Final: EOSC 352

8 December, 2010

This exam consists of four questions worth ten marks each. Available marks for each part of a question are indicated in brackets. Attempt THREE questions. You have 2 hours 20 minutes.

1. Consider one-dimensional heat conduction forced by an oscillating surface temperature. Also assume that heat flux tends to zero at infinity

$$\rho c_p \frac{\partial T}{\partial t} - k\frac{\partial^2 T}{\partial x^2} = 0 \qquad \text{for } 0 < x \qquad (1a)$$

$$T = T_0 \cos(\omega t) \qquad \text{at } x = 0 \qquad (1b)$$

$$-k\frac{\partial T}{\partial x} \to 0 \qquad \text{as } x \to \infty \qquad (1c)$$

ρ, c_p and k are constants.

   (a) (4 points) Let

$$T(x,t) = \mathrm{Re}\left[T_0 \exp(i\omega t + \lambda x)\right]$$

   Substitute this into (1a). Derive and solve an equation for λ so that (1a) is satisfied.

   (b) (1 point) You should have two possible values for λ. Which one allows (1c) to be satisfied?

   (c) (2 points) Take the real part in $T(x,t) = \mathrm{Re}[T_0 \exp(i\omega t + \lambda x)]$ to find an expression for T that does not involve i.

   (d) (3 points) Without resorting to complex numbers, compute $\partial T/\partial t$ and $\partial^2 T/\partial x^2$ for the expression you have obtained (this will require the product rule). Check that the heat equation is indeed satisfied.

2. This question is about the heat equation with advection in steady state, mass conservation and scaling.

   (a) (1 point) From your equation sheet, write down the heat equation with advection, *assuming a steady state*

   (b) (1 point) Near the centre of an ice sheet, a possible velocity field is

$$\mathbf{u}(x, z) = ax/h\mathbf{i} - az/h\mathbf{k} \tag{2}$$

   where h is ice thickness and a is the rate at which ice accumulates at the surface (units of velocity). Assume a and h to be constant. Assuming ice to be incompressible, show that this velocity field ensures mass conservation (again consult your equation sheet for the equation for mass conservation).

   (c) (2 points) Assume that temperature T depends only on elevation z above the base of the ice sheet. Show that

$$-\frac{\rho c_p a z}{h}\frac{\mathrm{d}T}{\mathrm{d}z} - k\frac{\mathrm{d}^2 T}{\mathrm{d}z^2} = 0. \tag{3}$$

   (d) (5 points) Assume prescribed temperatures at the base of the ice, $z = 0$, and the surface, $z = h$:

$$T = 0 \qquad \text{at } z = 0 \tag{4a}$$
$$T = -T_s \qquad \text{at } z = h. \tag{4b}$$

   (These apply if the base of the ice is at the melting point, taken to be zero, while the surface is at some colder temperature $-T_s < 0$.) Define dimensionless variables

$$z = [z]z^*, \qquad T = [T]T^*$$

   and substitute these into (3) and (4). Choose your scales such that the scaled equations become

$$\begin{aligned} -P_e z^* \frac{\mathrm{d}T^*}{\mathrm{d}z^*} - \frac{\mathrm{d}^2 T^*}{\mathrm{d}z^{*2}} &= 0 \qquad && \text{for } 0 < z^* < 1 \\ T^* &= 0 && \text{at } z^* = 0 \\ T^* &= -1 && \text{at } z^* = 1. \end{aligned}$$

   How does the *Péclet* number P_e relate to a, h, ρ, c_p and k?

   (e) (1 point) If $a = 1$ m year^{-1}, $h = 3000$ m, $\rho = 900$ kg, $c_p = 2\times10^3$ J kg^{-1} K^{-1}, $k = 2.2$ W m^{-1} K^{-1}, give a value for P_e (1 year $= 365 \times 24 \times 3600$ s). Is advection important in ice sheets?

3. Solve

$$
\begin{aligned}
-P_e z T'(z) - T''(z) &= 0 \qquad && \text{for } 0 < z < 1 \qquad && \text{(5a)}\\
T &= 0 && \text{at } z = 0 && \text{(5b)}\\
T &= -1 && \text{at } z = 1. && \text{(5c)}
\end{aligned}
$$

by separation of variables:

(a) (3 points) Separating variables, show that

$$T'(z) = C \exp\left(-\frac{P_e}{2} z^2\right)$$

(Recall that $T' = \mathrm{d}T/\,\mathrm{d}z$, $T'' = \mathrm{d}T'/\,\mathrm{d}z$.)

(b) (4 points) The error function is defined as

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-x'^2)\,\mathrm{d}x'.$$

Note that this implies $\mathrm{erf}(0) = 0$. Show that

$$T(z) = C\sqrt{\frac{\pi}{2P_e}}\mathrm{erf}\left(\sqrt{\frac{P_e}{2}} z\right).$$

Which boundary condition does this automatically satisfy?

(c) (2 points) The constant of integration C still needs to be determined. Use the boundary condition you have not yet used to show that

$$C = -\sqrt{\frac{2P_e}{\pi}} \frac{1}{\mathrm{erf}\left(\sqrt{\frac{P_e}{2}}\right)}$$

In general, the right-hand side needs to be evaluated numerically. However, $\mathrm{erf}(x)$ tends to unity very rapidly as $x \to \infty$: For $x > 4$, $\mathrm{erf}(x) \approx 1$ to within an error of 10^{-7}. Find an approximate value of C when $P_e > 32$, and write down the solution for $T(z)$ not involving the constant of integration C.

(d) (1 point) You are given $\mathrm{erf}(1.163) = 0.9$. For the parameter values given in question 2(e), how far above the bed do I have to go (in metres) to find temperatures within 10 % of $-T_s$?

4. You are given a slanted triangular surface S with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

   (a) (2 points) Your are given a constant heat flux

   $$\mathbf{q} = (1,2,3)$$

   What is the rate at which heat passes from above S to below (this rate has dimensions of energy over time)?

   (b) (3 points) You have a stress tensor σ_{ij} given in matrix form by

   $$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

   What is the force exerted by the material above S on the material below S?

   (c) (1 point) Why is this matrix not a stress tensor:

   $$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

   (d) (4 points) Let $\rho = 1$ and $\mathbf{u} = (x_1, x_2, x_3)$. Compute the momentum contained in the volume bounded by S and the planes $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$.