Final: EOSC 352

8 December, 2010

This exam consists of four questions worth ten marks each. Available marks for each part of a question are indicated in brackets. Attempt THREE questions. You have 2 hours 20 minutes.

1. Consider one-dimensional heat conduction forced by an oscillating surface temperature. Also assume that heat flux tends to zero at infinity

$$\rho c_p \frac{\partial T}{\partial t} - k\frac{\partial^2 T}{\partial x^2} = 0 \qquad \text{for } 0 < x \qquad (1a)$$

$$T = T_0 \cos(\omega t) \qquad \text{at } x = 0 \qquad (1b)$$

$$-k\frac{\partial T}{\partial x} \to 0 \qquad \text{as } x \to \infty \qquad (1c)$$

ρ, c_p and k are constants.

(a) (4 points) Let

$$T(x,t) = \mathrm{Re}\left[T_0 \exp(i\omega t + \lambda x)\right]$$

Substitute this into (1a). Derive and solve an equation for λ so that (1a) is satisfied.

ANS: We have

$$\frac{\partial}{\partial t}\mathrm{Re}\left[T_0 \exp(i\omega t + \lambda x)\right] = \mathrm{Re}\left[i\omega T_0 \exp(i\omega t + \lambda x)\right]$$

and

$$\frac{\partial}{\partial x}\mathrm{Re}\left[T_0 \exp(i\omega t + \lambda x)\right] = \mathrm{Re}\left[\lambda T_0 \exp(i\omega t + \lambda x)\right]$$

$$\frac{\partial^2}{\partial x^2}\mathrm{Re}\left[T_0 \exp(i\omega t + \lambda x)\right] = \mathrm{Re}\left[\lambda^2 T_0 \exp(i\omega t + \lambda x)\right]$$

Substituting into the heat equation,

$$\mathrm{Re}\left[(\rho c_p i\omega - k\lambda^2) T_0 \exp(i\omega t + \lambda x)\right] = 0$$

This is satisfied if $\rho c_p i\omega - k\lambda^2 = 0$, or

$$\lambda = \sqrt{i}\sqrt{\frac{\rho c_p \omega}{k}} = \pm\frac{(1+i)}{2}\sqrt{\frac{\rho c_p \omega}{k}}.$$

(b) (1 point) You should have two possible values for λ. Which one allows (1c) to be satisfied?

ANS: When taking the real part, we get an exponential function behaving as $\exp(\mathrm{Re}\lambda x)$. This must not blow up as $x \to \infty$, so the real part of λ must be negative.

$$\lambda = -(1+i)\sqrt{\frac{\rho c_p \omega}{2k}}.$$

(c) (2 points) Take the real part in $T(x,t) = \mathrm{Re}[T_0 \exp(i\omega t + \lambda x)]$ to find an expression for T that does not involve i.

$$T(x,t) = T_0 \exp\left(-\sqrt{\frac{\rho c_p \omega}{2k}} x\right) \cos\left(\omega t - \sqrt{\frac{\rho c_p \omega}{2k}} x\right).$$

(d) (3 points) Without resorting to complex numbers, compute $\partial T/\partial t$ and $\partial^2 T/\partial x^2$ for the expression you have obtained (this will require the product rule). Check that the heat equation is indeed satisfied.

ANS: With the above,

$$\frac{\partial T}{\partial t} = -\omega T_0 \exp\left(-\sqrt{\frac{\rho c_p \omega}{2k}}x\right)\sin\left(\omega t - \sqrt{\frac{\rho c_p \omega}{2k}}x\right),$$

as well as, by the product rule

$$\begin{aligned}
\frac{\partial T}{\partial x} =& -\sqrt{\frac{\rho c_p \omega}{2k}}T_0 \exp\left(-\sqrt{\frac{\rho c_p \omega}{2k}}x\right)\cos\left(\omega t - \sqrt{\frac{\rho c_p \omega}{2k}}x\right)\\
&+\sqrt{\frac{\rho c_p \omega}{2k}}T_0 \exp\left(-\sqrt{\frac{\rho c_p \omega}{2k}}x\right)\sin\left(\omega t - \sqrt{\frac{\rho c_p \omega}{2k}}x\right),\\
\frac{\partial^2 T}{\partial x^2} =& \frac{\rho c_p \omega}{2k}T_0 \exp\left(-\sqrt{\frac{\rho c_p \omega}{2k}}x\right)\cos\left(\omega t - \sqrt{\frac{\rho c_p \omega}{2k}}x\right)\\
&+2\frac{\rho c_p \omega}{2k}T_0 \exp\left(-\sqrt{\frac{\rho c_p \omega}{2k}}x\right)\sin\left(\omega t - \sqrt{\frac{\rho c_p \omega}{2k}}x\right)\\
&-\frac{\rho c_p \omega}{2k}T_0 \exp\left(-\sqrt{\frac{\rho c_p \omega}{2k}}x\right)\cos\left(\omega t - \sqrt{\frac{\rho c_p \omega}{2k}}x\right)\\
=&\frac{\rho c_p \omega}{k}T_0 \exp\left(-\sqrt{\frac{\rho c_p \omega}{2k}}x\right)\sin\left(\omega t - \sqrt{\frac{\rho c_p \omega}{2k}}x\right)
\end{aligned}$$

Hence

$$\begin{aligned}
\rho c_p \frac{\partial T}{\partial t} - k\frac{\partial^2 T}{\partial x^2} =& \rho c_p \omega T_0 \exp\left(-\sqrt{\frac{\rho c_p \omega}{2k}}x\right)\sin\left(\omega t - \sqrt{\frac{\rho c_p \omega}{2k}}x\right)\\
&-k\times\frac{\rho c_p \omega}{k}T_0 \exp\left(-\sqrt{\frac{\rho c_p \omega}{2k}}x\right)\sin\left(\omega t - \sqrt{\frac{\rho c_p \omega}{2k}}x\right)\\
=&0.
\end{aligned}$$

2. This question is about the heat equation with advection in steady state, mass conservation and scaling.

   (a) (1 point) From your equation sheet, write down the heat equation with advection, *assuming a steady state.*
   ANS:
   $$\rho c_p \mathbf{u} \cdot \nabla T - \nabla \cdot (k \nabla T) = 0.$$

   (b) (1 point) Near the centre of an ice sheet, a possible velocity field is
   $$\mathbf{u}(x, z) = ax/h\mathbf{i} - az/h\mathbf{k} \tag{2}$$
   where h is ice thickness and a is the rate at which ice accumulates at the surface (units of velocity). Assume a and h to be constant. Assuming ice to be incompressible, show that this velocity field ensures mass conservation (again consult your equation sheet for the equation for mass conservation).
   ANS: Conservation of mass says
   $$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$
   If ice is incompressible, then ρ is constant. The equation becomes
   $$\nabla \cdot \mathbf{u} = 0.$$
   In the present case, we have
   $$\nabla \cdot \mathbf{u} = \frac{\partial}{\partial x}\left(\frac{ax}{h}\right) + \frac{\partial}{\partial z}\left(-\frac{az}{h}\right) = \frac{a}{h} - \frac{a}{h} = 0$$
   as required.

   (c) (2 points) Assume that temperature T depends only on elevation z above the base of the ice sheet. Show that
   $$-\frac{\rho c_p a z}{h}\frac{\mathrm{d}T}{\mathrm{d}z} - k\frac{\mathrm{d}^2 T}{\mathrm{d}z^2} = 0. \tag{3}$$
   If $T = T(z)$, then $\nabla T = \mathrm{d}T/\mathrm{d}z\mathbf{k}$, $\nabla \cdot (k\nabla T) - -k\,\mathrm{d}^2T/\mathrm{d}z^2$, assuming k to be constant. Also
   $$\mathbf{u} \cdot \nabla T = \left(\frac{ax}{h}\mathbf{i} - \frac{az}{h}\mathbf{k}\right) \cdot \frac{\mathrm{d}T}{\mathrm{d}z}\mathbf{k} = -\frac{az}{h}\frac{\mathrm{d}T}{\mathrm{d}z}.$$
   Substituting gives
   $$-\frac{\rho c_p a z}{h}\frac{\mathrm{d}T}{\mathrm{d}z} - k\frac{\mathrm{d}^2 T}{\mathrm{d}z^2} = 0.$$

(d) (5 points) Assume prescribed temperatures at the base of the ice, $z = 0$, and the surface, $z = h$:

$$T = 0 \qquad \text{at } z = 0 \tag{4a}$$
$$T = -T_s \qquad \text{at } z = h. \tag{4b}$$

(These apply if the base of the ice is at the melting point, taken to be zero, while the surface is at some colder temperature $-T_s < 0$.) Define dimensionless variables

$$z = [z]z^*, \qquad T = [T]T^*$$

and substitute these into (3) and (4). Choose your scales such that the scaled equations become

$$\begin{aligned} -P_e z^* \frac{\mathrm{d}T^*}{\mathrm{d}z^*} - \frac{\mathrm{d}^2T^*}{\mathrm{d}z^{*2}} &= 0 && \text{for } 0 < z^* < 1 \\ T^* &= 0 && \text{at } z^* = 0 \\ T^* &= -1 && \text{at } z^* = 1. \end{aligned}$$

How does the *Péclet* number P_e relate to a, h, ρ, c_p and k?
Substituting in the heat equation, we have

$$-\frac{\rho c_p a[z]z^*}{h}\frac{[T]}{[z]}\frac{\mathrm{d}T^*}{\mathrm{d}z^*} - k\frac{[T]}{[z]^2}\frac{\mathrm{d}^2T*}{\mathrm{d}z^{*2}} = 0.$$

Rearranging,

$$\frac{\rho c_p a[z]^2}{kh}z^*\frac{\mathrm{d}T^*}{\mathrm{d}z^*} - \frac{\mathrm{d}^2T^*}{\mathrm{d}z^{*2}} = 0$$

which is of the correct form if

$$P_e = \frac{\rho c_p a[z]^2}{kh}.$$

We don't know $[z]$ or $[T]$ yet, however. Substitute into the boundary conditions at $z = 0$:

$$[T]T^* = 0 \qquad \text{at } [z]z^* = 0.$$

Rearranging (divide by $[T]$ and $[z]$, respectively), this becomes

$$T^* = 0 \qquad \text{at } z^* = 0$$

as required. Lastly, the boundary condition at $z = 1$ becomes

$$[T]T^* = -T_s \qquad \text{at } [z]z^* = h.$$

Rearranging,

$$T^* = -\frac{T_s}{[T]} \qquad \text{at } z^* = \frac{h}{[z]}.$$

This is of the required form if

$$\frac{T_s}{[T]} = 1, \qquad \frac{h}{[z]} = 1.$$

Hence $[T] = T_2$, $[z] = h$.

(e) (1 point) If $a = 1$ m year^{-1}, $h = 3000$ m, $\rho = 900$ kg, $c_p = 2\times10^3$ J kg^{-1} K^{-1}, $k = 2.2$ W m^{-1} K^{-1}, give a value for P_e (1 year $= 365 \times 24 \times 3600$ s). Is advection important in ice sheets?
Convert u to SI units: $a = 4.4 \times 10^{-8}$ m s^{-1}. We have

$$P_e = \frac{\rho c_p a [z]^2}{kh} = P_e = \frac{\rho c_p a h}{k} = 77.83.$$

(actually, 2 significant figures would be plenty here...)

3. Solve

$$-P_e z T'(z) - T''(z) = 0 \qquad \text{for } 0 < z < 1 \qquad (5a)$$

$$T = 0 \qquad \text{at } z = 0 \qquad (5b)$$

$$T = -1 \qquad \text{at } z = 1. \qquad (5c)$$

by separation of variables:

(a) (3 points) Separating variables, show that

$$T'(z) = C \exp\left(-\frac{P_e}{2} z^2\right)$$

(Recall that $T' = \mathrm{d}T/\mathrm{d}z$, $T'' = \mathrm{d}T'/\mathrm{d}z$.)
ANS: We get

$$-P_e z = \frac{1}{T'} \frac{\mathrm{d}T'}{\mathrm{d}z}.$$

Integrating both sides

$$-\frac{P_e}{2} z^2 + K = \log(T').$$

Exponentiate and define $C = \exp(K)$:

$$T'(z) = C \exp\left(-\frac{P_e}{2} z^2\right).$$

(b) (4 points) The error function is defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-x'^2)\,\mathrm{d}x'.$$

Note that this implies $\operatorname{erf}(0) = 0$. Show that

$$T(z) = C \sqrt{\frac{\pi}{2P_e}} \operatorname{erf}\left(\sqrt{\frac{P_e}{2}} z\right).$$

Which boundary condition does this automatically satisfy?
Integrate again from $z = 0$:

$$\int_0^z T'(z')\,\mathrm{d}z' = T(z) - T(0) = C \int_0^z \exp\left(-\frac{P_e}{2} z'^2\right)\,\mathrm{d}z'.$$

Change variables on the right to $u = \sqrt{P_e/2}z'$, $du = \sqrt{P_e/2}\,\mathrm{d}z'$. Then

$$\begin{aligned}
\int_0^z \exp\left(-\frac{P_e}{2} z'^2\right)\,\mathrm{d}z' &= \sqrt{\frac{2}{P_e}} \int_0^{\sqrt{P_e/2}z} \exp(-u^2)\,\mathrm{d}u \\
&= \sqrt{\frac{2}{P_e}} \frac{\sqrt{\pi}}{2} \frac{2}{\sqrt{\pi}} \int_0^{\sqrt{P_e/2}z} \exp(-u^2)\,\mathrm{d}u \\
&= \sqrt{\frac{\pi}{2P_e}} \operatorname{erf}\left(\sqrt{\frac{P_e}{2}} z\right).
\end{aligned}$$

Substituting and rearranging

$$T(z) = T(0) + C\sqrt{\frac{\pi}{2P_e}}\text{erf}\left(\sqrt{\frac{P_e}{2}}z\right).$$

But $T(0) = 0$, so the desired result follows. (This is the boundary condition at $z = 0$.)

(c) (2 points) The constant of integration C still needs to be determined. Use the boundary condition you have not yet used to show that

$$C = -\sqrt{\frac{2P_e}{\pi}}\frac{1}{\text{erf}\left(\sqrt{\frac{P_e}{2}}\right)}$$

In general, the right-hand side needs to be evaluated numerically. However, $\text{erf}(x)$ tends to unity very rapidly as $x \to \infty$: For $x > 4$, $\text{erf}(x) \approx 1$ to within an error of 10^{-7}. Find an approximate value of C when $P_e > 32$, and write down the solution for $T(z)$ not involving the constant of integration C.
ANS: We need to have $T(1) = -1$. Substituting $z = 1$,

$$T(1) = C\sqrt{\frac{\pi}{2P_e}}\text{erf}\left(\sqrt{\frac{P_e}{2}}\right) = -1.$$

Rearranging gives the desired result. For $P_e > 32$, $\sqrt{P_e/2} > 4$, and $\text{erf}(\sqrt{P_e/2}) \approx 1$ to within an error of 10^{-7}, so we can put

$$C \approx \sqrt{\frac{2P_e}{\pi}}.$$

The solution $T(z)$ is then

$$T(z) = -\text{erf}\left(\sqrt{\frac{P_e}{2}}z\right).$$

(d) (1 point) You are given $\text{erf}(1.163) = 0.9$. For the parameter values given in question 2(e), how far above the bed do I have to go (in metres) to find temperatures within 10 % of $-T_s$?
ANS: This happens when $T(z)$ is within 10 % of -1, i.e. when $\text{erf}(\sqrt{Pe/2}z) > 0.9$, so $z > 1.163 \times \sqrt{2/P_e}$. With $P_e = 77.83$, this is $z > 0.1864$. But z is in units of $[z] = h = 3000$ m, so corresponds to 0.1864×3000 m $= 560$ m.

4. You are given a slanted triangular surface S with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

   (a) (2 points) Your are given a constant heat flux

$$\mathbf{q} = (1,2,3)$$

   What is the rate at which heat passes from above S to below (this rate has dimensions of energy over time)?
   ANS: Normally, this is done through an integral $\int_S \mathbf{q}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$. You can do that integral by brute force, see the EOS 250 notes on surface integrals. Here $\mathbf{q}$ and $\hat{\mathbf{n}}$ are constant, so we can express this as

$$S\mathbf{q}\cdot\hat{\mathbf{n}}$$

   We still need S and $\hat{\mathbf{n}}$. The latter is easy to find by symmetry,

$$\hat{\mathbf{n}} = -\frac{(1,1,1)}{\sqrt{3}}.$$

   (Note that it points from above to below, hence the negative sign.) To find S, note that the surface is an equilateral triangle with side length $\sqrt{2}$. The area of a triangle is one half base times height. For an equilateral triangle, height = base times cosine of 60°. The base length is $\sqrt{2}$ ans the cosine of 60° is $\sqrt{3}/2$ so

$$S = \frac{1}{2}\times\sqrt{2}\times\sqrt{2}\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}.$$

   Then

$$S\mathbf{q}\cdot\hat{\mathbf{n}} = -\frac{\sqrt{3}}{2}(1,2,3)\cdot(1,1,1)/\sqrt{3} = -3.$$

   (b) (3 points) You have a stress tensor σ_{ij} given in matrix form by

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

   What is the force exerted by the material above S on the material below S? The answer in general is again a surface integral $\sqrt{S}\sigma_{ij}n_j\,\mathrm{d}S$, where n_j points *away* from the material the force is exerted on.We can write this as $S\sigma_{ij}n_j$ for the same reasons as before, only that now $\hat{\mathbf{n}}$ points *up*,

$$\hat{\mathbf{n}} = \frac{(1,1,1)}{\sqrt{3}}.$$

You can either recognize $\sigma_{ij}n_j = \sum_{j=1}^{3} \sigma_{ij}n_j$ as the product of the matrix σ and the column vector n, or work your way through the indices $i = 1, 2, 3$ and do the sum separately. The answer is

$$S\sigma_{ij}n_j\,\mathrm{d}S = \frac{\sqrt{3}}{2}\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}/\sqrt{3}$$
$$= \begin{pmatrix} 1 \\ 1 \\ \frac{1}{2} \end{pmatrix}$$

(c) (1 point) Why is this matrix not a stress tensor:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

ANS: The matrix is not symmetric, i.e., we do not have $\sigma_{ij} = \sigma_{ji}$.

(d) (4 points) Let $\rho = 1$ and $\mathbf{u} = (x_1, x_2, x_3)$. Compute the momentum contained in the volume bounded by S and the planes $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$.
ANS: This time you do have to do a volume integral,

$$p_i = \int_V \rho u_i\,\mathrm{d}V.$$

By symmetry (the velocity field is symmetric under changes in the x_i's, as is the volume), we only have to do the calculation for one momentum component, however. So let's pick $i = 1$:

$$\begin{aligned}
p_1 &= \int_0^1 \int_0^{1-x_1} \int_0^{1-x_1-x_2} x_1\,\mathrm{d}x_3\,\mathrm{d}x_2\,\mathrm{d}x_1 \\
&= \int_0^1 \int_0^{1-x_1} x_1(1 - x_1 - x_2)\,\mathrm{d}x_2\,\mathrm{d}x_1 \\
&= \int_0^1 \left[x_1(1 - x_1)x_2 - x_1\frac{x_2^2}{2}\right]_0^{1-x_1} \mathrm{d}x_1 \\
&= \int_0^1 x_1(1 - x_1)^2 - x_1\frac{(1 - x_1)^2}{2}\,\mathrm{d}x_1 \\
&= \int_0^1 x_1\frac{(1 - x_1)^2}{2}\,\mathrm{d}x_1 \\
&= \left[x_1\frac{(1 - x_1)^3}{6}\right]_0^1 - \int_0^1 \frac{(1 - x_1)^3}{6}\,\mathrm{d}x_1 \\
&= \frac{1}{24}
\end{aligned}$$

The total momentum is
$$\frac{(1,1,1)}{24}.$$