Final: EOSC 352

21 April, 2011

This exam consists of four questions worth ten marks each. Available marks for each part of a question are indicated in brackets. Attempt THREE questions. You have 2 hours 20 minutes.

1. Consider one-dimensional heat conduction between two half-spaces that are initially at uniform but different temperatures. There is no heat flow 'at infinity', i,e., coming in from long distances from the contact between the two half-spaces at $x = 0$. In non-dimensional form, this situation can be described by

$$\frac{\partial T}{\partial t} - \frac{\partial^2 T}{\partial x^2} = 0 \qquad \text{everywhere for } t > 0 \tag{1a}$$

$$T(x,0) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases} \tag{1b}$$

$$-\frac{\partial T}{\partial x} \to 0 \qquad \text{as } x \to \pm\infty \tag{1c}$$

In this question, you will construct a similarity solution to the problem.

(a) (4 points) Let

$$T(x,t) = t^{-\alpha}\Theta(x/t^{\beta}). \tag{2}$$

and define

$$\xi = x/t^{\beta}.$$

Substitute this into (1a), converting partial derivatives with respect to x and t into ordinary derivatives with respect to ξ. Show that you get

$$-\alpha t^{-\alpha-1}\Theta(\xi) - \beta t^{-\alpha-1}\xi\Theta'(\xi) - t^{-\alpha-2\beta}\Theta''(\xi) = 0. \tag{3}$$

What value does β have to take in order for a similarity solution (2) to hold?

(b) (2 points) Next, show that the initial condition (1b) and boundary condition (1c) can be expressed as

$$x^{-\alpha/\beta}\xi^{\alpha/\beta}\Theta(\xi) \to \pm 1 \qquad \text{as } \xi \to \pm\infty \text{ at any fixed } x, \tag{4a}$$

$$t^{-\alpha-\beta}\Theta'(\xi) \to 0 \qquad \text{as } \xi \to \pm\infty \text{ at any fixed } t. \tag{4b}$$

Why does it follow that $\alpha = 0$?

(c) (3 points) Put the value of β you have deduced and $\alpha = 0$ into (3). Separate variables to show that

$$\Theta'(\xi) = C\exp\left(-\frac{\xi^2}{4}\right) \tag{5}$$

The definition of the error function is

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-x'^2)\,\mathrm{d}x',$$

which behaves as $\operatorname{erf}(x) \to \pm 1$ as $x \to \pm\infty$. Use this and the boundary conditions in (4) to show that

$$\Theta(\xi) = \operatorname{erf}\left(\frac{\xi}{2}\right). \tag{6}$$

(d) (1 point) On the same graph, sketch the solution $T(x,t)$ as a function of x for $t = 0$, 1, 2, 4.

2. This question is about seismic P-waves ('primary' or 'pressure' waves generated by earthquakes, explosions or impacts) in a viscous liquid. The equations of motion for a compressible fluid (mass and momentum conservation) are

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_i)}{\partial x_i} = 0, \tag{7a}$$

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right) = \frac{\partial \sigma_{ij}}{\partial x_j}, \tag{7b}$$

where

$$\sigma_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right) - p\delta_{ij}.$$

To transmit P-waves, the fluid must be at least slightly compressible, and we assume that

$$\rho = \rho_0(+cp) \tag{7c}$$

with ρ_0 a mean density and c a compressibility, both of which are constant. Assume also that μ is constant.

(a) (3 points) To model a P-wave propagating in the x_1-direction, assume that motion is only in the x_1-direction, with velocity and pressure dependent only on x_1,

$$u_1 = u_1(x_1, t), \qquad u_2 = 0 \qquad u_3 = 0, \qquad p = p(x_1, t),$$

and impose boundary conditions at $x_1 = 0$ in the form of a pressure oscillation

$$p(0, t) = p_0 \cos(\omega t), \tag{8a}$$

with (7) holding for $x_1 > 0$. Show that

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_1)}{\partial x_1} = 0 \tag{8b}$$

$$\rho\left(\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x_1}\right) = \frac{4}{3}\mu\frac{\partial^2 u_1}{\partial x_1^2} - \frac{\partial p}{\partial x_1} \tag{8c}$$

$$\rho = \rho_0(1 + cp) \tag{8d}$$

(b) (5 points) Define scales $[x]$, $[t]$, $[u]$, $[p]$ and dimensionless variables

$$x_1 = [x]x^*, \qquad t = [t]t^*, \qquad u_1 = [u]u^*, \qquad p = [p]p^*$$

such that the equations (8) can be written in the form

$$(1 + \alpha p^*)\left(\frac{\partial u^*}{\partial t^*} + \alpha u^*\frac{\partial u^*}{\partial x^*}\right) - \frac{4}{3}\gamma\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial p^*}{\partial x^*} = 0 \qquad \text{for } x^* > 0 \tag{9a}$$

$$\frac{\partial p^*}{\partial t^*} + \frac{\partial[(1 + \alpha p^*)u^*]}{\partial x^*} = 0 \qquad \text{for } x^* > 0 \tag{9b}$$

$$p^*(0, t^*) = \cos(t^*) \quad \text{at } x^* = 0. \tag{9c}$$

Find the dimensionless groups α and γ in terms of ρ_0, c, η, p_0, ω. (HINT: It may be easiest if you substitute for ρ in (8b) and (8c) before introducing dimensionless variables)

(c) (2 points) For a (deafening!) 90 dB sound wave in water at 1000 Hz, we have $p_0 = .045$ Pa, $\omega = 2000\pi$ s^{-1}, $\rho_0 = 1000$ kg m^{-3}, $c = 4.6 \times 10^{-10}$ Pa^{-1}, $\mu = 1.7 \times 10^{-3}$ Pa s. Find numerical values of α and γ. Show how these can be used to motivate the simplified model

$$\frac{\partial u^*}{\partial t^*} = \frac{2}{3}\gamma\frac{\partial^2 u^*}{\partial x^{*2}} - \frac{\partial p^*}{\partial x^*} = 0 \qquad \text{for } x^* > 0 \qquad (10a)$$

$$\frac{\partial p^*}{\partial t^*} + \frac{\partial u^*}{\partial x^*} = 0 \qquad \text{for } x^* > 0 \qquad (10b)$$

$$p^*(0, t^*) = \cos(t^*) \qquad \text{at } x^* = 0. \qquad (10c)$$

3. This question is about solving a seismic P- and S-wave problem in a viscous fluid. A simplified model for seismic P-waves in viscous fluid is

$$\rho_0 \frac{\partial u}{\partial t} = \frac{4}{3}\mu \frac{\partial^2 u}{\partial x^2} - \frac{\partial p}{\partial x} \qquad \text{for } x > 0 \tag{11a}$$

$$c\frac{\partial p}{\partial t} + \frac{\partial u}{\partial x} = 0 \qquad \text{for } x > 0 \tag{11b}$$

$$p(0,t) = p_0 \cos(\omega t) \qquad \text{at } x = 0 \tag{11c}$$

$$p \to 0 \qquad \text{as } x \to \infty \tag{11d}$$

You will solve this by complex variable methods, and compare with the solution of an S-wave problem.

(a) (2 points) By differentiating (11a) and substituting from another equation in (11), show that

$$\rho_0 c \frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial x^2} - \frac{4}{3}\mu c \frac{\partial^3 p}{\partial x^2 \partial t} = 0. \tag{12}$$

(b) (2 points) Assume that $p(x,t)$ can be written in the form

$$p(x,t) = \mathrm{Re}\left[p_0 \exp(i\omega t + \lambda x)\right].$$

For p satisfying (12), find the equation that must be satisfied by λ, and solve for λ in terms of ρ_0, c, μ and ω.

(c) (2 points) The answer you get should be in the form

$$\lambda = \pm ia/(1+ib)^{1/2} \tag{13}$$

The Taylor expansion of $(1+x)^{-1/2}$ for small x is

$$(1+x)^{-1/2} = 1 - \frac{1}{2}x + \ldots$$

For small b, find an approximation to (13) of the form

$$\lambda = \pm(\alpha + i\beta)$$

with α and β real quantities that depend on ρ_0, c, μ and ω. If $\omega = 2000\pi\ \mathrm{s}^{-1}$, $\rho_0 = 1000\ \mathrm{kg\ m}^{-3}$, $c = 4.6 \times 10^{-10}\ \mathrm{Pa}^{-1}$, $\mu = 1.7 \times 10^{-3}$ Pa s, is this approximation valid?

(d) (3 points) Given your answer to part c, express $p(x,t)$ in real terms in the form

$$p(x,t) = p_0 \cos\left[\omega(t - x/v)\right] \exp(-x/x_0),$$

making sure to give expressions for v and x_0 in terms of ρ_0, c, μ and ω, and justifying your choice of signs. What is the wave velocity? What is the wavelength? What is the distance over which the wave amplitude decreases by a factor of $1/e$ (this is the 'e-folding distance')? If $\omega = 2000\pi$ s^{-1}, $\rho_0 = 1000$ kg m^{-3}, $c = 4.6 \times 10^{-10}$ Pa^{-1}, give numerical values for velocity, wavelength and the e-folding distance.

(e) (1 point) The corresponding S-wave problem would be

$$\begin{aligned} \rho_0 \frac{\partial v}{\partial t} - \mu \frac{\partial^2 v}{\partial x^2} &= 0 && \text{for } x > 0, \\ v(0,t) &= v_0 \cos(\omega t) && \text{at } x = 0, \\ v &\to 0 && \text{as } x \to \infty, \end{aligned}$$

where v is velocity transverse to the x-axis. Using methods from the course, it can be shown that the solution is (no need to derive this yourself! — simply use this formula)

$$v = v_0 \cos\left(\omega t - \sqrt{\frac{\rho_0 \omega}{2\mu}} x\right) \exp\left(-\sqrt{\frac{\rho_0 \omega}{2\mu}} x\right)$$

With the values for ρ_0, ω and μ given above, calculate the distance over which v decays to $1/e$ of its value at $x = 0$.

4. You are given a slanted triangular surface S with vertices $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$.

   (a) (2 points) Your are given a temperature field

   $$T = x + 2y + 3z$$

   and a constant thermal conductivity $k = 1$. What is the rate at which heat passes from above S to below (this rate has dimensions of energy over time)?

   (b) (3 points) You have a stress tensor σ_{ij} given in matrix form by

   $$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

   What is the force exerted by the material above S on the material below S?

   (c) (1 point) What is the pressure p that corresponds to the stress tensor in part b?

   (d) (6 points) Let $\rho = 1$ and $\mathbf{u} = (x_2, -x_1, x_3)$. Compute the angular momentum contained in the volume bounded by S and the planes $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$.