = -\frac{Q_0}{4\pi k r} + C$$

where C is a constant, so that $q(r) = -k\,\mathrm{d}T/\,\mathrm{d}r$. We can re-write this temperature field as

$$T = -\frac{Q_0}{4\pi k|\mathbf{r}|} + C$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ is the position vector of the point at which we want to find T. Now imagine that the heat source is located not at the origin but at another point (x_0, y_0, z_0). If we write the position vector of that point as $\mathbf{r}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$, the distance from (x, y, z) to the source is no longer $r = |\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}$, but

$$|\mathbf{r} - \mathbf{r}_0| = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}.$$

It turns out that the temperature field to this differently point heat source can be written as

$$T(x,y,z) = -\frac{Q_0}{4\pi k|\mathbf{r}-\mathbf{r}_0|},$$

and the flux field can be written as

$$\mathbf{q}(x,y,z) = \frac{Q_0}{4\pi|\mathbf{r}-\mathbf{r}_0|^2}\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|}.$$

This simply corresponds to translating the original flux field

$$\mathbf{q} = \frac{Q_0}{4\pi r^2}\hat{\mathbf{r}} = \frac{Q_0}{4\pi|\mathbf{r}|^2}\frac{\mathbf{r}}{|\mathbf{r}|}$$

by a displacement $\mathbf{r}_0$. The next two exercises will confirm that this works.

Exercise 16 *Draw a diagram showing the origin and the points (x,y,z) as well as (x_0,y_0,z_0). Indicate the position vectors $\mathbf{r}$ and $\mathbf{r}_0$ as well as the vector $\mathbf{r}-\mathbf{r}_0$. Placing the tail of the vector at the point (x,y,z), also draw the* unit vector

$$\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|}.$$

Exercise 17 *To look at point sources for the steady state heat equation that are not located at the origin, let*

$$T(x,y,z) = -\frac{Q_0}{4\pi k|\mathbf{r}-\mathbf{r}_0|} + C,$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathbf{r}_0 = x_0\mathbf{i} + y_0\mathbf{j} + z_0\mathbf{k}$ is a fixed position vector. Let k be a constant. First use the definition of the gradient and divergence to show that

$$\nabla|\mathbf{r}-\mathbf{r}_0| = \frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|}$$

and

$$\nabla\cdot(\mathbf{r}-\mathbf{r}_0) = 3.$$

Next, use the chain rule for gradients to show that

$$\mathbf{q}_c = -k\nabla T = \frac{Q_0}{4\pi|\mathbf{r}-\mathbf{r}_0|^2}\frac{\mathbf{r}-\mathbf{r}_0}{|\mathbf{r}-\mathbf{r}_0|}.$$

provided $\mathbf{r} \neq \mathbf{r}_0$ (that is, we are not located at the point source itself, so that $(x,y,z) \neq (x_0,y_0,z_0)$). Then use the product and chain rules for divergences to show that

$$-k\nabla^2 T = \nabla\cdot\mathbf{q}_c = 0,$$

provided again that $\mathbf{r} \neq \mathbf{r}_0$. This shows that no heat is produced at any points away from the point source: if we substitute this into (27), we find $a = 0$ provided $\mathbf{r} \neq \mathbf{r}_0$.

Next, for any spherical surface S around the point (x_0, y_0, z_0), defined through

$$|\mathbf{r} - \mathbf{r}_0| = R,$$

we have

$$\int_S \mathbf{q}_c \cdot \hat{\mathbf{n}} \, \mathrm{d}S = Q_0.$$

To do this, first write down the normal vector. To construct the normal vector, remember that the unit normal to a surface defined through $f(x, y, z) = 0$ can be written in the form

$$\hat{\mathbf{n}} = \pm \frac{\nabla f}{|\nabla f|}.$$

Use the normal you have computed to show that

$$\mathbf{q}_c \cdot \hat{\mathbf{n}} = \frac{Q_0}{4\pi R^2}$$

and therefore that

$$\int_S \mathbf{q}_c \cdot \hat{\mathbf{n}} \, \mathrm{d}S = Q_0.$$

The results above hold if we still have spherical symmetry, except that symmetry is about the point (x_0, y_0, z_0) rather than the origin. We can go a step further and look at the case where we have multiple point sources, which breaks the spherical symmetry. Provided we have no boundaries at which any boundary conditions hold, we can show that, if we have point sources of strength Q_i $(i = 1, 2 \ldots n)$ at points (x_i, y_i, z_i) with position vectors $\mathbf{r}_i = x_i\mathbf{i} + y_i\mathbf{j} + z_i\mathbf{k}$, then the temperature field is simply the sum of temperature fields due to individual point sources

$$T(x, y, z) = \sum_i \frac{Q_i}{4\pi k |\mathbf{r} - \mathbf{r}_i|} + \text{constant} \tag{28}$$

and the flux field is

$$\mathbf{q} = \sum_i \frac{Q_i}{4\pi |\mathbf{r} - \mathbf{r}_i|^2} \frac{\mathbf{r} - \mathbf{r}_i}{|\mathbf{r} - \mathbf{r}_i|}.$$

The following exercise will confirm this.

Exercise 18 *Let $T(x, y, z) = \sum_i T(x, y, z) + C$, where*

$$T_i(x, y, z) = \frac{Q_i}{4\pi k |\mathbf{r} - \mathbf{r}_i|}$$

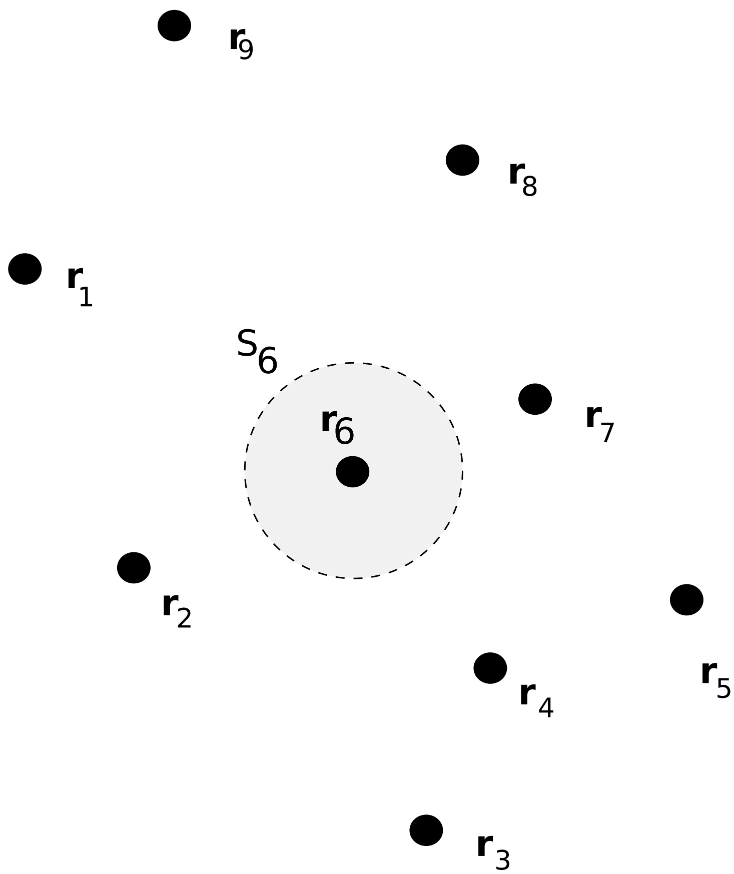

Figure 7: An example of multiple sources. The surface S_6 would be used to show that the flux solution (28) does correspond to a source of strength Q_6 at position $\mathbf{r}_6$ (meaning $j = 6$). The sum $\sum_{i \neq j}$ would correspond to the sum over the point sources at $\mathbf{r}_1$, $\mathbf{r}_2$, $\mathbf{r}_3$, $\mathbf{r}_4$, $\mathbf{r}_5$, $\mathbf{r}_7$ and $\mathbf{r}_8$.

is the temperature field due to an individual point source. Also let

$$\mathbf{q}_i = \frac{Q_i}{4\pi|\mathbf{r}-\mathbf{r}_i|^2}\frac{\mathbf{r}-\mathbf{r}_i}{|\mathbf{r}-\mathbf{r}_i|}$$

be the flux field due to an individual point source, and define $\mathbf{q}(x,y,z) = \sum_i \mathbf{q}_i(x,y,z)$ *Following exercise 17, show that*

$$-k\nabla T_i = \mathbf{q}_i,$$

with

$$\nabla\cdot\mathbf{q}_i = 0$$

provided $(x,y,z) \neq (x_i,y_i,z_i)$. *Using the properties of gradients, show that*

$$-k\nabla T = -\sum_i k\nabla T_i = \sum_i \mathbf{q}_i,$$

and using the properties of divergences, show that

$$-k\nabla^2 T = \nabla\cdot\mathbf{q} = \sum_i \nabla\cdot\mathbf{q}_i = 0$$

provided $(x,y,z) \neq (x_i,y_i,z_i)$ *for* all *the point source locations. This part should be straightforward. Note that we are using something called the* linearity *of the gradient and divergence, and therefore of the differential operator* ∇^2*:*

$$\nabla\left(\sum_i T_i\right) = \sum_i \nabla T, \qquad \nabla\cdot\left(\sum_i \mathbf{q}_i\right) = \sum_i \nabla\cdot\mathbf{q}_i, \qquad \nabla^2\left(\sum_i T_i\right) = \sum_i \nabla^2 T_i.$$

This ensures that, if each T_i *satisfies Poisson's equation with zero production rate density,*

$$-k\nabla^2 T_i = 0,$$

then the sum over the T_i*'s satisfies the same equation.*

Next, let S_j *be a spherical surface centered on* (x_j,y_j,z_j). *Let the radius of* S_j *be small enough that none of the other point sources* (x_i,y_i,z_i) $(i = 1,\ldots,n$ *and* $i \neq j)$ *are inside* S_j. *Show that*

$$\int_{S_i} \mathbf{q}_c\cdot\hat{\mathbf{n}}\,\mathrm{d}S = Q_i.$$

To do so, write

$$\begin{aligned}\int_{S_j} \mathbf{q}_c\cdot\hat{\mathbf{n}}\,\mathrm{d}S &= \int_{S_j}\sum_i \mathbf{q}_i\cdot\hat{\mathbf{n}}\,\mathrm{d}S \\ &= \sum_i \int_{S_j} \mathbf{q}_i\cdot\hat{\mathbf{n}}\,\mathrm{d}S\end{aligned}$$

The sum can be splot into $\int_{S_j} \mathbf{q}_j \cdot \hat{\mathbf{n}}\,\mathrm{d}S$ *and all the remaining terms* $\sum_{i\neq j}\int_{S_j} \mathbf{q}_i \cdot \hat{\mathbf{n}}\,\mathrm{d}S$*,*

$$\begin{aligned}\int_{S_j} \mathbf{q}_c \cdot \hat{\mathbf{n}}\,\mathrm{d}S &= \int_{S_j} \mathbf{q}_j \cdot \hat{\mathbf{n}}\,\mathrm{d}S + \sum_{i\neq j}\int_{S_j} \mathbf{q}_i \cdot \hat{\mathbf{n}}\,\mathrm{d}S \\ &= \int_{S_j} \mathbf{q}_j \cdot \hat{\mathbf{n}}\,\mathrm{d}S + \sum_{i\neq j}\int_{V_j} \nabla \cdot \mathbf{q}_i\,\mathrm{d}V,\end{aligned}$$

where we have applied the divergence theorem to the terms in the sum over i*, but not to the first term. Use this and exercise 17 to show that*

$$\int_{S_j} \mathbf{q}_c \cdot \hat{\mathbf{n}}\,\mathrm{d}S = \int_{S_j} \mathbf{q}_j \cdot \hat{\mathbf{n}}\,\mathrm{d}S = Q_j.$$

This means that the amount of heat that flows out through a spherical surface that encloses only the j*th heat source is given by the strength* Q_j *of that heat source.*

Now, how does this relate to heat flow with volume heat sources, i.e., to equation (5) with a non-zero rate of heat production per unit volume $a(x, y, z)$, but no point sources? The answer is that we can pretend that we can turn these volume heat sources into point sources, as follows: Suppose that a is non-zero in some volume V. Split V up into small volumes ΔV in the usual way. Heat is produced in each of these volumes at a rate $a(x_i, y_i, z_i)\Delta V$, where (x_i, y_i, z_i) denotes the location of each small volume. Now all we have to do is pretend that there is in fact a point source of strength $Q_i = a(x_i, y_i, z_i)\Delta V$ at the point (x_i, y_i, z_i), in which case the formula (28) gives

$$T(x, y, z) = \sum_i \frac{a(x_i, y_i, z_i)\Delta V}{4\pi k|\mathbf{r} - \mathbf{r}_i|} + \text{constant}.$$

In the usual way, we can then recognize the sum over small volumes ΔV as a volume integral, so

$$T(x, y, z) = \int_V \frac{a(x', y'z')\,\mathrm{d}V'}{4\pi k\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}} + \text{constant},$$

where $\mathrm{d}V' = \mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z'$. Now, we have taken a large number of mathematical liberties to arrive at this result, but it turns out that this really does work provided that there are no boundary conditions to worry about — the domain in which heat conduction occurs is infinite. The integral on the right is known as a *Green's function* representation of the temperature field. The solution T is given by the integral of the rate of heat production $a(x', y', z')$ times a function

$$G(x, y, z, x', y', z') = \frac{1}{4\pi k\sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}},$$

known as a Green's function.

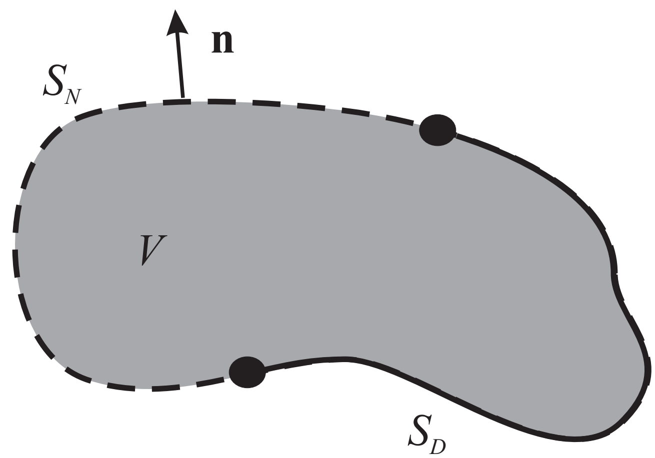

Figure 8: The geometry on which Poisson's equation is solved.

More: uniqueness of solutions

We finish these notes of with the following problem: When we computed steady states for systems with symmetries, we made arguments like the following: suppose we have a sphere of radius R and the rate of heat production in the sphere $a(r)$ at a point in the sphere depends only on the distance r of that point from the centre of the sphere, and that the temperature on the surface of the sphere is uniform. It is then reasonable to assume that temperature inside the sphere also depends only on distance r from the centre of the sphere, but not on angular position. Based on that assumption, we can then proceed to calculate a valid solution $T(r)$ of the steady-state heat equation (Poisson's equation, (5)). This however leaves a question: just because surface temperature and rate of heat production do not depend on angular position, does it follow that temperature also does not depend on angular position?

It may seem obvious that temperature should not depend on angular position, but there are lots of examples of physical processes in which there is *symmetry breaking*. For instance, a steady wind blowing over a sand surface can spontaneously cause ripples and dunes with three-dimensional structure to form, so interesting three-dimensional structures can emerge even if a process is driven in a way that does not contain any structure to start with. Is it similarly possible that the temperature field in our conducting sphere could have some much more interesting patterned appearance than the simple radius-dependent temperature field $T(r)$ we have so far been able to compute? The answer to this question is no: for a given set of sufficiently simple boundary conditions, Poisson's equation (5) admits only one solution. This can be shown relatively easily using the divergence theorem, so we give a demonstration here.

Let temperature T satisfy Poisson's equation in some volume V, and suppose that the surface S of this volume can be split into two parts, S_N and S_D. On S_N, assume that the heat flux $\mathbf{q} \cdot \hat{\mathbf{n}}$ out of the volume is prescribed value q_n (this is known as Neumann boundary conditions) and that on S_D, the temperature $T = T_S$ is prescribed (also known as Dirichlet boundary conditions)

We want to show that there can be only one solution T for this problem. The easiest way to do this is to suppose that there could be two different solutions, T_1 and T_2, both satisfying the problem:

$$\begin{aligned}
-k\nabla^2 T_1 &= a && \text{in } V \\
-k\nabla^2 T_2 &= a && \text{in } V \\
-k\nabla T_1 \cdot \hat{\mathbf{n}} &= q_n && \text{on } S_N \\
-k\nabla T_2 \cdot \hat{\mathbf{n}} &= q_n && \text{on } S_N \\
T_1 &= T_S && \text{on } S_D \\
T_2 &= T_S && \text{on } S_D
\end{aligned}$$

Now take the difference between these two solutions,[4]

$$T' = T_2 - T_1.$$

If we can show that $T' = 0$, we will have succeeded. But subtracting equations above, we get

$$\begin{aligned}
-k\nabla^2 T' &= 0 && \text{on } V && (29) \\
-k\nabla T' \cdot \hat{\mathbf{n}} &= 0 && \text{on } S_N && (30) \\
-k\nabla T' \cdot \hat{\mathbf{n}} &= 0 && \text{on } S_D && (31)
\end{aligned}$$

Next, we multiply (29) by T' itself, giving

$$T'\nabla^2 T' = 0.$$

But we know from the product rule for divergences that

$$\nabla \cdot (T'\nabla T') = T'\nabla \cdot \nabla T' + \nabla T' \cdot \nabla T'$$

so that

$$T'\nabla^2 T' = \nabla \cdot (T'\nabla T') - |\nabla T'|^2 = 0$$

Integrating this over the volume V gives

$$\int_V \nabla \cdot (T'\nabla T')\, \mathrm{d}V - \int_V |\nabla T'|^2\, \mathrm{d}V = 0. \tag{32}$$

[4]Again, do not be intimidated by the prime on T': this does not indicate anything deep like a derivative. T' is simply a difference between two temperatures as defined here

But we can apply the divergence to the first term,

$$\int_V \nabla \cdot (T' \nabla T')\,\mathrm{d}V = \int_S T' \nabla T' \cdot \hat{\mathbf{n}}\,\mathrm{d}S$$
$$= \int_{S_N} T' \nabla T' \cdot \hat{\mathbf{n}}\,\mathrm{d}S + \int_{S_D} T' \nabla T' \cdot \hat{\mathbf{n}}\,\mathrm{d}S$$

But from (30), we see that the surface integral over S_N must be zero, and similarly, the surface integral over S_D is zero from (31). Hence we are left with the second term in (32):

$$\int_V |\nabla T'|^2\,\mathrm{d}V = 0.$$

But $|\nabla T'|^2$ is never negative, so this integral will not be zero unless $|\nabla T'| = 0$ everywhere in V. (If $|\nabla T'|$ were positive in some part of V, the contribution of that region to the volume integral could not be cancelled out by a negative contribution from some other part of V.)

But if $\nabla T' = 0$, then T' has zero partial derivatives with respect to x, y and z. Hence T' must be a constant everywhere in V.[5] But we know that $T' = 0$ on part of the boundary of V (i.e., on S_D) so T' must be zero everywhere in V. Hence $T_1 = T_2$, and there can be only one solution.

[5] There is actually a technical complication here: V must be *connected*, i.e., consist of a single piece. Otherwise, T' must be constant on each part of V but could differ between them.