Final: EOSC 352

18 December, 2012

This exam consists of four questions worth ten marks each. Available marks for each part of a question are indicated in brackets. Attempt THREE questions. You have 2 hours 20 minutes.

1. Consider two-dimensional heat conduction with cylindrical symmetry about the origin, so the termperature field $T(r,t)$ depends only on time t and distance r from the origin. There is no heat flow 'at infinity', but there is a heat source at the origin that is switched on at $t=0$. Prior to $t=0$, the temperature is uniform at $T=0$. In non-dimensional form, this situation can be described by

$$\frac{\partial T}{\partial t} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = 0 \qquad \text{everywhere for } t>0 \tag{1a}$$

$$T(r,0) = 0 \qquad \text{for all } r>0. \tag{1b}$$

$$\lim_{r\to 0}\left(-2\pi r\frac{\partial T}{\partial r}\right) = 1 \qquad \text{for all } t>0 \tag{1c}$$

$$-\frac{\partial T}{\partial r} \to 0 \qquad \text{as } r\to\infty. \tag{1d}$$

In this question, you will construct a similarity solution to the problem.

(a) (4 points) Let

$$T(r,t) = t^{-\alpha}\Theta(r/t^{\beta}). \tag{2}$$

and define

$$\xi = r/t^{\beta}.$$

Substitute this into (1a), converting partial derivatives with respect to r and t into ordinary derivatives with respect to ξ. Show that you get

$$-\alpha t^{-\alpha-1}\Theta(\xi) - \beta t^{-\alpha-1}\xi\frac{\mathrm{d}\Theta}{\mathrm{d}\xi} - t^{-\alpha-2\beta}\frac{1}{\xi}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi\frac{\mathrm{d}\Theta}{\mathrm{d}\xi}\right) = 0. \tag{3}$$

What value does β have to take in order for a similarity solution (2) to hold?
ANS: We have

$$\begin{aligned}\frac{\partial T}{\partial t} &= -\alpha t^{-\alpha-1}\Theta(\xi) - \beta r t^{-\alpha-\beta-1}\Theta'(\xi)\\ &= -\alpha t^{-\alpha-1}\Theta(\xi) - \beta t^{-\alpha-1}\xi\Theta'(\xi)\end{aligned}$$

as well as

$$
\begin{aligned}
\frac{\partial T}{\partial r} &= t^{-\alpha-\beta}\Theta'(\xi) \\
r\frac{\partial T}{\partial r} &= rt^{-\alpha-\beta}\Theta'(\xi) \\
&= t^{-\alpha}\xi\Theta'(\xi) \\
\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) &= \frac{\partial}{\partial r}\left(t^{-\alpha}\xi\Theta'(\xi)\right) \\
&= t^{-\alpha}\frac{\partial}{\partial r}\left(\xi\Theta'(\xi)\right) \\
&= t^{-\alpha}\frac{\partial \xi}{\partial r} \times \left(\xi\Theta'(\xi)\right)' \\
&= t^{-\alpha-\beta}\left(\xi\Theta'(\xi)\right)' \\
\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) &= t^{-\alpha-\beta}r^{-1}\left(\xi\Theta'(\xi)\right)' \\
&= t^{-\alpha-2\beta}(r/t^{-\beta})^{-1}\left(\xi\Theta'(\xi)\right)' \\
&= t^{-\alpha-2\beta}\frac{1}{\xi}\left(\xi\Theta'(\xi)\right)'
\end{aligned}
$$

Substituting these into (1a) gives (3). The factors $t^{-\alpha-1}$ and $t^{-\alpha-2\beta}$ must cancel as Θ must not depend explicitly on time t, so their exponents must be the same. It follows that $\beta = 1/2$.

(b) (2 points) Next, show that the initial condition (1b) and boundary condition (1d) can be expressed as

$$r^{-\alpha/\beta}\xi^{\alpha/\beta}\Theta(\xi) \to 0 \qquad \text{as } \xi \to \infty \text{ at any fixed } r > 0, \tag{4a}$$

$$t^{-\alpha-\beta}\Theta'(\xi) \to 0 \qquad \text{as } \xi \to \infty \text{ at any fixed } t > 0. \tag{4b}$$

ANS: We have $\xi = r/t^{\beta}$ so that $t = (r/\xi)^{1/\beta}$ and $t^{-\alpha} = \xi^{\alpha/\beta}/r^{\alpha/\beta}$. The limit $t \to 0$ at fixed r therefore corresponds to $\xi \to \infty$ while r remains fixed. (1b) needs to be interpreted as $\lim_{t\to 0} T(r,t) = 0$ for any fixed r, which we can re-write as

$$\lim_{t\to 0} T(r,t) = \lim_{t\to 0}\left(t^{-\alpha}\Theta(r/t^{\beta})\right) = \lim_{\xi\to\infty} \xi^{\alpha/\beta}/r^{\alpha/\beta}\Theta(\xi) = 0.$$

Similarly, we have from above that

$$\frac{\partial T}{\partial r} = t^{-\alpha-\beta}\Theta'(\xi) \to 0$$

as $\xi \to \infty$ at fixed t.

(c) (1 point) Show that the heat source condition (1c) takes the form

$$\lim_{\xi\to 0}\left(-t^{-\alpha}2\pi\xi\Theta'(\xi)\right)=1 \qquad \text{for any fixed } t>0. \tag{4c}$$

Why does it follow that $\alpha = 0$?
ANS: From above, we have that

$$r\frac{\partial T}{\partial r}=t^{-\alpha}\xi\Theta'(\xi)$$

and the limit $r\to 0$ corresponds to $\xi\to 0$ at fixed t. The condition (1c) therefore takes the required form. The only way that the left-hand side above can take the value of 1 regardless of time t is if the exponent $-\alpha$ is zero,

(d) (2 points) Put the value of β you have deduced and $\alpha = 0$ into (3). Exapnd the last term in (3) using the product rule. Separate variables to show that

$$\Theta'(\xi)=\frac{C}{\xi}\exp\left(-\frac{\xi^2}{4}\right) \tag{5}$$

Show that $C=-1/2\pi$.
ANS: Substituting and cancelling the factors $t^{-\alpha-1}=t^{-\alpha-\beta}$ gives

$$-\frac{1}{2}\xi\Theta'(\xi)-\frac{1}{\xi}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi\Theta'(\xi)\right)'=0$$

Expanding the second term gives

$$\frac{1}{\xi}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi\Theta'(\xi)\right)=\frac{1}{\xi}\Theta'(\xi)+\Theta''(\xi)$$

so that

$$-\frac{1}{2}\xi\Theta'(\xi)-\frac{1}{\xi}\Theta'(\xi)-\Theta''(\xi)=0.$$

Separating variables gives

$$\frac{\Theta''(\xi)}{\Theta'(\xi)}=-\frac{1}{2}\xi-\frac{1}{\xi}.$$

Integrating both sides, we get

$$\log(\Theta'(\xi))=-\frac{1}{4}\xi^2-\log(\xi)+K.$$

Exponentiating gives

$$\begin{aligned}\Theta'(\xi) &= \exp\left(-\frac{1}{4}\xi^2 - \log(\xi) + K\right)\\ &= \frac{\exp(K)}{\exp(\log(\xi))}\exp\left(-\frac{\xi^2}{4}\right)\\ &= \frac{C}{\xi}\exp\left(-\frac{\xi^2}{4}\right)\end{aligned}$$

To find C, use the boundary condtion (1c) in transformed form with $\alpha = 0$,

$$\lim_{\xi\to 0}\left(-2\pi\xi\Theta'(\xi)\right) = 1.$$

In the solution above, this requires

$$-\xi\Theta'(\xi) = -C\exp\left(-\frac{\xi^2}{4}\right)$$

and so $\lim_{\xi\to 0}(-2\pi\xi\Theta'(\xi)) = -2\pi C = 1$. Hence $C = -1/(2\pi)$.

(e) (2 points) Show that

$$T(r,t) = \Theta(\xi) = \int_\xi^\infty \frac{1}{2\pi\xi'}\exp\left(-\frac{\xi'^2}{4}\right)\,\mathrm{d}\xi'.$$

The *upper incomplete gamma function* is defined as

$$\Gamma(s,x) = \int_x^\infty t^{s-1}\exp(-t)\,\mathrm{d}t.$$

Show that

$$T(r,t) = \frac{1}{4\pi}\Gamma\left(0,\frac{r^2}{4t}\right).$$

ANS: Integrate $\Theta'(\xi)$ from ξ to ∞, taking care not to use the same symbol ξ as a limit of integration and as an integration variable. By the fundamental theorem of calculus,

$$\Theta(\infty) - \Theta(\xi) = \int_\xi^\infty \Theta'(\xi')\,\mathrm{d}\xi'$$

But from (4a) with $\alpha = 0$, we have $\lim_{\xi\to\infty}\Theta(\xi) = 0$, so

$$\begin{aligned}\Theta(\xi) &= -\int_\xi^\infty \Theta'(\xi')\,\mathrm{d}\xi'\\ &= \int_\xi^\infty \frac{1}{2\pi\xi'}\exp\left(-\frac{\xi'^2}{4}\right)\,\mathrm{d}\xi'\end{aligned}$$

To transform this to an upper incomplete gamma function, put $t = \xi'^2/4$, $\xi' = 2t^{1/2},\ \mathrm{d}\xi' = t^{-1/2}\,\mathrm{d}t$. Substituting gives

$$\begin{aligned}
\Theta(\xi) &= \int_{\xi^2/4}^{\infty} \frac{1}{4\pi t^{1/2}} \exp\left(-t\right) t^{-1/2}\,\mathrm{d}t \\
&= \frac{1}{4\pi} \int_{\xi^2/4}^{\infty} \frac{1}{t} \exp(-t)\,\mathrm{d}t \\
&= \frac{1}{4\pi} \Gamma\left(0, \xi^2/4\right) \\
&= \frac{1}{4\pi} \Gamma\left(0, \frac{r^2}{4t}\right).
\end{aligned}$$

2. The following is a model for temperature waves in the ground, driven by seasonally varying solar radiation intensity. The mean temperature $\bar{T}$ of the ground surface is given by a balanace of incoming radiation and outgoing radiation,

$$\bar{q} = \sigma \bar{T}^4, \tag{6}$$

where $\bar{q} = 1350$ W m^{-2} is the *solar constant* and $\sigma = 5.67 \times 10^{-8}$ W m^{-2} K^{-4} is the Stefan-Boltzmann constant, and $\bar{T}$ is expressed in Kelvins. Variations in incoming radiation about their mean $\bar{q}$ are balance by radiation back in space and conduction of heat into the ground. This leads to the following energy conservation model in the ground ($x > 0$):

$$\rho c \frac{\partial T}{\partial t} - k \frac{\partial^2 T}{\partial x^2} = 0 \qquad \text{for } x > 0, \tag{7a}$$

$$q_0 \cos(\omega t) = -k \frac{\partial T}{\partial x} + 4\sigma \bar{T}^3 T \qquad \text{at } x = 0, \tag{7b}$$

$$-k \frac{\partial T}{\partial x} \to 0 \qquad \text{as } x \to \infty. \tag{7c}$$

Let $k = 2$ W m^{-1} K^{-1}, $\rho = 2000$ kg m^{-3}, $c = 8 \times 10^2$ J kg^{-1} K^{-1}, and $q_0 = 0.2 \times \bar{q}$.

(a) (3 points) Define dimensionless variables through $T = [T]T^*$, $t = [t]t^*$, $x = [x]x^*$. Show that the problem can be rendered in the form

$$\frac{\partial T^*}{\partial t^*} - \frac{\partial^2 T^*}{\partial x^{*2}} = 0 \qquad \text{for } x^* > 0, \tag{8a}$$

$$\cos(t^*) = -\alpha \frac{\partial T^*}{\partial x^*} + T^* \qquad \text{at } x^* = 0, \tag{8b}$$

$$-\frac{\partial T^*}{\partial x^*} \to 0 \qquad \text{as } x^* \to \infty. \tag{8c}$$

Define the scales $[x]$, $[t]$ and $[T]$ in terms of ρ, c, k, $\bar{q}$, q_0, ω and σ. For the values given above, give numerical values for these scales and for the parameter α. Is there a useful approximation you could make to (8)?
ANS: We have

$$\frac{\partial T}{\partial t} = \frac{[T]}{[t]} \frac{\partial T^*}{\partial t^*}$$

$$\frac{\partial T}{\partial x} = \frac{[T]}{[x]} \frac{\partial T^*}{\partial x^*}$$

$$\frac{\partial^2 T}{\partial x^2} = \frac{[T]}{[x]^2} \frac{\partial^2 T^*}{\partial x^{*2}}$$

Substitute this into (7) to get

$$
\begin{aligned}
\frac{\rho c[T]}{[t]}\frac{\partial T^*}{\partial t^*} - \frac{k[T]}{[x]^2}\frac{\partial^2 T^*}{\partial x^{*2}} &= 0 && \text{for } [x]x^* > 0,\\
q_0\cos(\omega[t]t^*) &= -\frac{k[T]}{[x]}\frac{\partial T^*}{\partial x^*} + 4\sigma\bar{T}^3[T]T^* && \text{at } [x]x^* = 0,\\
-\frac{k[T]}{[x]}\frac{\partial T^*}{\partial x^*} &\to 0 && \text{as } [x]x^* \to \infty.
\end{aligned}
$$

Divide to give

$$
\begin{aligned}
\frac{\rho c[x]^2}{k[t]}\frac{\partial T^*}{\partial t^*} - \frac{\partial^2 T^*}{\partial x^{*2}} &= 0 && \text{for } x^* > 0,\\
\cos(\omega[t]t^*) &= -\frac{k[T]}{q_0[x]}\frac{\partial T^*}{\partial x^*} + \frac{4\sigma\bar{T}^3[T]}{q_0}T^* && \text{at } x^* = 0,\\
-\frac{\partial T^*}{\partial x^*} &\to 0 && \text{as } x^* \to \infty.
\end{aligned}
$$

To get the desired form, we need

$$
\frac{\rho c[x]^2}{k[t]} = 1, \qquad \omega[t] = 1 \qquad \frac{4\sigma\bar{T}^3[T]}{q_0} = 1,
$$

which leads to

$$
[t] = \frac{1}{\omega}, \qquad [x] = \frac{k}{\rho c\omega}, \qquad [T] = \frac{q_0}{4\sigma\bar{T}^3}.
$$

To express $[T]$ not in terms of $\bar{T}$ but $\bar{q}$, we need $\bar{T} = (\bar{q}/\sigma)^{1/4}$, so

$$
[T] = \frac{q_0}{4\ \sigma(\bar{q}/\sigma)^{3/4}} = \frac{q_0}{4\sigma^{1/4}\bar{q}^{3/4}}.
$$

The dimensionless group α is

$$
\alpha = \frac{k[T]}{q_0[x]}
$$

Numerical values with the parameter values above are

$$
[t] = 5.02^6\,\mathrm{s}, \qquad [x] = 6.27\,\mathrm{m}, \qquad [T] = 19.6\,\mathrm{K}
$$

and

$$
\alpha = 0.0232.
$$

An obvious approximation *would* be to set $\alpha = 0$.

(b) (4 points) Look for a solution

$$T^*(x^*, t^*) = \mathrm{Re}[A \exp(it^* + \lambda x^*)]$$

Find λ so that (8a) and (8c) are satisifed simultaneously. Show that you can write (8b) in the form

$$\mathrm{Re}\{[A(1-\alpha\lambda) - 1] \exp(it^*)\} = 0$$

for any t^*. Give a solution for A in terms of α.
ANS: Substituting in (8a) and performing standard manipulations gives

$$\mathrm{Re}\left(A(i-\lambda^2) \exp(it^* + \lambda x^*)\right) = 0,$$

and to ensure that this holds regardless of the value of x^* and t^* requires

$$\lambda^2 = i.$$

Hence

$$\lambda = \sqrt{i} = \pm\frac{1+i}{\sqrt{2}}$$

In order to pick which root to use, can look at (8c). If we pick the '+' sign, the solution will have oscillations whose size increases exponentially as $x^* \to \infty$, which cannot satisfy (8c). Hence we must pick the minus sign, and the oscillations will decay exponentially as $x^* \to \infty$. We still have to satisfy (8b). Substitute $\cos(t^*) = \mathrm{Re}[\exp(it^*)]$ and the solution above to get

$$\mathrm{Re}[\exp(it^*)] = \mathrm{Re}[-\alpha\lambda A \exp(it^*)] + \mathrm{Re}[A \exp(it^*)].$$

Combine into one:

$$\mathrm{Re}\{[A(1-\alpha\lambda) - 1] \exp(it^*)\} = 0.$$

To satisfy this for all t^*, set the factor $A(1-\alpha\lambda) - 1$ to zero:

$$A = \frac{1}{1-\alpha\lambda} = \frac{1}{1+\alpha(1+i)/\sqrt{2}}.$$

(c) (2 points) Write A in polar form

$$A = |A| \exp(i\theta),$$

and find $|A|$ and θ in terms of α. You may either find an exact version of the polar form, or try approximating it using $\alpha \ll 1$. Note that, for a small number θ, $\cos(\theta) \approx 1$ and $\sin(\theta) \approx \theta$. For the value of α you have computed above, how big is θ? (Note that the approximation $\theta = 0$ will not get you

any points.)
ANS: We have
$$A = \frac{1}{1 + \alpha/\sqrt{2} + i\alpha/\sqrt{2}}$$
so
$$A^{-1} = |A|^{-1}\exp(-i\theta) = 1 + \alpha/\sqrt{2} + i\alpha/\sqrt{2}.$$
Using Euler's formula
$$|A|^{-1}\cos(\theta) = 1 + \alpha/\sqrt{2}$$
$$-|A|^{-1}\sin(\theta) = \alpha/\sqrt{2}$$
Squaring both equations and adding gives
$$|A|^{-2} = (1 + \alpha/\sqrt{2})^2 + \alpha^2/2,$$
so
$$|A| = 1/\sqrt{(1 + \alpha/\sqrt{2})^2 + \alpha^2/2}.$$
Divinding the two equations, we have
$$\tan(\theta) = -\alpha/(\sqrt{2} + \alpha).$$
or
$$\theta = \tan^{-1}\left(\frac{\alpha}{\alpha + \sqrt{2}}\right).$$
An approximate solution can be found if we assume α is small, so Taylor expanding in α,
$$A \approx 1 - \frac{\alpha(1+i)}{\sqrt{2}}.$$
At leading order, this is $A \approx 1$, so θ is small. Doing slightly better than this,
$$|A|\exp(i\theta) = |A|\cos(\theta) + i|A|\sin(\theta) \approx |A| + i|A|\theta.$$
Comparing real and imaginary parts,
$$|A| \approx 1 - \alpha/\sqrt{2}, \qquad |A|\theta \approx -\alpha/\sqrt{2}$$
so
$$\theta \approx \frac{\alpha}{\alpha + \sqrt{2}} \approx -\frac{\alpha}{\sqrt{2}}.$$

(d) (1 point) What is the dimensional amplitude of temperature variations? By how many days does the surface temperature peak lag the surface insolation peak? In the northern hemisphere, which date does this correspond to?
ANS: The amplitude of oscillations in dimensional terms is $|A|[T]$ With $[T] = 19.64$ and $\alpha = .0232$, we have an amplitude of 19.31 K. Also, the phase lag is $\theta/(2 * pi)$ times one period of oscillation (365 days), which gives 9.4 days. In the northern hemisphere, this is about June 30th.

3. This question is about flow in a pipe. Assume the pipe is aligned with the x_1-axis and has a cross-section that does not change along the length of the pipe. Consider a fluid of constant density ρ with an *unidirectional* velocity field $\mathbf{u} = (u_1, 0, 0)$ (meaning that velocity components that point across the pipe are zero) that depends only on time t and position (x_2, x_3) in the cross-section but not on distance x_1 along the pipe. Assume also that the body force f_i does not depend on position or time. In addition, the velocity is zero at the wall of the pipe, so $u_1 = 0$ there. Start with the Navier-Stokes equations for fluid flow

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_i)}{\partial x_i} = 0, \tag{9a}$$

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = \frac{\partial \sigma_{ij}}{\partial x_j} + f_i, \tag{9b}$$

where

$$\sigma_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right) - p\delta_{ij}.$$

(a) (4 points) For the geometry assumed above, show that the Navier-Stokes equations (9) reduce to

$$\rho\frac{\partial u_1}{\partial t} - \mu\left(\frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_2^2}\right) = -\frac{\partial p}{\partial x_1} + f_1 \tag{10a}$$

$$0 = -\frac{\partial p}{\partial x_2} + f_2 \tag{10b}$$

$$0 = -\frac{\partial p}{\partial x_3} + f_3 \tag{10c}$$

$$\tag{10d}$$

ANS: We have $u_2 = u_3 = 0$ and $\partial u_1/\partial x_1 = 0$. The mass conservation equation (9a) is always satisfied because ρ is constant and $\partial u_1/\partial x_1 = \partial u_2/\partial x_2 = \partial u_3/\partial x_3 = 0$. The only non-zero components of σ_{ij} are

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = -p, \qquad \sigma_{12} = \sigma_{21} = \mu\frac{\partial u_1}{\partial x_2}, \qquad \sigma_{13} = \sigma_{31} = \mu\frac{\partial u_1}{\partial x_3}.$$

Also, all momentum advection terms $u_j \partial u_i/\partial x_j$ vanish: u_i is only non-zero if $i = 1$, but we have $u_1 \partial u_1/\partial x_1 = 0$ because u_1 does not depend on x_1. Substituting into (9b) for $i = 1$ gives

$$\begin{aligned}\rho\frac{\partial u_1}{\partial t} &= \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + f_1 \\ &= -\frac{\partial p}{\partial x_1} + \frac{\partial}{\partial x_2}\left(\mu\frac{\partial u_1}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\mu\frac{\partial u_1}{\partial x_3}\right)\end{aligned}$$

which can be rearranged into

$$\rho\frac{\partial u_1}{\partial t} - \mu\left(\frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2}\right) = -\frac{\partial p}{\partial x_1} + f_1. \tag{11}$$

Substituting into (9b) for $i = 2$ gives

$$\begin{aligned} 0 &= \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + f_2 \\ &= \frac{\partial}{\partial x_1}\left(\mu\frac{\partial u_1}{\partial x_2}\right) - \frac{\partial p}{\partial x_2} + f_2 \\ &= -\frac{\partial p}{\partial x_2} + f_2 \end{aligned}$$

as u_1 and hence $\partial u_1/\partial x_2$ do not depend on x_1. Following exactly the same procedure, we can also find

$$-\frac{\partial p}{\partial x_3} + f_3 = 0.$$

(b) (2 points) Recall that f_1, f_2 and f_3 are assumed to be constant. Show that we *must* have

$$p = f_2x_2 + f_3x_3 - Cx_1 + D$$

for some constants C and D.
ANS: From

$$-\frac{\partial p}{\partial x_3} + f_3 = 0$$

we know that

$$\frac{\partial(p - f_3x_3)}{\partial x_3} = 0,$$

and hence $p - f_3x_3$ cannot depend on x_3. Similarly, from

$$-\frac{\partial p}{\partial x_2} + f_2 = 0,$$

we know that

$$\frac{\partial(p - f_2x_2 - f_3x_3)}{\partial x_3} = 0,$$

and hence $p - f_3x_3 - f_2x_2$ can depend neither on x_3 nor on x_2. In other words,

$$p = f_2x_2 + f_3x + 3 + p_0(x_1),$$

where p_0 is some function of x_1. But we can differentiate (11) with respect to x_1 and use $\partial u_1/\partial x_1 = 0$ to find

$$\frac{\partial^2 u_1}{\partial x_1 \partial t} - \mu\left(\frac{\partial^3 u_1}{\partial x_1\partial x_2^2} + \frac{\partial^3 u_1}{\partial x_1\partial x_3^2}\right) = -\frac{\partial^2 p}{\partial x_1^2} = 0$$

Substitute $p = f_2x_2 + f_3x + 3 + p_0(x_1)$ to find

$$\frac{\mathrm{d}^2 p_0}{\mathrm{d}x_1^2} = 0,$$

from which it follows, integrating twice, that

$$p_0 = -Cx_1 + D,$$

where C and D are constants of integration.

(c) (3 points) Assume that $C = 0$, and that the flow is in steady state, so u_1 is independent of t. Assume also that the pipe has a circular cross-section with radius R centered on $(x_2, x_3) = (0, 0)$. We can then define plane polar coordinates (r, θ) through $x_2 = r\cos(\theta)$, $x_3 = r\sin(\theta)$. If the velocity $u_1 = u_1(r)$ depends only on distance r from the centre of the pipe, then the Laplacian of u_1 can be written as

$$\frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2} = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u_1}{\mathrm{d}r}\right) \tag{12}$$

From this, derive that

$$u_1(r) = \frac{f_1}{4\mu}(R^2 - r^2).$$

Make sure to justify the choice of any constants of integration.
ANS: We have

$$-\mu\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u_1}{\mathrm{d}r}\right) = f_1.$$

Separating variables,

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u_1}{\mathrm{d}r}\right) = -\frac{f_1 r}{\mu}.$$

Integrate

$$r\frac{\mathrm{d}u_1}{\mathrm{d}r} = -\frac{f_1 r^2}{2\mu} + C.$$

Separate variables

$$\frac{\mathrm{d}u_1}{\mathrm{d}r} = -\frac{f_1 r}{2\mu} + \frac{C}{r}.$$

Integrate again

$$u_1 = -\frac{f_1 r^2}{4\mu} + C\log(r) + D.$$

We want a bounded velocity field, so the logarithm term needs to go away with $C = 0$.[1] We need to have zero velocity at the wall of the pipe $r = R$,

[1] It could also be shown that $C \neq 0$ corresponds to a force exerted along a line at the centre of the pipe, which is unphysical.

so

$$0 = -\frac{f_1 R^2}{4\mu} + D,$$

and hence

$$D = \frac{f_1 R^2}{4\mu},$$

and

$$u_1 = \frac{f_1(R^2 - r^2)}{4\mu}.$$

(d) (1 point) Show that the rate at which mass passes through any given cross-section of the pipe is given by

$$2\pi\rho \int_0^R \frac{f_1}{4\mu}(R^2 - r^2) r \, \mathrm{d}r.$$

ANS: From basic continuum physics, the rate at which mass passes through the cross-section is

$$\int_S \rho \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}S.$$

With a cross-section at right-angles to the x_1-axis, we have $\mathbf{n} = (1, 0, 0)$ and $\mathbf{u} \cdot \mathbf{n} = u_1$. Also, the cross section S is the circle $r < R$, with surface element $\mathrm{d}S = r \, \mathrm{d}r \, \mathrm{d}\theta$. Hence

$$\begin{aligned}
\int_S \rho \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}S &= \int_0^{2\pi} \int_0^R \rho u_1 r \, \mathrm{d}r \, \mathrm{d}\theta \\
&= \int_0^R \rho\rho \frac{f_1(R^2 - r^2)}{4\mu} r \, \mathrm{d}r \times \int_0^{s\pi} \mathrm{d}\theta \\
&= 2\pi\rho \int_0^R \rho\rho \frac{f_1(R^2 - r^2)}{4\mu} r \, \mathrm{d}r.
\end{aligned}$$

4. This question is about stresses in a unidirectional flow. Consider a semicircular channel of radius R, aligned with the x_1-axis and open to the atmosphere at the top (see figure 1). The region occupied by the fluid is given by

$$x_2^2 + x_3^2 < R^2 \qquad \text{and} \qquad x_3 < 0.$$

For a fluid of constant viscosity μ and density ρ, the velocity field in this region is given by $\mathbf{u} = (u_1, 0, 0)$, where

$$u_1 = \frac{\rho g \sin(\alpha)}{4\mu}(R^2 - x_2^2 - x_3^2)$$

where g is acceleration due to gravity and α the angle of inclination of the channel to the horizontal. The pressure field is

$$p = -\rho g \cos(\alpha) x_3.$$

(a) (3 points) For an incompressible viscous fluid, stress σ_{ij} is given by

$$\sigma_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - p\delta_{ij}.$$

For the velocity field given above, compute σ_{ij} as a function of x_2, x_3, R, μ, ρ, g and α. Write your answer as a matrix.

ANS: We have $u_2 = u_3 = 0$ and hence the only non-zero stress components are

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = -p, \qquad \sigma_{12} = \sigma_{21} = \mu \frac{\partial u_1}{\partial x_2}, \qquad \sigma_{13} = \sigma_{31} = \mu \frac{\partial u_1}{\partial x_3}.$$

Forming the relevant derivatives, we get

$$\boldsymbol{\sigma} = \begin{pmatrix} \rho g \cos \alpha x_3 & -\rho g \sin(\alpha) x_2/2 & -\rho g \sin(\alpha) x_3/2 \\ -\rho g \sin(\alpha) x_2/2 & \rho g \cos \alpha x_3 & 0 \\ -\rho g \sin(\alpha) x_3/2 & 0 & \rho g \cos \alpha x_3 \end{pmatrix}$$

(b) (1 point) Show that, at the upper surface at $x_3 = 0$, we have $\sigma_{ij} n_j = 0$, where n_i is the normal to that surface.

ANS: As a column vector, we have $\mathbf{n} = (0, 0, 1)^{\mathrm{T}}$ and the product

$$\begin{aligned} \boldsymbol{\sigma}\mathbf{n} &= \begin{pmatrix} \rho g \cos \alpha x_3 & -\rho g \sin(\alpha) x_2/2 & -\rho g \sin(\alpha) x_3/2 \\ -\rho g \sin(\alpha) x_2/2 & \rho g \cos \alpha x_3 & 0 \\ -\rho g \sin(\alpha) x_3/2 & 0 & \rho g \cos \alpha x_3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} \rho g \sin(\alpha) x_3/2 \\ 0 \\ \rho g \cos(\alpha) x_3 \end{pmatrix} \\ &= \mathbf{0} \end{aligned}$$

as $x_3 = 0$ at the surface.

(c) (1 point) Next, consider a small area element ΔS just below the the upper surface of the flow at $(0, x_2, 0)$, but with normal $(\cos(\theta), \sin(\theta), 0)$. *Note that this is not the normal to the upper surface, so ΔS is not parallel to the upper surface.* Compute the force $\Delta F_i = \sigma_{ij} n_j \Delta S$ exerted by the fluid flow on the surface ΔS as a function of x_2, R, μ, ρ, g and α.

ANS: A similar computation to the above gives, with $x_3 = 0$,

$$\begin{aligned}\boldsymbol{\sigma}\mathbf{n}\Delta S &= \begin{pmatrix} 0 & -\rho g \sin(\alpha) x_2/2 & 0 \\ -\rho g \sin(\alpha) x_2/2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \end{pmatrix} \Delta S \\ &= \begin{pmatrix} -\rho g \sin(\alpha) \sin(\theta) x_2/2 \\ -\rho g \sin(\alpha) \cos(\theta) x_2/2 \\ 0 \end{pmatrix} \Delta S.\end{aligned}$$

(d) (1 point) Show that the component $\Delta F_n = \Delta F_i n_i$ of ΔF_i normal to ΔS is given by

$$\Delta F_n = -\rho g \sin(\alpha) \cos(\theta) \sin(\theta) x_2 \Delta S.$$

The normal component is

$$\Delta S \begin{pmatrix} -\rho g \sin(\alpha) \sin(\theta) x_2/2 & -\rho g \sin(\alpha) \cos(\theta) x_2/2 & 0 \end{pmatrix} \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \end{pmatrix} =$$

$$= -\rho g \sin(\alpha) \sin(\theta) \cos(\theta) x_2 \Delta S.$$

(e) (2 points) Keep the size of the area element ΔS and its position x_2 fixed, but allow its orientation (given by the angle θ) to vary. If $x_2 > 0$, what angle maximizes ΔF_n? What angle maximizes ΔF_n if $x_2 < 0$? (*Hint. Recall that* $\cos^2(\theta) - \sin^2(\theta) = \cos(2\theta)$.)

ANS: Two possiblities here. First, we have $\sin(\theta)\cos(\theta) = \sin(2\theta)/2$. This is maximized if $2\theta = \pi/2 + 2n\pi$, $\theta = (n + 1/4)\pi$, and minimized for $2\theta = -\pi/2 + 2n\pi$, $\theta = (n - 1/4)\pi$, where n is an integer. We can limit ourselves to $-\pi < \theta \leq \pi$. If $x_2 > 0$, the maximum value of ΔF_n occurs when $\sin(\theta)\cos(\theta)$ attains its minimum, so $\pi = -\theta/4$ or $\pi = 3/4$. Note that $\theta = 3\pi/4$ corresponds to the same orientation of ΔS as $\theta = -\pi/4$, but n_i pointing in the opposite direction. When $x_2 < 0$, the maximum occurs when $\sin(\theta)\cos(\theta)$ attains a maximum, so $\pi = \theta/4$ or $\theta = -3\pi/4$; again these correspond to the same orientation for ΔS.

The second possiblity, using the hint, would be the following: A necessary (but not sufficient!) condition for a maximum is that $\partial \Delta F_n / \partial \theta = 0$. We have

$$\begin{aligned}\frac{\partial \Delta F_n}{\partial \theta} &= -\rho g \sin(\alpha) \Delta S x_2 [\cos^2(\theta) - \sin^2(\theta)] \\ &= -\rho g \sin(\alpha) \Delta S x_2 \cos(2\theta)\end{aligned}$$

Hence we must have $\cos(2\theta) = 0$, $\theta = (1 + 2n)\pi/4$ where n is an integer. In addition to maxima, these may also correspond to minima (or less likely, inflection points). We can restrict ourselves to $-\pi < \theta \leq \pi$, so we have $\theta = -3\pi/4. -\pi/4, \pi/4, 3\pi/4$. But for these, we have

$$\Delta F_n = -\rho g \sin(\alpha) x_2 \Delta S/2$$

if $\theta = \pi/4$ or $\theta = -3\pi/4$, and

$$\Delta F_n = -\rho g \sin(\alpha) x_2 \Delta S/2$$

for $\theta = -\pi/4$ or $\theta = 3\pi/4$. This leads to the same answer as above.

(f) (2 points) Glacier ice flows as an incompressible viscous fluid, but can crack to form *crevasses* when subjected to high enough stresses. Specifically, crevasses form along surfaces in the ice that experience large enough normal forces $\Delta F_n/\Delta S$. The orientation of crevasses when they first form is therefore such as to maximize ΔF_n, and they form first at those positions on the surface where ΔF_n is biggest. Consider a glacier flowing down a semicircular channel. Where will crevasses first form? Sketch the channel and indicate the flow direction as in figure 1, and indicate the pattern of crevasses you expect to form.
ANS: The maximum value of ΔF_n is always $\rho g \sin(\alpha) \Delta S |x_2|/2$, so is greatest at the edge of the channel where $|x_2| = R$ is largest. The orientation of the crevasses are inclined at 45 degress to the upstream direction as shown in figure 1

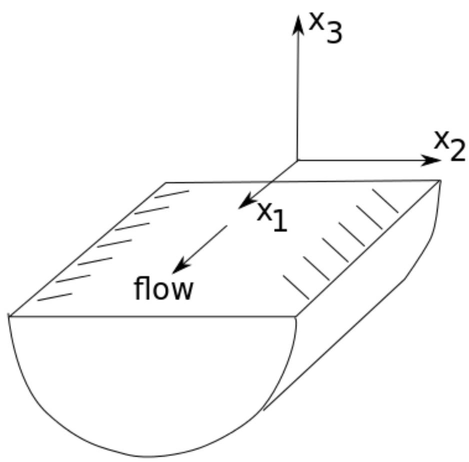

Figure 1: Sketch of the semicircular channel of question 4. The radius of the semicircular cross-section is R.