Final Practice: EOSC 352

30 November, 2009

1. Consider heat conduction with sinusoidally varying surface heat flux. Mathematically, this can be written as

$$\rho c_p \frac{\partial T}{\partial t} - k\frac{\partial^2 T}{\partial x^2} = 0 \qquad \text{for } x > 0 \tag{1a}$$

$$-k\frac{\partial T}{\partial x}\bigg|_{x=0} = q_0 \cos(\omega t) \qquad \text{at } x = 0 \tag{1b}$$

$$T \to 0 \qquad \text{as } x \to \infty \tag{1c}$$

where q_0 is constant.

(a) (4 points) Assume the solution can be written in the form

$$T(x,t) = \mathrm{Re}\left[T_0 \exp(i\omega t + \lambda x)\right] \tag{2}$$

Substitute this into the heat equation to find λ. Explain carefully the choice of signs in λ.

(b) (4 points) At this point, you do not know T_0. In fact, you cannot assume that T_0 is real. Instead, substitute T from (2) into (1b), and re-write this in the form

$$\mathrm{Re}\left\{\left[(a+ib)T_0 - q_0\right]\exp(i\omega t)\right\} = 0$$

where A is real. Use this to deduce T_0.

(c) (2 points) Use the fact that $i = \exp(i\pi/2)$ to rewrite T in (2) in the form

$$T(x,t) = \mathrm{Re}\left[A \exp(i\omega t - \lambda x + i\theta)\right].$$

Use this to write $T(x,t)$ in terms of real quantities only.

2. (a) (1 point) Write the differential equation that represents conservation of mass in subscript notation. What does this simplify into if density ρ is constant?

(b) (4 points) Conservation of momentum requires

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j} = \frac{\partial \sigma_{ij}}{\partial x_j} + f_i. \tag{3}$$

For a Newtonian viscous fluid, ρ is constant and stress takes the form

$$\sigma_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - p\delta_{ij},$$

with viscosity μ also constant. Substitute σ_{ij} into (3), and *show* that this can be simplified into the form

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right) = \mu\frac{\partial^2 u_i}{\partial x_j \partial x_j} - \frac{\partial p}{\partial x_i} + f_i.$$

(c) (1 point) Suppose that you are told that you have a flow in which velocity $\mathbf{u}$ is everywhere parallel to the x_1-axis, and that the form of the velocity field depends only on the coordinates transverse to the x_1 axis and on time. Mathematically, this means that

$$u_1 = u_1(x_2, x_3, t), \qquad u_2 = 0, \qquad u_3 = 0. \tag{4}$$

Show that this satisfies the mass conservation equation you stated in part a

(d) (4 points) Assume that you have the velocity field in (4) and that body force $f_i = 0$. Show that

$$\frac{\partial p}{\partial x_2} = \frac{\partial p}{\partial x_3} = 0$$

From this it follows that $p = p(x_1)$. Next, show that the momentum equation can be reduced to

$$\frac{\partial u_1}{\partial t} - \frac{\mu}{\rho}\left(\frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2}\right) = -\frac{\partial p}{\partial x_1}. \tag{5}$$

Why does it follow that $\partial p/\partial x_1$ is actually constant? What is equation (5) called?

3. (a) (1 point) Take equation (5) above. Assume that $\partial p/\partial x_1 = 0$, and that u_1 depends on x_2 and x_3 as $u_1 = u(r,t)$, where $r = \sqrt{x_2^2 + x_3^2}$. In cylidnrical polar coordinates, where

$$x_1 = z, \qquad x_2 = r\cos\theta, \qquad x_3 = r\sin\theta,$$

the Laplacian can be written as

$$\nabla^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2}.$$

Show that (5) becomes

$$\frac{\partial u}{\partial t} - \frac{\mu}{\rho}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = 0.$$

(b) (3 points) Suppose you have a fluid initially at rest, and that a small amount of fluid is injected at time $t = 0$ at high velocity along the line $r = 0$. A mathematical model for this is

$$\frac{\partial u}{\partial t} - \frac{\mu}{\rho}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = 0 \qquad \text{for } r > 0,\ t > 0 \qquad (6a)$$

$$u(r, 0) = 0 \qquad \text{for all } r > 0 \qquad (6b)$$

$$u(r, t) \to 0 \qquad \text{as } r \to \infty \text{ for all } t > 0 \qquad (6c)$$

$$\int_0^\infty 2\pi u(r, t)\, r\, \mathrm{d}r = P_0 \qquad \text{for all } t > 0 \qquad (6d)$$

where P_0 is a constant related to the amount of momentum contained in the fluid injected at $t = 0$. Consider a similarity solution of the form

$$u(r, t) = t^{-\alpha}\theta\left(\frac{r}{t^\beta}\right).$$

Substitute this into (6), and derive a differential equation for θ in terms of the similarity variable $\xi = x/t^\beta$. What do α and β have to be to make a similarity solution work?

(c) (3 points) The ordinary differential equation for θ in terms of the similarity variable $\xi = x/t^\beta$ can be re-written in the form

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^a \theta\right) + b\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = 0. \qquad (7)$$

What are a and b?

(d) (3 points) Use separation of variables to solve for θ as a function of ξ with μ, ρ and P_0 as parameters. You may assume that $\mathrm{d}\theta/\,\mathrm{d}\xi = 0$ at $\xi = 0$ (and consequently also that θ remains finite at $\xi = 0$).