Final Practice: EOSC 352

30 November, 2009

1. Consider heat conduction with sinusoidally varying surface heat flux. Mathematically, this can be written as

$$\rho c_p \frac{\partial T}{\partial t} - k\frac{\partial^2 T}{\partial x^2} = 0 \qquad \text{for } x > 0 \tag{1a}$$

$$-k\frac{\partial T}{\partial x}\bigg|_{x=0} = q_0 \cos(\omega t) \qquad \text{at } x = 0 \tag{1b}$$

$$T \to 0 \qquad \text{as } x \to \infty \tag{1c}$$

where q_0 is constant.

(a) (4 points) Assume the solution can be written in the form

$$T(x,t) = \mathrm{Re}\left[T_0 \exp(i\omega t + \lambda x)\right] \tag{2}$$

Substitute this into the heat equation to find λ. Explain carefully the choice of signs in λ.
ANS: This should be familiar by now.

$$\frac{\partial T}{\partial t} = \mathrm{Re}\left[i\omega T_0 \exp(i\omega t + \lambda x)\right]$$
$$\frac{\partial^2 T}{\partial x^2} = \mathrm{Re}\left[\lambda^2 T_0 \exp(i\omega t + \lambda x)\right]$$

Substituting into the heat equation

$$\mathrm{Re}\left[T_0 \exp(i\omega t + \lambda x)\right] = 0$$

which is the case if

$$rhoc_p i\omega - k\lambda^2 = 0.$$

Rearranging,

$$\lambda = \pm\sqrt{i\frac{\rho c_p \omega}{k}} = \sqrt{i}\sqrt{\frac{\rho c_p \omega}{k}}.$$

But $\sqrt{i} = \sqrt{\exp(i\pi/2)} = \exp(i\pi/4) = \cos(\pi/4) + i\sin(\pi/4) = (1+i)/\sqrt{2}$. Hence

$$\lambda = \pm(1+i)\sqrt{\frac{\rho c_p \omega}{2k}}.$$

If we choose the plus sign, the answer ends up being

$$T(x,t) \propto \exp\left(\sqrt{\frac{\rho c_p \omega}{2k}}x\right)\cos\left(\omega t + \sqrt{\frac{\rho c_p \omega}{2k}}x\right).$$

This does not satisfy the boundary condition $T \to \infty$ as $x \to 0$. Hence we must pick the negative root,

$$\lambda = -(1+i)\sqrt{\frac{\rho c_p \omega}{2k}}.$$

(b) (4 points) At this point, you do not know T_0. In fact, you cannot assume that T_0 is real. Instead, substitute T from (2) into (1b), and re-write this in the form

$$\mathrm{Re}\left\{[(a+ib)T_0 - q_0]\exp(i\omega t)\right\} = 0$$

where A is real. Use this to deduce T_0.
ANS Substituting, we get

$$-k\frac{\partial T}{\partial x}\bigg|_{x=0} = \mathrm{Re}\left[\lambda T_0 \exp(i\omega t)\right] = q_0\cos(\omega t) = \mathrm{Re}\left[q_0\exp(i\omega t)\right].$$

Rearranging,

$$\mathrm{Re}\left\{\left[\left(\sqrt{\frac{\rho c_p \omega}{2k}} + i\sqrt{\frac{\rho c_p \omega}{2k}}\right)T_0 - q_0\right]\exp(i\omega t)\right\} = 0.$$

which is the case if

$$T_0 = \frac{q_0}{1+i}\sqrt{\frac{2k}{\rho c_p \omega}}.$$

(c) (2 points) Use the fact that $i+1 = \sqrt{2}\exp(i\pi/4)$ to rewrite T in (2) in the form

$$T(x,t) = \mathrm{Re}\left[A\exp(i\omega t - \lambda x + i\theta)\right].$$

Use this to write $T(x,t)$ in terms of real quantities only.
ANS. We have

$$T_0 = \frac{q_0}{\sqrt{2}\exp(i\pi/4)}\sqrt{\frac{2k}{\rho c_p \omega}} = q_0\exp(-i\pi/4)\sqrt{\frac{2k}{\rho c_p \omega}}.$$

Therefore

$$
\begin{aligned}
T(x,t) &= \mathrm{Re}\left\{ q_0 \sqrt{\frac{2k}{\rho c_p \omega}} \exp i\left(\omega t - \sqrt{\frac{\rho c_p \omega}{2k}} x - \pi/4\right) \exp\left(-\sqrt{\frac{\rho c_p \omega}{2k}} x\right)\right\} \\
&= q_0 \sqrt{\frac{2k}{\rho c_p \omega}} \exp\left(-\sqrt{\frac{\rho c_p \omega}{2k}} x\right) \cos\left(\omega t - \sqrt{\frac{\rho c_p \omega}{2k}} x - \pi/4\right).
\end{aligned}
$$

2. (a) (1 point) Write the differential equation that represents conservation of mass in subscript notation. What does this simplify into if density ρ is constant? ANS:

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_i)}{\partial x_i} = 0,$$

which reduces to

$$\frac{\partial u_i}{\partial x_i} = 0$$

when ρ is constant.

(b) (4 points) Conservation of momentum requires

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j} = \frac{\partial \sigma_{ij}}{\partial x_j} + f_i. \tag{3}$$

For a Newtonian viscous fluid, ρ is constant and stress takes the form

$$\sigma_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - p\delta_{ij},$$

with viscosity μ also constant. Substitute σ_{ij} into (3), and *show* that this can be simplified into the form

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} - \frac{\partial p}{\partial x_i} + f_i. \tag{4}$$

ANS: We can pull ρ out of the derivatives because it is constant:

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j} = \rho\left(\frac{\partial u_i}{\partial t} + \frac{\partial u_i u_j}{\partial x_j}\right).$$

Further, we have by the product rule

$$\frac{\partial(u_i u_j)}{\partial x_j} = u_i \frac{\partial u_j}{\partial x_j} + u_j \frac{\partial u_i}{\partial x_j} = u_j \frac{\partial u_i}{\partial x_j}$$

where we have used the answer to part a to get rid of $\partial u_j/\partial x_j$. Also, we have

$$\begin{aligned}\frac{\partial \sigma_{ij}}{\partial x_j} &= \frac{\partial}{\partial x_j}\left(\mu \frac{\partial u_i}{\partial x_j}\right) + \frac{\partial}{\partial x_j}\left(\mu \frac{\partial u_j}{\partial x_i}\right) - \delta_{ij}\frac{\partial p}{\partial x_j} \\ &= \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \mu \frac{\partial^2 u_j}{\partial x_j \partial x_i} - \frac{\partial p}{\partial x_i} \\ &= \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \mu \frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_j}\right) - \frac{\partial p}{\partial x_i} \\ &= \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} - \frac{\partial p}{\partial x_i}\end{aligned}$$

where we have again used $\partial u_j/\partial x_j = 0$. Substituting these three results back into (3) gives the required result.

(c) (1 point) Suppose that you are told that you have a flow in which velocity $\mathbf{u}$ is everywhere parallel to the x_1-axis, and that the form of the velocity field depends only on the coordinates transverse to the x_1 axis and on time. Mathematically, this means that

$$u_1 = u_1(x_2, x_3, t), \qquad u_2 = 0, \qquad u_3 = 0. \tag{5}$$

Show that this satisfies the mass conservation equation you stated in part a.
ANS: We need to have $\partial u_i/\partial x_i = 0$. But this is

$$\frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \frac{\partial u_1}{\partial x_1} = 0$$

because $u_2 = u_3 = 0$ and u_1 does not depend on x_1.

(d) (4 points) Assume that you have the velocity field in (5) and that body force $f_i = 0$. Show that

$$\frac{\partial p}{\partial x_2} = \frac{\partial p}{\partial x_3} = 0$$

From this it follows that $p = p(x_1)$. Next, show that the momentum equation can be reduced to

$$\frac{\partial u_1}{\partial t} - \frac{\mu}{\rho}\left(\frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2}\right) = -\frac{\partial p}{\partial x_1}. \tag{6}$$

Why does it follow that $\partial p/\partial x_1$ is actually constant? What is equation (6) called?
ANS: First, let us look at (4) with $i = 2$ or 3. Then $u_i = 0$, and all the derivatives of u_i are also zero. Hence the only term left is $\partial p/\partial x_i$, which must therefore equal zero. So

$$\frac{\partial p}{\partial x_2} = \frac{\partial p}{\partial x_3} = 0$$

So p depends only on x_1, as does $\partial p/\partial x_1$. Next, look at $i = 1$ in (4). We have

$$\frac{\partial u_1}{\partial t} + u_1 \frac{\partial u_1}{\partial x_1} + u_2 \frac{\partial u_1}{\partial x_2} + u_3 \frac{\partial u_1}{\partial x_3} = \frac{\mu}{\rho}\left(+ \frac{\partial^2 u_1}{\partial x_1^2} + \frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2} \right) - \frac{\partial p}{\partial x_1}.$$

But $u_2 = u_3 = 0$ and $\partial u_1/\partial x_1 = 0$. This gives us, taking the viscous stress term onto the left-hand side

$$\frac{\partial u_1}{\partial t} - \frac{\mu}{\rho}\left(\frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2} \right) = -\frac{\partial p}{\partial x_1}$$

Now the left-hand side depends only on x_2 and x_3 (because u_2 depends only x_2 and x_3) while the right-hand side depends only on x_1. This is only possible if both sides equal a constant. This equation for u_1 takes the form of the heat equation in 2D, as

$$\frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2} = \nabla^2 u_1$$

is the Laplacian of u_1.

3. (a) (1 point) Take equation (6) above. Assume that $\partial p/\partial x_1 = 0$, and that u_1 depends on x_2 and x_3 as $u_1 = u(r, t)$, where $r = \sqrt{x_2^2 + x_3^2}$. In cylidnrical polar coordinates, where

$$x_1 = z, \qquad x_2 = r\cos\theta, \qquad x_3 = r\sin\theta,$$

the Laplacian can be written as

$$\nabla^2 \phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2}.$$

Show that (6) becomes

$$\frac{\partial u}{\partial t} - \frac{\mu}{\rho}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r} \right) = 0.$$

ANS: As discussed above,

$$\frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2} = \nabla^2 u_1$$

is the Laplacian of u_1. But u_1 is now assumed depend on x_2 and x_3 only through r (but not through θ, and u_1 also does not depend on $z = x_3$. Hence $\partial u_1/\partial \theta = \partial u_1/\partial z = 0$, and

$$\frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2} = \nabla^2 u_1 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_1}{\partial r} \right).$$

The required form of (6) follows.

(b) (3 points) Suppose you have a fluid initially at rest, and that a small amount of fluid is injected at time $t = 0$ at high velocity along the line $r = 0$. A mathematical model for this is

$$\frac{\partial u}{\partial t} - \frac{\mu}{\rho}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = 0 \qquad \text{for } r > 0,\ t > 0 \tag{7a}$$

$$u(r,0) = 0 \qquad \text{for all } r > 0 \tag{7b}$$

$$u(r,t) \to 0 \text{ as } r \to \infty \qquad \text{for all } t > 0 \tag{7c}$$

$$\int_0^\infty 2\pi u(r,t) r\,\mathrm{d}r = P_0 \qquad \text{for all } t > 0 \tag{7d}$$

where P_0 is a constant related to the amount of momentum contained in the fluid injected at $t = 0$. Consider a similarity solution of the form

$$u(r,t) = t^{-\alpha}\theta\left(\frac{r}{t^\beta}\right).$$

Substitute this into (7), and derive a differential equation for θ in terms of the similarity variable $\xi = x/t^\beta$. What do α and β have to be to make a similarity solution work?

ANS: Noting that

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} = t^{-\beta}\frac{\partial}{\partial \xi},$$

(the $\partial/\partial\xi$ meaning a partial derivative with respect to ξ while t is held constant), we get

$$\begin{aligned}
\frac{\partial u}{\partial t} &= -\alpha t^{-\alpha-1}\theta\left(\frac{x}{t^\beta}\right) - \beta t^{-\alpha-\beta-1}x\theta'\left(\frac{x}{t^\beta}\right) \\
&= -\alpha t^{-\alpha-1}\theta\left(\xi t\right) - \beta t^{-\alpha-1}\xi\theta'\left(\xi\right) \\
\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) &= \frac{1}{t^\beta\xi}t^{-\beta}\frac{\partial}{\partial\xi}\left(t^\beta\xi t^{-\beta}\frac{\partial(t^{-\alpha}\theta)}{\partial\xi}\right) \\
&= t^{-\alpha-2\beta}\frac{1}{\xi}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)
\end{aligned}$$

Putting this into the momentum equation equation, we get

$$-\alpha t^{-\alpha-1}\theta\left(\xi\right) - \beta t^{-\alpha-1}\xi\theta'\left(\xi\right) - \frac{\mu}{\rho}t^{-\alpha-2\beta}\frac{1}{\xi}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right),$$

or, rearranging,

$$-\alpha\theta\left(\xi\right) - \beta\xi\theta'\left(\xi\right) - \frac{k}{\rho c_p}t^{1-2\beta}\frac{1}{\xi}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = 0. \tag{8}$$

To get rid of t in this equation, we need $1 - 2\beta = 0$, or $\beta = 1/2$. Putting the similarity solution into the rest of (7), we get, putting $x = t^\beta \xi$ into the integral,

$$\theta(\xi) \to 0 \text{ as } \xi \to \infty$$

$$\int_0^\infty 2\pi t^{-\alpha}\theta(\xi) t^\beta \xi t^\beta \, \mathrm{d}\xi = P_0$$

The last equation can be rewritten as

$$t^{2\beta-\alpha} \int_0^\infty \theta(\xi)\xi \, \mathrm{d}\xi = \frac{P_0}{2\pi}. \tag{9}$$

To get rid of t on the left-hand side, we must have

$$\alpha = 2\beta = 1.$$

(c) (3 points) The ordinary differential equation for θ in terms of the similarity variable $\xi = x/t^\beta$ can be re-written in the form

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^a\theta\right) + b\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = 0. \tag{10}$$

What are a and b?
ANS: Putting $\alpha = 1$ and $\beta = 1/2$ into (8) gives

$$-\theta\left(\xi\right) - \frac{1}{2}\xi\theta'\left(\xi\right) - \frac{\mu}{\rho}\frac{1}{\xi}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = 0$$

Guided by (10), we rearrange this into

$$2\xi\theta'(\xi) + \xi^2\theta(\xi) + \frac{2\mu}{\rho}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = 0.$$

The first two terms can be combined (using the product rule in reverse) to give

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\theta)\right) + \frac{2\mu}{\rho}\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = 0$$

which is of the required form ($a = 2$, $b = 2\mu/\rho$).

(d) (3 points) Use separation of variables to solve for θ as a function of ξ with μ, ρ and P_0 as parameters. You may assume that $\mathrm{d}\theta/\,\mathrm{d}\xi = 0$ at $\xi = 0$ (and consequently also that θ remains finite at $\xi = 0$).
ANS: Integrate once to find

$$\xi^2\theta + \frac{2\mu}{\rho}\xi\frac{\mathrm{d}\theta}{\mathrm{d}\xi} = C,$$

or

$$\frac{\mathrm{d}\theta}{\mathrm{d}\xi} = -\frac{\rho}{2\mu}\xi\theta + \frac{\rho}{2\mu}\frac{C}{\xi^2}.$$

Now, at $\xi = 0$, the left-hand side and first term on the right-hand side are finite, so the last term on the right-hand side cannot be infinite. This is only possible if $C = 0$. So

$$\frac{\mathrm{d}\theta}{\mathrm{d}\xi} = -\frac{\rho}{2\mu}\xi\theta.$$

Separating variables,

$$\frac{1}{\theta}\frac{\mathrm{d}\theta}{\mathrm{d}\xi} = -\frac{\rho}{2\mu}\xi.$$

Integrating both sides,

$$\log(\theta) = -\frac{\rho}{4\mu}\xi^2 + \text{ constant.}$$

Rearranging,

$$\theta = C\exp\left(-\frac{\rho}{4\mu}\xi^2\right).$$

To find C, use the last relation in (7), which we have already rewritten in (9) as

$$\int_0^\infty \theta(\xi)\xi\,\mathrm{d}\xi = \frac{P_0}{2\pi}.$$

Substituting,

$$\begin{aligned}\frac{P_0}{2\pi} &= \int_0^\infty C\exp\left(-\frac{\rho}{4\mu}\xi^2\right)\xi\,\mathrm{d}\xi \\ &= \left[-\frac{2\mu}{\rho}C\exp\left(-\frac{\rho}{4\mu}\xi^2\right)\right]_0^\infty \\ &= \frac{2\mu}{\rho}C\end{aligned}$$

So

$$C = \frac{P_0\rho}{4\pi\mu}$$

and

$$\theta(\xi) = \frac{P_0\rho}{4\pi\mu}\exp\left(-\frac{\rho}{4\mu}\xi^2\right).$$