Midterm: EOSC 250

2 March, 2016

This exam consists of two questions worth ten marks each. Available marks for each part of a question are indicated in brackets. Attempt both questions. Read the questions carefully. You have 45 minutes.

1. This question is about differential equations. If you put a straw in a cup of water, you will notice that water flows up into the straw. This is caused by surface tension. This question is about a model for the motion of water in the straw. There are three forces acting on the water in the straw: gravity, surface tension, and a drag force due to the viscosity of the water resisting flow. Let $h(t)$ be the water level in the straw. The gravitational force on the water in the straw is

$$F_1 = \rho g \pi r^2 h$$

acting in a *downward* direction, where ρ, g and r are constants (r is the radius of the straw). The surface tension force is

$$F_2 = 2\pi r c$$

acting in the *upward* direction, where c is another constant related to surface tension. The drag force is given by

$$F_3 = 8\mu\pi h \frac{\mathrm{d}h}{\mathrm{d}t}$$

in the downward direction, where μ is the (constant) viscosity of water.

(a) (1 point) A model for how fast the water level h in the tube rises is

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{r^2}{8\mu}\left(\frac{2c}{r}\frac{1}{h} - \rho g\right) \tag{1}$$

What assumption about forces does this model make?

(b) (6 points) Use separation of variables to solve for t as a function of h, assuming that $h = 0$ at $t = 0$. *Note you will not be able to solve analytically for h as a function of t. You may also need to use the fact that*

$$\frac{h}{a-h} = -1 + \frac{a}{a-h}.$$

You can use this fact, and do not need to prove it.

(c) (1 point) What value does the water level approach as $t \to \infty$? *Note that this is the steady state value of h. You can assume that $h > 0$.*

(d) (2 points) Let h_∞ denote the limit $h_\infty = \lim_{t\to\infty} h(t)$ that you computed in the previous part. How long does it take h to attain a value $h_\infty/2$?

2. This question is about volume integrals. Let V be the tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$ and $(1, 0, 1)$. Let

$$\rho = x^2 + y^2 + z^2.$$

   (a) (2 points) Sketch V.

   (b) (6 points) Compute $\int_V \rho \, \mathrm{d}V$.

   (c) (2 points) Compute $\int_V 1 \, \mathrm{d}V$,

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Equation Summary

$$\mathbf{q} = -k\nabla T$$

$$\nabla T = \mathbf{i}\frac{\partial T}{\partial x} + \mathbf{j}\frac{\partial T}{\partial y} + \mathbf{k}\frac{\partial T}{\partial z}$$

$$A = \pi r^2$$

$$A = 2\pi r h$$

$$A = 4\pi r^2$$

$$A = \frac{1}{2} \times \text{base} \times \text{height}$$

$$V = \pi r^2 h$$

$$V = \frac{4}{3}\pi r^3$$

$$V = \frac{1}{3} \times \text{base} \times \text{height}$$

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(k\frac{\mathrm{d}T}{\mathrm{d}x}\right) = a(x), \qquad q(x) = -k\frac{\mathrm{d}T}{\mathrm{d}x}$$

$$-\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(rk\frac{\mathrm{d}T}{\mathrm{d}r}\right) = a(r)$$

$$-\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2k\frac{\mathrm{d}T}{\mathrm{d}r}\right) = a(r)$$

$$\frac{\mathrm{d}q}{\mathrm{d}x} = a(x)$$

$$\frac{1}{r}\frac{\mathrm{d}(rq)}{\mathrm{d}r} = a(r)$$

$$\frac{1}{r^2}\frac{\mathrm{d}(r^2q)}{\mathrm{d}r} = a(r)$$

$$\rho c\frac{\partial T}{\partial t} - \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) = a$$

$$\mathbf{q} = -k\nabla T$$

$$\nabla \cdot \mathbf{q} = \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}, \qquad \mathbf{q} = q_x\mathbf{i} + q_y\mathbf{j} + q_z\mathbf{k}$$

$$\nabla \cdot \mathbf{q} = a$$

$$-\nabla \cdot (k\nabla T) = a$$

$$\nabla \cdot \nabla T = \nabla^2 T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}$$

$$\hat{\mathbf{n}} = \pm \frac{\mathbf{k} - \frac{\partial h}{\partial x}\mathbf{i} - \frac{\partial h}{\partial y}\mathbf{j}}{\sqrt{1 + \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2}}$$

$$\mathrm{d}S = \sqrt{1 + \left(\frac{\partial h}{\partial x}\right)^2 + \left(\frac{\partial h}{\partial y}\right)^2}\,\mathrm{d}x\,\mathrm{d}y$$

$$\int \mathbf{q} \cdot \hat{\mathbf{n}}\,\mathrm{d}S = \int\int q_z - q_x \frac{\partial h}{\partial x} - q_y \frac{\partial h}{\partial y}\,\mathrm{d}y\,\mathrm{d}x$$

$$\int_S \mathbf{q} \cdot \hat{\mathbf{n}}\,\mathrm{d}S = \int_V \nabla \cdot \mathbf{q}\,\mathrm{d}V$$