

Calculus and differential equations: EOSC 352

due 20th January, 2023

1. Calculate the gradient of the following scalar fields:

(a)

$$\phi(x, y, z) = y \cos(x) + x \cos(y),$$

(b)

$$\phi(x, y, z) = x^2 - y^2,$$

2. Compute the divergence of the following vector fields:

(a)

$$\mathbf{q}(x, y, z) = (x + y)\mathbf{i} + (x - y)\mathbf{j} + z\mathbf{k},$$

(b)

$$\mathbf{q}(x, y, z) = -(x^2 + y^2)y\mathbf{i} + (x^2 + y^2)x\mathbf{j},$$

3. Let

$$\Phi(x, y, z) = x \exp(x) \cos(y) - y \exp(x) \sin(y).$$

Calculate $\nabla^2 \Phi$.

4. Let $\rho = \sin(x) \cos(y)$. Compute $\int_V \rho \, dV$, if V is the triangular prism given by $0 < x < 1 - y$, $0 < y < 1$, $0 < z < 1$.

5. Compute the integral $\int_S \mathbf{q} \cdot \hat{\mathbf{n}} \, dS$, where S is the part of the surface $z = 2 - x^2 - y^2$ that lies above the triangle $0 < y < 1$, $-y < x < y$, $\hat{\mathbf{n}}$ is the upward-pointing unit normal to the surface, and

$$\mathbf{q}(x, y, z) = x\mathbf{i} + y\mathbf{j} - z\mathbf{k}.$$

6. Let

$$\mathbf{q} = x^2\mathbf{i} + y^2\mathbf{j} - 2zx\mathbf{k},$$

and let S be the surface of the unit cube ($0 < x < 1$, $0 < y < 1$, $0 < z < 1$), with $\hat{\mathbf{n}}$ its outward-pointing unit normal. Calculate

$$\int_S \mathbf{q} \cdot \hat{\mathbf{n}} \, dS.$$

7. Let

$$\frac{dN}{dT} = \lambda N(N_c - N)$$

with λ and N_c constant. Define $n = \alpha N$, $t = \beta T$ with α, β constant. Substitute for N and T , using the chain rule to convert the derivative with respect to T into a derivative with respect to t . Find α and β such that the resulting equation for n as a function of t takes the form.

$$\frac{dn}{dt} = n(1 - n)$$

Let the initial condition be $n(0) = n_0$. Find $n(t)$. You can use the fact that

$$\frac{1}{n} + \frac{1}{1 - n} = \frac{1}{n(1 - n)}.$$

8. Find a general solution to

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0.$$