# Final: EOSC 352

21 April, 2011

This exam consists of four questions worth ten marks each. Available marks for each part of a question are indicated in brackets. Attempt THREE questions. You have 2 hours 20 minutes.

1. Consider one-dimensional heat conduction between two half-spaces that are initially at uniform but different temperatures. There is no heat flow 'at infinity', i,e., coming in from long distances from the contact between the two half-spaces at $x = 0$. In non-dimensional form, this situation can be described by

$$\frac{\partial T}{\partial t} - \frac{\partial^2 T}{\partial x^2} = 0 \qquad \text{everywhere for } t > 0 \tag{1a}$$

$$T(x,0) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases} \tag{1b}$$

$$-\frac{\partial T}{\partial x} \to 0 \qquad \text{as } x \to \pm\infty \tag{1c}$$

In this question, you will construct a similarity solution to the problem.

(a) (4 points) Let

$$T(x,t) = t^{-\alpha}\Theta(x/t^{\beta}). \tag{2}$$

and define

$$\xi = x/t^{\beta}. \tag{3}$$

Substitute this into (1a), converting partial derivatives with respect to $x$ and $t$ into ordinary derivatives with respect to $\xi$. Show that you get

$$-\alpha t^{-\alpha-1}\Theta(\xi) - \beta t^{-\alpha-1}\xi\Theta'(\xi) - t^{-\alpha-2\beta}\Theta''(\xi) = 0. \tag{4}$$

What value does $\beta$ have to take in order for a similarity solution (2) to hold?
ANSWER: Using the product and chain rules, we have

$$\begin{aligned}\frac{\partial}{\partial t}\left(t^{-\alpha}\Theta(\xi)\right) &= -\alpha t^{-\alpha-1}\Theta(\xi) + t^{-\alpha}\Theta'(\xi)\frac{\partial \xi}{\partial t} \\ &= -\alpha t^{-\alpha-1}\Theta(\xi) + t^{-\alpha}\Theta'(\xi)(-\beta)xt^{-\beta-1} \\ &= -\alpha t^{-\alpha-1}\Theta(\xi) - \beta t^{-\alpha-1}\xi\Theta'(\xi)\end{aligned}$$

Also,

$$\frac{\partial^2}{\partial x^2}\left(t^{-\alpha}\Theta(\xi)\right) = t^{-\alpha-2\beta}\Theta''(\xi).$$

Substituting in (1a)

$$-\alpha t^{-\alpha-1}\theta(\xi) - \beta t^{-\alpha-1}\xi\Theta'(\xi) - t^{-\alpha-2\beta}\Theta''(\xi) = 0.$$

Rearranging

$$-t^{2\beta-1}\left(\alpha\theta(\xi) + \beta\xi\Theta'(\xi)\right) - \Theta''(\xi) = 0. \tag{5}$$

This must not contain $t$ explicitly in order to ensure that $\theta$ depends only on $\xi$, so

$$\beta = \frac{1}{2}.$$

(b) (2 points) Next, show that the initial condition (1b) and boundary condition (1c) can be expressed as

$$x^{-\alpha/\beta}\xi^{\alpha/\beta}\Theta(\xi) \to \pm 1 \qquad \text{as } \xi \to \pm\infty \text{ at any fixed } x, \tag{6a}$$

$$t^{-\alpha-\beta}\Theta'(\xi) \to 0 \qquad \text{as } \xi \to \pm\infty \text{ at any fixed } t. \tag{6b}$$

Why does it follow that $\alpha = 0$?

ANSWER: To transform the boundary condition (1c), note that

$$\frac{\partial T}{\partial x} = t^{-\alpha-\beta}\theta'(\xi)$$

and that the limit $x \to \pm\infty$ at fixed time $t > 0$ corresponds to $\xi \to \pm\infty$. Hence (1c) becomes

$$\lim_{\xi\to\infty} t^{-\alpha-\beta}\Theta'(\xi) = 0 \qquad \text{at any fixed } t > 0.$$

Because the limit is taken at fixed $t$, we can simplify to

$$\lim_{\xi\to\infty} \Theta'(\xi) = 0 \qquad \text{at any fixed } t > 0.$$

Next, the initial condition (1b) should be interpreted as

$$\lim_{t\to 0} T(x,t) = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases} \tag{7}$$

for any fixed $x > 0$. But the limit $t \to 0$ corresponds to $\xi \to \infty$ for $x > 0$, and $\xi \to -\infty$ for $x < 0$. Similarly, for fixed $x$, $t = (x/\xi)^{1/\beta}$ from (3). Hence, substituting from (2) in (7)

$$\lim_{\xi\to\pm\infty} (x/\xi)^{\alpha/\beta}\Theta(\xi) = \pm 1$$

at any fixed $x$. Now, $x$ and $\xi$ have the same sign for $t > 0$, so $(x/\xi) = |x|/|\xi|$, and we can re-write this as

$$\lim_{\xi\to\pm\infty} |\xi|^{-\alpha/\beta}\Theta(\xi) = \pm|x|^{\alpha/\beta}$$

for any fixed $x$. But none of the quantities on the left depend on $x$, so neither should the right-hand side. It follows that we must have $\alpha/\beta = 0$, or $\alpha = 0$.

(c) (3 points) Put the value of $\beta$ you have deduced and $\alpha = 0$ into (4). Separate variables to show that

$$\Theta'(\xi) = C \exp\left(-\frac{\xi^2}{4}\right) \tag{8}$$

The definition of the error function is

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-x'^2)\,\mathrm{d}x',$$

which behaves as $\mathrm{erf}(x) \to \pm 1$ as $x \to \pm\infty$. Use this and the boundary conditions in (6) to show that

$$\Theta(\xi) = \mathrm{erf}\left(\frac{\xi}{2}\right). \tag{9}$$

ANSWER: Putting $\alpha = 0$, $\beta = 1/2$ into (5),

$$-\frac{1}{2}\xi\Theta'(\xi) - \theta''(\xi) = 0.$$

Use $\Theta''(\xi) = \mathrm{d}\Theta'/\,\mathrm{d}\xi$ and separate variables as

$$\frac{1}{\Theta'(\xi)}\frac{\mathrm{d}\Theta'}{\mathrm{d}\xi} = -\frac{1}{2}\xi$$

Integrate both sides with respect to $\xi$ to get

$$\int \frac{1}{\Theta'}\frac{\mathrm{d}\Theta'}{\mathrm{d}\xi}\,\mathrm{d}\xi = -\int \frac{1}{2}\xi\,\mathrm{d}\xi.$$

so

$$\log(\Theta') = -\frac{\xi^2}{4} + C'.$$

or

$$\Theta' = C\exp\left(-\frac{\xi^2}{4}\right).$$

Integrate again

$$\Theta(\xi) = \int C\exp\left(-\frac{\xi^2}{4}\right)\mathrm{d}\xi.$$

Change variables to $u = \xi/2$, so $\mathrm{d}\xi = 2\,\mathrm{d}u$, we get

$$\begin{aligned}
\Theta &= 2C\int \exp(-u^2)\,\mathrm{d}u \\
&= 2C\frac{\sqrt{\pi}}{2}\frac{2}{\sqrt{\pi}}\int \exp(-u^2)\,\mathrm{d}u \\
&= C\sqrt{\pi}\mathrm{erf}(u) + C_2 \\
&= C\sqrt{\pi}\mathrm{erf}\left(\frac{\xi}{2}\right) + C_2.
\end{aligned}$$

But $\Theta \to \pm 1$ as $\xi \to \pm\infty$, so (taking $\xi \to -\infty$ first)

$$-1 = -C\sqrt{\pi} + C_2$$

and (taking $\xi \to \infty$)

$$1 = C\sqrt{\pi} + C_2,$$

from which it follows that $C_2 = 0$ and

$$C = \frac{1}{\sqrt{\pi}}.$$

Hence

$$\Theta(\xi) = \operatorname{erf}\left(\frac{\xi}{2}\right).$$

(d) (1 point) On the same graph, sketch the solution $T(x,t)$ as a function of $x$ for $t = 0,\ 1,\ 2,\ 4$.

2. This question is about seismic P-waves ('primary' or 'pressure' waves generated by earthquakes, explosions or impacts) in a viscous liquid. The equations of motion for a compressible fluid (mass and momentum conservation) are

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_i)}{\partial x_i} = 0, \tag{10a}$$

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right) = \frac{\partial \sigma_{ij}}{\partial x_j}, \tag{10b}$$

where

$$\sigma_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right) - p\delta_{ij}.$$

To transmit P-waves, the fluid must be at least slightly compressible, and we assume that

$$\rho = \rho_0(1 + cp) \tag{10c}$$

with $\rho_0$ a mean density and $c$ a compressibility, both of which are constant. Assume also that $\mu$ is constant.

(a) (3 points) To model a $P$-wave propagating in the $x_1$-direction, assume that motion is only in the $x_1$-direction, with velocity and pressure dependent only on $x_1$,

$$u_1 = u_1(x_1, t), \qquad u_2 = 0 \qquad u_3 = 0, \qquad p = p(x_1, t),$$

and impose boundary conditions at $x_1 = 0$ in the form of a pressure oscillation

$$p(0, t) = p_0\cos(\omega t), \tag{11a}$$

with (10) holding for $x_1 > 0$. Show that

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_1)}{\partial x_1} = 0 \tag{11b}$$

$$\rho\left(\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x_1}\right) = \frac{4}{3}\mu\frac{\partial^2 u_1}{\partial x_1^2} - \frac{\partial p}{\partial x_1} \tag{11c}$$

$$\rho = \rho_0(1 + cp) \tag{11d}$$

ANSWER: With $u_2 = u_3 = 0$ and $u_1$ dependent only on $x_1$ and $t$, we have

$$\begin{aligned}\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j} &= \frac{\partial u_i}{\partial t} + u_1\frac{\partial u_i}{\partial x_1} + u_2\frac{\partial u_i}{\partial x_2} + u_3\frac{\partial u_i}{\partial x_3} \\ &= \begin{cases} \frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x_1} & \text{if } i = 1 \\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

We also have

$$\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} = \begin{cases} 2\frac{\partial u_1}{\partial x_1} & \text{if } i = j = 1 \\ 0 & \text{otherwise}\end{cases}$$

and

$$\frac{\partial u_k}{\partial x_k} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \frac{\partial u_1}{\partial x_1}.$$

so the only non-zero components of $\sigma_{ij}$ are

$$\begin{aligned}\sigma_{11} &= \frac{4}{3}\frac{\partial u_1}{\partial x_1} - p\\ \sigma_{22} &= -\frac{2}{3}\frac{\partial u_1}{\partial x_1} - p\\ \sigma_{33} &= -\frac{2}{3}\frac{\partial u_1}{\partial x_1} - p,\end{aligned}$$

and, with $u_1$ and $p$ independent of $x_2$ and $x_3$, we have

$$\begin{aligned}\frac{\partial \sigma_{ij}}{\partial x_j} &= \frac{\partial \sigma_{i1}}{\partial x_1} + \frac{\partial \sigma_{i2}}{\partial x_2} + \frac{\partial \sigma_{i3}}{\partial x_3}\\ &= \begin{cases} \frac{\partial \sigma_{11}}{\partial x_1} & \text{if } i = 1\\ 0 & \text{otherwise}\end{cases}\end{aligned}$$

Putting this into the second equation in (10), we get for $i = 1$

$$\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x_1} = \frac{\partial \sigma_{11}}{\partial x_1} = \frac{\partial}{\partial x_1}\left(\frac{4}{3}\mu\frac{\partial u_1}{\partial x_1} - p\right) = \frac{4}{3}\mu\frac{\partial^2 u_1}{\partial x_1^2} - \frac{\partial p}{\partial x_1}$$

and $0 = 0$ for $i = 2, 3$. Similarly, $\rho$ depends only on $x_1$ and so

$$\frac{\partial(\rho u_i)}{\partial x_i} = \frac{\partial(\rho u_1)}{\partial x_1} + \frac{\partial(\rho u_2)}{\partial x_2} + \frac{\partial(\rho u_3)}{\partial x_3} = \frac{\partial(\rho u_1)}{\partial x_1},$$

and the first equation in (10) becomes

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_1)}{\partial x_1} = 0.$$

(b) (5 points) Define scales $[x]$, $[t]$, $[u]$, $[p]$ and dimensionless variables

$$x_1 = [x]x^*, \qquad t = [t]t^*, \qquad u_1 = [u]u^*, \qquad p = [p]p^*$$

such that the equations (11) can be written in the form

$$\begin{aligned}(1+\alpha p^*)\left(\frac{\partial u^*}{\partial t^*} + \alpha u^*\frac{\partial u^*}{\partial x^*}\right) - \frac{4}{3}\gamma\frac{\partial^2 u^*}{\partial x^{*2}} + \frac{\partial p^*}{\partial x^*} &= 0 \qquad \text{for } x^* > 0 \quad \text{(12a)}\\ \frac{\partial p^*}{\partial t^*} + \frac{\partial[(1+\alpha p^*)u^*]}{\partial x^*} &= 0 \qquad \text{for } x^* > 0 \quad \text{(12b)}\\ p^*(0, t^*) = \cos(t^*) & \quad \text{at } x^* = 0. \quad \text{(12c)}\end{aligned}$$

Find the dimensionless groups $\alpha$ and $\gamma$ in terms of $\rho_0$, $c$, $\eta$, $p_0$, $\omega$. (HINT: It may be easiest if you substitute for $\rho$ in (11b) and (11c) before introducing dimensionless variables)
ANSWER: Substituting for $\rho$ in (11b) and bearing in mind that $\rho_0$ and $c$ are constants, we get

$$\rho_0 c \frac{\partial p}{\partial t} + \frac{\partial[\rho_0(1+cp)u_1]}{\partial x} = 0. \tag{13}$$

We can similarly substitute in (11c)

$$\rho_0(1+cp)\left(\frac{\partial u_1}{\partial t} + u_1\frac{\partial u_1}{\partial x_1}\right) = \frac{4}{3}\mu\frac{\partial^2 u_1}{\partial x_1^2} - \frac{\partial p}{\partial x}. \tag{14}$$

Define dimensionless variables as follows

$$x_1 = [x_1]x^*, \qquad t = [t]t^*, \qquad u_1 = [u_1]u^*, \qquad p = [p]p^*.$$

Substituting, we get

$$\frac{\rho_0 c[p]}{[t]}\frac{\partial p^*}{\partial t^*} + \frac{\rho_0[u_1]}{[x_1]}\frac{\partial[(1+c[p]p^*)u^*]}{\partial x^*} = 0.$$

and

$$\frac{\rho_0[u_1]}{[t]}\left(\frac{\partial u^*}{\partial t^*} + \frac{[u_1][t]}{[x_1]}u^*\frac{\partial u^*}{\partial x^*}\right) = \frac{4}{3}\mu\frac{[u_1]}{[x_1]^2}\frac{\partial^2 u^*}{\partial x^{*2}} - \frac{[p]}{[x_1]}\frac{\partial p^*}{\partial x^*}.$$

while the boundary condition (11a) becomes

$$[p]p^*(0,t^*) = p_0\cos(\omega[t]t^*)$$

Rearranging,

$$\begin{aligned}
\frac{c[p][x_1]}{[u_1][t]}\frac{\partial p^*}{\partial t^*} + \frac{\partial[(1+c[p]p^*)u^*]}{\partial x^*} &= 0,\\
\frac{\rho_0[u_1][x_1]}{[p][t]}\left(\frac{\partial u^*}{\partial t^*} + \frac{[u_1][t]}{[x_1]}u^*\frac{\partial u^*}{\partial x^*}\right) &= \frac{4}{3}\frac{\mu[u_1]}{[p][x_1]}\frac{\partial^2 u^*}{\partial x^{*2}} - \frac{\partial p^*}{\partial x^*}\\
p^*(0,t^*) &= \frac{p_0}{[p]}\cos(\omega[t]t^*)
\end{aligned}$$

We obtain the desired form if we equate the following dimensionless groups to unity:

$$\frac{c[p][x_1]}{[u_1][t]} = 1, \qquad \frac{\rho_0[u_1][x_1]}{[p][t]} = 1, \qquad \frac{p_0}{[p]} = 1 \qquad \omega[t] = 1. \tag{15}$$

The dimensionless parameters $\alpha$ and $\gamma$ are then given by

$$\alpha = c[p], \qquad \gamma = \frac{\mu[u_1]}{[p][x_1]}.$$

Note that an alternative form for $\alpha$ is

$$\alpha = \frac{[u_1][t]}{[x_1]};$$

this follows from the first equation in (15).

(c) (2 points) For a (deafening!) 90 dB sound wave in water at 1000 Hz, we have $p_0 = .045$ Pa, $\omega = 2000\pi$ s$^{-1}$, $\rho_0 = 1000$ kg m$^{-3}$, $c = 4.6 \times 10^{-10}$ Pa$^{-1}$, $\mu = 1.7 \times 10^{-3}$ Pa s. Find numerical values of $\alpha$ and $\gamma$. Show how these can be used to motivate the simplified model

$$\begin{aligned}
\frac{\partial u^*}{\partial t^*} = \frac{2}{3}\gamma\frac{\partial^2 u^*}{\partial x^{*2}} - \frac{\partial p^*}{\partial x^*} &= 0 && \text{for } x^* > 0 && \text{(16a)}\\
\frac{\partial p^*}{\partial t^*} + \frac{\partial u^*}{\partial x^*} &= 0 && \text{for } x^* > 0 && \text{(16b)}\\
p^*(0, t^*) &= \cos(t^*) && \text{at } x^* = 0. && \text{(16c)}
\end{aligned}$$

ANSWER: We have $\alpha = c[p] = cp_0 = 2.1 \times 10^{-11}$. In addition, we have $[t] = 1/\omega = 1.6 \times 10^{-4}$ s. Manipulating the first equation in (15), we also get

$$\frac{[u_1]}{[x_1]} = \frac{c[p]}{[t]},$$

and substituting in the definition of $\gamma$,

$$\gamma = \frac{\mu[u_1]}{[p][x_1]} = \frac{\mu c}{[t]} = 4.6 \times 10^{-9}.$$

The simplified equations above result if we ignore terms multiplied by $\alpha$ (on the basis that $\alpha \ll 1$) while retaining $\gamma$ (which is somewhat larger than $\alpha$, though still small).

3. This question is about solving a seismic P- and S-wave problem in a viscous fluid. A simplified model for seismic P-waves in viscous fluid is

$$\rho_0 \frac{\partial u}{\partial t} = \frac{4}{3}\mu \frac{\partial^2 u}{\partial x^2} - \frac{\partial p}{\partial x} \qquad \text{for } x > 0 \tag{17a}$$

$$c\frac{\partial p}{\partial t} + \frac{\partial u}{\partial x} = 0 \qquad \text{for } x > 0 \tag{17b}$$

$$p(0, t) = p_0 \cos(\omega t) \qquad \text{at } x = 0 \tag{17c}$$

$$p \to 0 \qquad \text{as } x \to \infty \tag{17d}$$

You will solve this by complex variable methods, and compare with the solution of an $S$-wave problem.

(a) (2 points) By differentiating (17a) and substituting from another equation in (17), show that

$$\rho_0 c \frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial x^2} - \frac{4}{3}\mu c \frac{\partial^3 p}{\partial x^2 \partial t} = 0. \tag{18}$$

ANSWER: Differentiate (17a) with respect to $x$:

$$\rho_0 \frac{\partial^2 u}{\partial t \partial x} = \frac{4}{3}\mu \frac{\partial^3 u}{\partial x^3} - \frac{\partial^2 p}{\partial x^2}$$

But, from (17b),

$$\frac{\partial u}{\partial x} = -c\frac{\partial p}{\partial t}.$$

Substituting therefore gives

$$-\rho_0 c \frac{\partial^2 p}{\partial t^2} = -\frac{4}{3}\mu c \frac{\partial^3 p}{\partial t \partial x^2} - \frac{\partial^2 p}{\partial x^2}$$

which gives the desired form on re-arranging.

(b) (2 points) Assume that $p(x, t)$ can be written in the form

$$p(x, t) = \mathrm{Re}\left[p_0 \exp(i\omega t + \lambda x)\right].$$

For $p$ satisfying (18), find the equation that must be satisfied by $\lambda$, and solve for $\lambda$ in terms of $\rho_0$, $c$, $\mu$ and $\omega$.
ANSWER: We have

$$\frac{\partial^2 p}{\partial t^2} = \mathrm{Re}\left[-\omega^2 p_0 \exp(i\omega t + \lambda x)\right]$$

$$\frac{\partial^2 p}{\partial x^2} = \mathrm{Re}\left[\lambda^2 p_0 \exp(i\omega t + \lambda x)\right]$$

$$\frac{\partial^3 p}{\partial t \partial^2 x} = \mathrm{Re}\left[i\lambda^2 \omega p_0 \exp(i\omega t + \lambda x)\right]$$

Substitute

$$\mathrm{Re}\left[\left(-\rho_0 c\omega^2 - \lambda^2 - \frac{4}{3}\mu ci\omega\lambda^2\right) p_0 \exp(i\omega t + \lambda x)\right] = 0.$$

To make this zero at all times $t$ and positions $x$, the expression inside the round brackets must be zero:

$$-\rho_0 c\omega^2 - \lambda^2 - \frac{4}{3}\mu ci\omega\lambda^2 = 0.$$

Hence

$$\lambda = \left(\frac{-\rho_0 c\omega^2}{1 + i\frac{4}{3}\mu c\omega}\right)^{1/2} \qquad = \pm\frac{i\sqrt{\rho c}\omega}{\left(1 + i\frac{4}{3}\mu c\omega\right)^{1/2}}.$$

(c) (2 points) The answer you get should be in the form

$$\lambda = \pm ia/(1 + ib)^{1/2} \tag{19}$$

The Taylor expansion of $(1 + x)^{-1/2}$ for small $x$ is

$$(1 + x)^{-1/2} = 1 - \frac{1}{2}x + \ldots$$

For small $b$, find an approximation to (19) of the form

$$\lambda = \pm(\alpha + i\beta)$$

with $\alpha$ and $\beta$ real quantities that depend on $\rho_0$, $c$, $\mu$ and $\omega$. If $\omega = 2000\pi\ \mathrm{s}^{-1}$, $\rho_0 = 1000\ \mathrm{kg\ m}^{-3}$, $c = 4.6 \times 10^{-10}\ \mathrm{Pa}^{-1}$, $\mu = 1.7 \times 10^{-3}$ Pa a, is this approximation valid?
ANSWER: We have

$$\left(1 + i\frac{4}{3}\mu c\omega\right)^{-1/2} = 1 - i\frac{2}{3}\mu c\omega + \ldots$$

and hence

$$\begin{aligned}
\lambda &= \pm\left[i\sqrt{\rho_0 c}\omega\left(1 + i\frac{4}{3}\mu c\omega\right)^{-1/2}\right] \\
&\approx \pm\left[i\sqrt{\rho_0 c}\omega\left(1 - i\frac{2}{3}\mu c\omega\right)\right] \\
&\approx \pm\left[i\sqrt{\rho_0 c}\omega + \frac{2}{3}\mu c\sqrt{\rho_0 c}\omega^2\right]
\end{aligned}$$

To satisfy the boundary condition $p \to \infty$, $\lambda$ must have negative real part. Hence we have to choose the $-$ sign out of $\pm$, and

$$\lambda \approx -i\sqrt{\rho_0 c}\omega - \frac{2}{3}\mu c\sqrt{\rho_0 c}\omega^2.$$

To determine whether the approximation above is valid, calcuate

$$\frac{4}{3}\mu c\omega = 6.5 \times 10^{-9}$$

Clearly, this is small so

(d) (3 points) Given your answer to part c, express $p(x,t)$ in real terms in the form

$$p(x,t) = p_0 \cos\left[\omega(t - x/v)\right] \exp(-x/x_0),$$

making sure to give expressions for $v$ and $x_0$ in terms of $\rho_0$, $c$, $\mu$ and $\omega$, and justifying your choice of signs. What is the wave velocity? What is the wavelength? What is the distance over which the wave amplitude decreases by a factor of $1/e$ (this is the '$e$-folding distance')? If $\omega = 2000\pi$ s$^{-1}$, $\rho_0 = 1000$ kg m$^{-3}$, $c = 4.6 \times 10^{-10}$ Pa$^{-1}$, give numerical values for velocity, wavelength and the $e$-folding distance.

Taking the real part,

$$\begin{aligned} p(x,t) &= \mathrm{Re}\left[p_0 \exp(i\omega t + \lambda x)\right] \\ &\approx \mathrm{Re}\left[p_0 \exp\left(i\omega t - i\sqrt{\rho_0 c}\omega x - \frac{2}{3}\mu c\sqrt{\rho_0 c}\omega^2 x\right)\right] \\ &= p_0 \exp\left(-\frac{2}{3}\mu c\sqrt{\rho_0 c}\omega^2 x\right)\cos\left[\omega(t - \sqrt{\rho_0 c}x)\right] \end{aligned}$$

The wave velocity is therefore

$$\frac{1}{\sqrt{\rho_0 c}} = 1470 \text{ m s}^{-1}$$

while the $e$-folding distance is

$$\frac{1}{\frac{2}{3}\mu c\sqrt{\rho c}\omega^2} = 7.2 \times 10^{11}\text{m}$$

(e) (1 point) The corresponding S-wave problem would be

$$\begin{aligned} \rho_0\frac{\partial v}{\partial t} - \mu\frac{\partial^2 v}{\partial x^2} &= 0 && \text{for } x > 0, \\ v(0,t) &= v_0\cos(\omega t) && \text{at } x = 0, \\ v &\to 0 && \text{as } x \to \infty, \end{aligned}$$

where $v$ is velocity transverse to the $x$-axis. Using methods from the course, it can be shown that the solution is (no need to derive this yourself! — simply use this formula)

$$v = v_0 \cos\left(\omega t - \sqrt{\frac{\rho_0 \omega}{2\mu}} x\right) \exp\left(-\sqrt{\frac{\rho_0 \omega}{2\mu}} x\right)$$

With the values for $\rho_0$, $\omega$ and $\mu$ given above, calculate the distance over which $v$ decays to $1/e$ of its value at $x = 0$.
ANSWER: The $e$-folding distance in this example is

$$\sqrt{\frac{2\mu}{\rho_0 \omega}} = 2.3 \times 10^{-5}\mathrm{m}.$$

4. You are given a slanted triangular surface $S$ with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

   (a) (2 points) Your are given a temperature field

   $$T = x + 2y + 3z$$

   and a constant thermal conductivity $k = 1$. What is the rate at which heat passes from above $S$ to below (this rate has dimensions of energy over time)? ANSWER: The heat flux is

   $$\mathbf{q} = -\nabla T = -\mathbf{i} - 2\mathbf{j} - 3\mathbf{k}.$$

   The normal that points from above the surface to below is

   $$\mathbf{n} = -\frac{\mathbf{i} + \mathbf{j} + \mathbf{k}}{\sqrt{3}}.$$

   The surface is an equilateral triangle with side length $\sqrt{2}$, and so has area one half base times height $= 1/2 \times \sqrt{2} \times \sqrt{2}\cos(\pi/3) = \sqrt{3}/2$. The surface being flat (with constant normal) and the flux being constant, we do not need to integrate. The rate of heat transfer is simply

   $$\mathbf{q} \cdot \mathbf{n} S = 3.$$

   (b) (3 points) You have a stress tensor $\sigma_{ij}$ given in matrix form by

   $$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

   What is the force exerted by the material above $S$ on the material below $S$? ANSWER: The surface is flat, so the normal is a constant vector. Similarly, the stress tensor is a constant. Hence no intergration is necessary. The force is therefore

   $$F_i = \sigma_{ij} n_j S$$

   where $\mathbf{n}$ now points from *below* the surface to *above*, so $n_1 = n_2 = n_3 = 1/\sqrt{3}$. Hence, using the fact that $\sigma_{ij} n_j = \sum_{j=1}^{3} \sigma_{ij} n_j$ signifies the $i$th component of the product of the matrix $\sigma$ with the vector $\mathbf{n}$, we have

   $$\begin{aligned} \mathbf{F} &= \sigma \mathbf{n} \times S \\ &= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix} \times \frac{\sqrt{3}}{2} \\ &= \begin{pmatrix} 1 \\ 1/2 \\ 1 \end{pmatrix} \end{aligned}$$

(c) (1 point) What is the pressure $p$ that corresponds to the stress tensor in part b?
ANSWER: We have $p = -\sigma_{ii}/3 = -(\sigma_{11} + \sigma_{22} + \sigma_{33})/3 = -1/3$.

(d) (6 points) Let $\rho = 1$ and $\mathbf{u} = (x_2, -x_1, x_3)$. Compute the angular momentum contained in the volume bounded by $S$ and the planes $x_1 = 0$, $x_2 = 0$ and $x_3 = 0$.
ANSWER: We have $L_{ij} = \int_V \rho(x_i u_j - x_j u_i)\,\mathrm{d}V$. We only need to compute $L_{12}$, $L_{13}$ and $L_{23}$. With the values given for $\rho$, $u_1$, $u_2$ and $u_3$, we get

$$\begin{aligned}
L_{12} &= \int -x_1^2 - x_2^2 \,\mathrm{d}V \\
L_{13} &= \int_V x_1 x_3 - x_3 x_2 \,\mathrm{d}V \\
L_{23} &= \int_V x_2 x_3 + x_3 x_1 \,\mathrm{d}V.
\end{aligned}$$

By symmetry (the volume looks the same in the $x_1$- $x_2$ and $x_3$-directions), we expect $\int_V x_1 x_3 \,\mathrm{d}V = \int_V x_2 x_3 \,\mathrm{d}V$ and $\int_V x_1^2 \,\mathrm{d}V = \int_V x_2^2 \,\mathrm{d}V$. Immediately we have that $L_{13} = 0$. Also

$$\begin{aligned}
\int_V x_1^2 \,\mathrm{d}V &= \int_0^1 \int_0^{1-x_1} \int_0^{1-x_1-x_2} x_1^2 \,\mathrm{d}x_3 \,\mathrm{d}x_2 \,\mathrm{d}x_1 \\
&= \int_0^1 \int_0^{1-x_1} x_1^2 (1 - x_1 - x_2) \,\mathrm{d}x_2 \,\mathrm{d}x_1 \\
&= \int_0^1 x_1^2 (1 - x_1)^2 - x_1^2 (1 - x_1)^2/2 \,\mathrm{d}x_1 \\
&= \int_0^1 x_1^2 (1 - x_1)^2/2 \,\mathrm{d}x_1 \\
&= \left[-x_1^2 (1 - x_1)^3/6\right]_0^1 - \int_0^1 -2x_1 (1 - x_1)^3/6 \,\mathrm{d}x_1 \\
&= \int_0^1 x_1 (1 - x_1)^3/3 \,\mathrm{d}x_1 \\
&= \left[-x_1^2 (1 - x_1)^4/12\right]_0^1 - \int_0^1 -(1 - x_1)^4/12 \,\mathrm{d}x_1 \\
&= \left[-\frac{1}{6}(1 - x_1)^5\right]_0^1 \\
&= \frac{1}{60}.
\end{aligned}$$

Hence

$$L_{12} = -2 \int_V x_1^2 \,\mathrm{d}V = -\frac{1}{30}.$$

Also

$$\begin{aligned}
\int_V x_1 x_2 \,\mathrm{d}V &= \int_0^1 \int_0^{1-x_1} \int_0^{1-x_1-x_2} x_1 x_2 \,\mathrm{d}x_3 \,\mathrm{d}x_2 \,\mathrm{d}x_1 \\
&= \int_0^1 \int_0^{1-x_1} x_1 x_2 (1 - x_1 - x_2) \,\mathrm{d}x_2 \,\mathrm{d}x_1 \\
&= \int_0^1 \int_0^{1-x_1} x_1 x_2 \,\mathrm{d}x_2 \,\mathrm{d}x_1 - \int_0^1 \int_0^{1-x_1} x_1^2 x_2 \,\mathrm{d}x_2 \,\mathrm{d}x_1 - \int_0^1 \int_0^{1-x_1} x_1 x_2^2 \,\mathrm{d}x_2 \,\mathrm{d}x_1 \\
&= \int_0^1 x_1 (1-x_1)^2/2 \,\mathrm{d}x_1 - \int_0^1 x_1^2 (1-x_1)^2/2 \,\mathrm{d}x_1 - \int_0^1 x_1 (1-x_1)^3/3 \,\mathrm{d}x_1
\end{aligned}$$

Do each of these integrals in turn:

$$\begin{aligned}
\int_0^1 x_1 (1-x_1)^2/2 \,\mathrm{d}x_1 &= \left[-x_1(1-x_1)^3/6\right]_0^1 - \int_0^1 -(1-x_1)^3/6 \,\mathrm{d}x_1 \\
&= \left[-\frac{1}{24}(1-x_1)^4\right]_0^1 \\
&= \frac{1}{24}
\end{aligned}$$

and

$$\begin{aligned}
\int_0^1 x_1^2 (1-x_1)^2/2 \,\mathrm{d}x_1 &= \left[-x_1^2(1-x_1)^3/6\right]_0^1 - \int_0^1 -x_1(1-x_1)^3/3 \,\mathrm{d}x_1 \\
&= \left[-x_1(1-x_1)^4/12\right]_0^1 - \int_0^1 -(1-x_1)^4/12 \,\mathrm{d}x_1 \\
&= \left[-\frac{1}{60}(1-x_1)^5\right]_0^1 \\
&= \frac{1}{60}
\end{aligned}$$

as well as

$$\begin{aligned}
\int_0^1 x_1 (1-x_1)^3/3 \,\mathrm{d}x_1 &= \left[-x_1^2(1-x_1)^4/12\right]_0^1 - \int_0^1 -(1-x_1)^4/12 \,\mathrm{d}x_1 \\
&= \frac{1}{60}
\end{aligned}$$

which could also have been deduced from the previous calculation by symmetry. Combining the last three results,

$$\int_V x_1 x_2 \,\mathrm{d}V = \int_0^1 \int_0^{1-x_1} \int_0^{1-x_1-x_2} x_1 x_2 \,\mathrm{d}x_3 \,\mathrm{d}x_2 \,\mathrm{d}x_1 = \frac{1}{24} - \frac{1}{30} = \frac{1}{120}.$$

Hence

$$L_{23} = 2 \times \int_V x_1 x_2 \, \mathrm{d}V = \frac{1}{60}.$$