Final: EOSC 352

18 December, 2012

This exam consists of four questions worth ten marks each. Available marks for each part of a question are indicated in brackets. Attempt THREE questions. You have 2 hours 20 minutes.

1. Consider two-dimensional heat conduction with cylindrical symmetry about the origin, so the termperature field $T(r,t)$ depends only on time t and distance r from the origin. There is no heat flow 'at infinity', but there is a heat source at the origin that is switched on at $t=0$. Prior to $t=0$, the temperature is uniform at $T=0$. In non-dimensional form, this situation can be described by

$$\frac{\partial T}{\partial t} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = 0 \qquad \text{everywhere for } t>0 \tag{1a}$$

$$T(r,0) = 0 \qquad \text{for all } r>0. \tag{1b}$$

$$\lim_{r\to 0}\left(-2\pi r\frac{\partial T}{\partial r}\right) = 1 \qquad \text{for all } t>0 \tag{1c}$$

$$-\frac{\partial T}{\partial r} \to 0 \qquad \text{as } r\to\infty. \tag{1d}$$

In this question, you will construct a similarity solution to the problem.

(a) (4 points) Let

$$T(r,t) = t^{-\alpha}\Theta(r/t^{\beta}). \tag{2}$$

and define

$$\xi = r/t^{\beta}.$$

Substitute this into (1a), converting partial derivatives with respect to r and t into ordinary derivatives with respect to ξ. Show that you get

$$-\alpha t^{-\alpha-1}\Theta(\xi) - \beta t^{-\alpha-1}\xi\frac{\mathrm{d}\Theta}{\mathrm{d}\xi} - t^{-\alpha-2\beta}\frac{1}{\xi}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi\frac{\mathrm{d}\Theta}{\mathrm{d}\xi}\right) = 0. \tag{3}$$

What value does β have to take in order for a similarity solution (2) to hold?

(b) (2 points) Next, show that the initial condition (1b) and boundary condition (1d) can be expressed as

$$r^{-\alpha/\beta}\xi^{\alpha/\beta}\Theta(\xi) \to 0 \qquad \text{as } \xi\to\pm\infty \text{ at any fixed } r>0, \tag{4a}$$

$$t^{-\alpha-\beta}\Theta'(\xi) \to 0 \qquad \text{as } \xi\to\pm\infty \text{ at any fixed } t>0. \tag{4b}$$

(c) (1 point) Show that the heat source condition (1c) takes the form

$$\lim_{\xi\to 0}\left(-t^{-\alpha}2\pi\xi\Theta'(\xi)\right) = 1 \qquad \text{for any fixed } t>0. \tag{4c}$$

Why does it follow that $\alpha = 0$?

(d) (2 points) Put the value of β you have deduced and $\alpha=0$ into (3). Exapnd the last term in (3) using the product rule. Separate variables to show that

$$\Theta'(\xi) = \frac{C}{\xi}\exp\left(-\frac{\xi^2}{4}\right) \tag{5}$$

Show that $C = -1/2\pi$.

(e) (2 points) Show that

$$T(r,t) = \Theta(\xi) = \int_{\xi}^{\infty} \frac{1}{2\pi\xi'} \exp\left(-\frac{\xi'^2}{4}\right) \mathrm{d}\xi'.$$

The *upper incomplete gamma function* is defined as

$$\Gamma(s,x) = \int_{x}^{\infty} t^{s-1} \exp(-t)\, \mathrm{d}t.$$

Show that

$$T(r,t) = \frac{1}{4\pi}\Gamma\left(0, \frac{r^2}{4t}\right).$$

2. The following is a model for temperature waves in the ground, driven by seasonally varying solar radiation intensity. The mean temperature $\bar{T}$ of the ground surface is given by a balanace of incoming radiation and outgoing radiation,

$$\bar{q} = \sigma \bar{T}^4, \tag{6}$$

where $\bar{q} = 1350$ W m^{-2} is the *solar constant* and $\sigma = 5.67 \times 10^{-8}$ W m^{-2} K^{-4} is the Stefan-Boltzmann constant, and $\bar{T}$ is expressed in Kelvins. Variations in incoming radiation about their mean $\bar{q}$ are balance by radiation back in space and conduction of heat into the ground. This leads to the following energy conservation model in the ground ($x > 0$):

$$\rho c \frac{\partial T}{\partial t} - k \frac{\partial^2 T}{\partial x^2} = 0 \qquad \text{for } x > 0, \tag{7a}$$

$$q_0 \cos(\omega t) = -k \frac{\partial T}{\partial x} + 4\sigma \bar{T}^3 T \qquad \text{at } x = 0, \tag{7b}$$

$$-k \frac{\partial T}{\partial x} \to 0 \qquad \text{as } x \to \infty. \tag{7c}$$

Let $k = 2$ W m^{-1} K^{-1}, $\rho = 2000$ kg m^{-3}, $c = 8 \times 10^2$ J kg^{-1} K^{-1}, and $q_0 = 0.2 \times \bar{q}$.

(a) (3 points) Define dimensionless variables through $T = [T]T^*$, $t = [t]t^*$, $x = [x]x^*$. Show that the problem can be rendered in the form

$$\frac{\partial T^*}{\partial t^*} - \frac{\partial^2 T^*}{\partial x^{*2}} = 0 \qquad \text{for } x > 0, \tag{8a}$$

$$\cos(t^*) = -\alpha \frac{\partial T^*}{\partial x^*} + T^* \qquad \text{at } x^* = 0, \tag{8b}$$

$$-\frac{\partial T^*}{\partial x^*} \to 0 \qquad \text{as } x^* \to \infty. \tag{8c}$$

Define the scales $[x]$, $[t]$ and $[T]$ in terms of ρ, c, k, $\bar{q}$, q_0, ω and σ. For the values given above, give numerical values for these scales and for the parameter α. Is there a useful approximation you could make to (8)?

(b) (4 points) Look for a solution

$$T^*(x^*, t^*) = \mathrm{Re}[A \exp(it^* + \lambda x^*)]$$

Find λ so that (8a) and (8c) are satisifed simultaneously. Show that you can write (8b) in the form

$$\mathrm{Re}\{[A(1 - \alpha\lambda) - 1] \exp(it^*)\} = 0$$

for any t^*. Give a solution for A in terms of α.

(c) (2 points) Write A in polar form

$$A = |A| \exp(i\theta),$$

and find $|A|$ and θ in terms of α. You may either find an exact version of the polar form, or try approximating it using $\alpha \ll 1$. Note that, for a small number θ, $\cos(\theta) \approx 1$ and $\sin(\theta) \approx \theta$. For the value of α you have computed above, how big is θ? (Note that the approximation $\theta = 0$ will not get you any points.)

(d) (1 point) What is the dimensional amplitude of temperature variations? By how many days does the surface temperature peak lag the surface insolation peak? In the northern hemisphere, which date does this correspond to?

3. This question is about flow in a pipe. Assume the pipe is aligned with the x_1-axis and has a cross-section that does not change along the length of the pipe. Consider a fluid of constant density ρ with an *unidirectional* velocity field $\mathbf{u} = (u_1, 0, 0)$ (meaning that velocity components that point across the pipe are zero) that depends only on time t and position (x_2, x_3) in the cross-section but not on distance x_1 along the pipe. Assume also that the body force f_i does not depend on position or time. In addition, the velocity is zero at the wall of the pipe, so $u_1 = 0$ there. Start with the Navier-Stokes equations for fluid flow

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_i)}{\partial x_i} = 0, \tag{9a}$$

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right) = \frac{\partial \sigma_{ij}}{\partial x_j} + f_i, \tag{9b}$$

where

$$\sigma_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right) - p\delta_{ij}.$$

(a) (4 points) For the geometry assumed above, show that the Navier-Stokes equations (9) reduce to

$$\rho\frac{\partial u_1}{\partial t} - \mu\left(\frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_2^2}\right) = -\frac{\partial p}{\partial x_1} + f_1 \tag{10a}$$

$$0 = -\frac{\partial p}{\partial x_2} + f_2 \tag{10b}$$

$$0 = -\frac{\partial p}{\partial x_3} + f_3 \tag{10c}$$

$$\tag{10d}$$

(b) (2 points) Recall that f_1, f_2 and f_3 are assumed to be constant. Show that we *must* have

$$p = f_2 x_2 + f_3 x_3 - C x_1 + D$$

for some constants C and D.

(c) (3 points) Assume that $C = 0$, and that the flow is in steady state, so u_1 is independent of t. Assume also that the pipe has a circular cross-section with radius R centered on $(x_2, x_3) = (0, 0)$. We can then define plane polar coordinates (r, θ) through $x_2 = r\cos(\theta)$, $x_3 = r\sin(\theta)$. If the velocity $u_1 = u_1(r)$ depends only on distance r from the centre of the pipe, then the Laplacian of u_1 can be written as

$$\frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2} = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u_1}{\mathrm{d}r}\right) \tag{11}$$

From this, derive that

$$u_1(r) = \frac{f_1}{4\mu}(R^2 - r^2).$$

Make sure to justify the choice of any constants of integration.

(d) (1 point) Show that the rate at which mass passes through any given cross-section of the pipe is given by

$$2\pi \int_0^R \frac{f_1}{4\mu}(R^2 - r^2) r \, \mathrm{d}r.$$

4. This question is about stresses in a unidirectional flow. Consider a semicircular channel of radius R, aligned with the x_1-axis and open to the atmosphere at the top (see figure 1). The region occupied by the fluid is given by

$$x_2^2 + x_3^2 < R^2 \qquad \text{and} \qquad x_3 < 0.$$

For a fluid of constant viscosity μ and density ρ, the velocity field in this region is given by $\mathbf{u} = (u_1, 0, 0)$, where

$$u_1 = \frac{\rho g \sin(\alpha)}{4\mu}(R^2 - x_2^2 - x_3^2)$$

where g is acceleration due to gravity and α the angle of inclination of the channel to the horizontal. The pressure field is

$$p = -\rho g \cos(\alpha) x_3.$$

(a) (3 points) For an incompressible viscous fluid, stress σ_{ij} is given by

$$\sigma_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - p\delta_{ij}.$$

For the velocity field given above, compute σ_{ij} as a function of x_2, x_3, R, μ, ρ, g and α. Write your answer as a matrix.

(b) (1 point) Show that, at the upper surface at $x_3 = 0$, we have $\sigma_{ij}n_j = 0$, where n_i is the normal to that surface.

(c) (1 point) Next, consider a small area element ΔS just below the the upper surface of the flow at $(0, x_2, 0)$, but with normal $(\cos(\theta), \sin(\theta), 0)$. *Note that this is not the normal to the upper surface, so ΔS is not parallel to the upper surface.* Compute the force $\Delta F_i = \sigma_{ij}n_j\Delta S$ exerted by the fluid flow on the surface ΔS as a function of x_2, R, μ, ρ, g and α.

(d) (1 point) Show that the component $\Delta F_n = \Delta F_i n_i$ of ΔF_i normal to ΔS is given by

$$\Delta F_n = -\rho g \sin(\alpha)\cos(\theta)\sin(\theta) x_2 \Delta S.$$

(e) (2 points) Keep the size of the area element ΔS and its position x_2 fixed, but allow its orientation (given by the angle θ) to vary. If $x_2 > 0$, what angle maximizes ΔF_n? What angle maximizes ΔF_n if $x_2 < 0$? (*Hint. Recall that* $\cos^2(\theta) - \sin^2(\theta) = \cos(2\theta)$.)

(f) (2 points) Glacier ice flows as an incompressible viscous fluid, but can crack to form *crevasses* when subjected to high enough stresses. Specifically, crevasses form along surfaces in the ice that experience large enough normal forces $\Delta F_n/\Delta S$. The orientation of crevasses when they first form is therefore such as to maximize ΔF_n, and they form first at those positions on the surface where ΔF_n is biggest. Consider a glacier flowing down a semicircular channel. Where will crevasses first form? Sketch the channel and indicate the flow direction as in figure 1, and indicate the pattern of crevasses you expect to form.

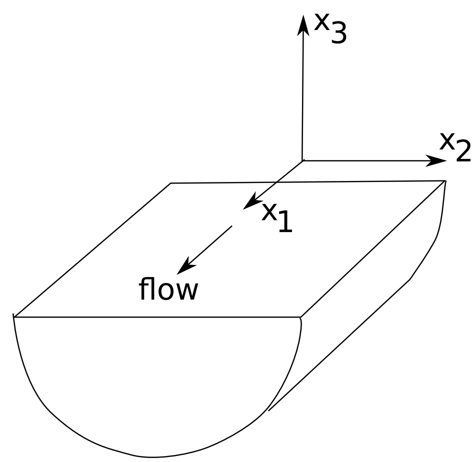

Figure 1: Sketch of the semicircular channel of question 4. The radius of the semicircular cross-section is R.