Calculus and differential equations: EOSC 352

due 16th January, 2024

1. Calculate the gradient of the following scalar fields:

   (a)
   $$\phi(x, y, z) = y\cos(x) + x\cos(y),$$

   (b)
   $$\phi(x, y, z) = x^2 - y^2,$$

2. Compute the divergence of the following vector fields:

   (a)
   $$\mathbf{q}(x, y, z) = (x + y)\mathbf{i} + (x - y)\mathbf{j} + z\mathbf{k},$$

   (b)
   $$\mathbf{q}(x, y, z) = -(x^2 + y^2)y\mathbf{i} + (x^2 + y^2)x\mathbf{j},$$

3. Let
   $$\Phi(x, y, z) = x\exp(x)\cos(y) - y\exp(x)\sin(y).$$
   Calculate $\nabla^2\Phi$.

4. Let $\rho = \sin(x)\cos(y)$. Compute $\int_V \rho \, \mathrm{d}V$, if V is the triangular prism given by $0 < x < 1 - y$, $0 < y < 1$, $0 < z < 1$.

5. Compute the integral $\int_S \mathbf{q} \cdot \hat{\mathbf{n}} \, \mathrm{d}S$, where S is the part of the surface $z = 2 - x^2 - y^2$ that lies above the triangle $0 < y < 1$, $-y < x < y$, $\hat{\mathbf{n}}$ is the upward-pointing unit normal to the surface, and
   $$\mathbf{q}(x, y, z) = x\mathbf{i} + y\mathbf{j} - z\mathbf{k}.$$

6. Let
   $$\mathbf{q} = x^2\mathbf{i} + y^2\mathbf{j} - 2zx\mathbf{k},$$
   and let S be the surface of the unit cube ($0 < x < 1$, $0 < y < 1$, $0 < z < 1$), with $\hat{\mathbf{n}}$ its outward-pointing unit normal. Calculate
   $$\int_S \mathbf{q} \cdot \hat{\mathbf{n}} \, \mathrm{d}S.$$

7. Let
$$\frac{\mathrm{d}N}{\mathrm{d}T} = \lambda N(N_c - N)$$
with λ and N_c constant. Define $n = \alpha N$, $t = \beta T$ with α, β constant. Substitute for N and T, using the chain rule to convert the derivative with respect to T into a derivative with respect to t. Find α and β such that the resulting equation for n as a function of t takes the form.
$$\frac{\mathrm{d}n}{\mathrm{d}t} = n(1-n)$$
Let the initial condition be $n(0) = n_0$. Find $n(t)$. You can use the fact that
$$\frac{1}{n} + \frac{1}{1-n} = \frac{1}{n(1-n)}.$$

8. Find a general solution to
$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} + y = 0.$$