EOS 352 Continuum Dynamics
Conservation of angular momentum

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Overview

These notes cover the following

- Angular momentum for point particles
- Cross products in subscript notation
- Conservation of angular momentum for continua
- Torques due to surface and body forces
- Reduction to a symmetric stress tensor

Angular momentum for point particles

Angular momentum for a collection of point particles is a distinct quantity from linear momentum. It is easily possible to have zero total linear momentum but a finite angular momentum. The definition of angular momentum $\mathbf{L}$ for a single particle of mass m, velocity $\mathbf{u}$ and position vector $\mathbf{r} = (x, y, z) = (x_1, x_2, x_3)$ relative to some fixed coordinate system is

$$\mathbf{L} = \mathbf{r} \times m\mathbf{u} = \mathbf{r} \times \mathbf{p}.$$

where $\mathbf{p}$ is the ordinary linear momentum of the particle. For a collection of particles A, B, C etc, with positions $\mathbf{r}_A$, $\mathbf{r}_B$, ... and linear momenta $\mathbf{p}_A$, $\mathbf{p}_B$, ... total angular momentum is

$$\mathbf{L}_{\text{tot}} = \mathbf{r}_A \times \mathbf{p}_A + \mathbf{r}_B \times \mathbf{p}_B + \ldots = \sum_P \mathbf{r}_P \times \mathbf{p}_P$$

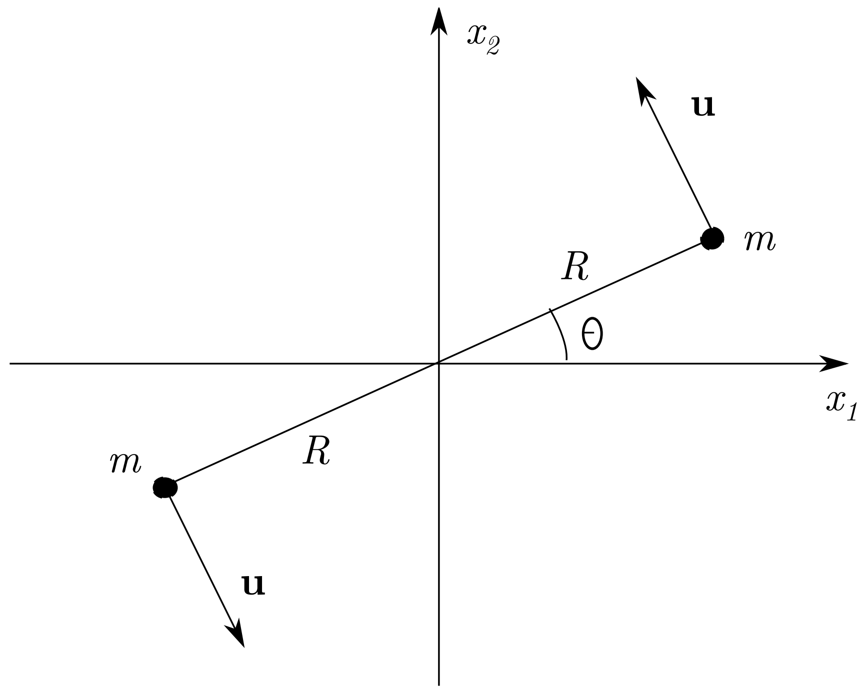

Figure 1: Rotation of two equal masses about a their common of centre of mass. The system has zero total linear momentum but a finite angular momentum.

Often, angular momentum is associated with rotations. Take two particles A and B of equal mass m connected by a string of length $2R$, rotating in the x_1x_2-plane about their common centre of mass at $(0,0,0)$. When the string is at an angle θ to the x_1-axis, the particles will have positions and linear velocities

$$\mathbf{r}_A = R(\cos(\theta), \sin(\theta), 0), \qquad \mathbf{u}_A = R\omega(-\sin(\theta), \cos(\theta), 0),$$

and

$$\mathbf{r}_B = R(-\cos(\theta), -\sin(\theta), 0), \qquad \mathbf{u}_B = R\omega(\sin(\theta), -\cos(\theta), 0).$$

where $\omega = \mathrm{d}\theta/\,\mathrm{d}t$ is rotation rate or *angular velocity*.

Using the definition above, their angular momenta are

$$\mathbf{r}_A \times m\mathbf{u}_A = mR^2\omega(0,0,\cos(\theta)^2 + \sin(\theta)^2) = mR^2\omega(0,0,1),$$

and

$$\mathbf{r}_B \times m\mathbf{u}_B = mR^2\omega(0,0,\cos(\theta)^2 + \sin(\theta)^2) = mR^2\omega(0,0,1).$$

so total angular momentum is

$$\mathbf{L}_{\text{tot}} = 2mR^2\omega(0,0,1).$$

We see that angular momentum increases with angular velocity and the length of the string. Through its definition as a cross product, it is also perpendicular to the plane of rotation: the vector $\mathbf{L}$ is aligned with the axis of rotation (in this case, the z- or x_3-axis).

Lastly, we can compute the total *linear* momentum of the system as

$$m\mathbf{u}_A + m\mathbf{u}_B = \mathbf{0},$$

so the system has zero total linear momentum but non-zero angular momentum. This is because the system overall has no linear motion (i.e., its centre of mass is not moving), but it does have a non-zero rotation, which is what angular momentum describes.

Conservation of angular momentum for point particles

Newton's second law states that, for a particle P

$$\frac{\mathrm{d}\mathbf{p}_P}{\mathrm{d}t} = \mathbf{F}_P \tag{1}$$

where $\mathbf{F}_P$ is the total force on particle P. We can think of this as the sum of forces $\mathbf{F}_{PP'}$ exerted on P by other particles P',

$$\frac{\mathrm{d}\mathbf{p}_P}{\mathrm{d}t} = \sum_{P'} \mathbf{F}_{PP'}. \tag{2}$$

Newton's third law, which really ensures conservation of momentum, requires that the forces between a particle P and a particle P' are equal and opposite, or

$$\mathbf{F}_{PP'} = -\mathbf{F}_{P'P}, \tag{3}$$

which incidentally (with $P' = P$) ensures that $\mathbf{F}_{PP} = \mathbf{0}$, so a particle cannot exert a net force on itself. The rate of change of total momentum of the system is then

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p}_{\text{tot}} &= \frac{\mathrm{d}}{\mathrm{d}t}\sum_P \mathbf{p}_P \\
&= \sum_P \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{p}_P \\
&= \sum_P \sum_{P'} \mathbf{F}_{PP'} \\
&= (\mathbf{F}_{AB} + \mathbf{F}_{AC} + \ldots) + (\mathbf{F}_{BA} + \mathbf{F}_{BC} + \ldots) + (\mathbf{F}_{CA} + \mathbf{F}_{CB} + \ldots) + \ldots \\
&= \mathbf{0}
\end{aligned}$$

as all the terms in the sum can be paired up with terms that are equal and opposite, for instance $\mathbf{F}_{AC} + \mathbf{F}_{CA} = \mathbf{0}$ etc.

A more mathematically elegant way of doing this would be to realize that P and P' are dummy indices in the sum, so we must have

$$\sum_{P} \sum_{P'} \mathbf{F}_{PP'} = \sum_{Q} \sum_{Q'} \mathbf{F}_{QQ'}$$

swapping the index P for Q and P' for Q'. But equally, we should have

$$\sum_{Q} \sum_{Q'} \mathbf{F}_{QQ'} = \sum_{P'} \sum_{P} \mathbf{F}_{P'P}$$

swapping Q for P' and Q' for P. Hence, noting also that the order of summation does not matter,

$$\sum_{P} \sum_{P'} \mathbf{F}_{PP'} = \sum_{P'} \sum_{P} \mathbf{F}_{P'P} = \sum_{P} \sum_{P'} \mathbf{F}_{P'P},$$

that is, we have just swapped the dummy indices P and P'. But also

$$\sum_{P} \sum_{P'} \mathbf{F}_{P'P} = -\sum_{P} \sum_{P'} \mathbf{F}_{PP'}$$

as $\mathbf{F}_{PP'} = -\mathbf{F}_{P'P}$. But the last two equations taken together imply that $\sum_P \sum_{P'} \mathbf{F}_{PP'} = -\sum_P \sum_{P'} \mathbf{F}_{PP'}$, which is possible only if $\sum_P \sum_{P'} \mathbf{F}_{PP'} = \mathbf{0}$.

It follows from the above that the total linear momentum $\mathbf{p}_{\text{tot}}$ cannot change over time and is therefore conserved. Total angular momentum is also a conserved quantity. The reason for this is that forces between these particles are generally parallel to the line joining them.

To understand this, consider the equivalent of Newton's second law for angular momentum. For a single particle, have

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \frac{\mathrm{d}(\mathbf{r} \times m\mathbf{u})}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} \times m\mathbf{u} + \mathbf{r} \times \frac{\mathrm{d}(m\mathbf{u})}{\mathrm{d}t}$$

But $\mathbf{r}$ is the position vector of the particle, so the rate of change of $\mathbf{r}$ is the particle velocity $\mathbf{u}$,

$$\mathbf{u} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}.$$

Hence

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \mathbf{u} \times m\mathbf{u} + \mathbf{r} \times \frac{\mathrm{d}(m\mathbf{u})}{\mathrm{d}t}.$$

But the cross product of $\mathbf{u}$ with itself vanishes while $\mathrm{d}(m\mathbf{u})/\mathrm{d}t = \mathbf{F}$ is the force on the particle, so

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \mathbf{r} \times \mathbf{F}. \tag{4}$$

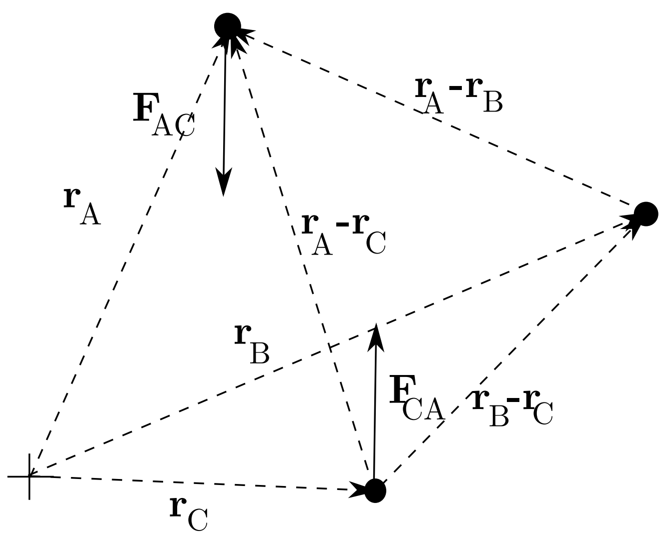

Figure 2: Multiple particles A, B, C, their position vectors and the vectors joining them. If the force $\mathbf{F}_{AC}$ exerted by C on A is *not* aligned with the vector $\mathbf{r}_A - \mathbf{r}_C$ joining them (as suggested by the image), then neither is the equal and opposite force $\mathbf{F}_{CA}$ exerted by C on A, the two forces generate a net torque. As shown in this figure, they would cause an anticlockwise rotation and thereby change the angular momentum of the system. However, angular momentum is conserved if the forces one particle exerts on another are aligned with the vector joining them.

The quantity on the right-hand side of this equation is usually known as the *torque* acting on the particle, denoted by

$$\mathbf{T} = \mathbf{r} \times \mathbf{F}.$$

Now, for a particle P in a collection of particles, we can write

$$\begin{aligned}\frac{\mathrm{d}\mathbf{L}_P}{\mathrm{d}t} &= \mathbf{r}_P \times \mathbf{F}_P \\ &= \mathbf{r}_P \times \sum_{P'} \mathbf{F}_{PP'} \\ &= \sum_{P'} \mathbf{r}_P \times \mathbf{F}_{PP'}\end{aligned}$$

The rate of change of total angular momentum is therefore

$$\begin{aligned}\frac{\mathrm{d}\mathbf{L}_{\mathrm{tot}}}{\mathrm{d}t} &= \sum_P \frac{\mathrm{d}\mathbf{L}_P}{\mathrm{d}t} \\ &= \sum_P \mathbf{r}_P \times \mathbf{F}_P \\ &\sum_P \sum_{P'} \mathbf{r}_P \times \mathbf{F}_{PP'}\end{aligned}$$

where the right-hand side is the sum of torques on the particles.

But, by the same process of swapping the dummy indices P and P' as above, we can show that

$$\sum_P \sum_{P'} \mathbf{r}_P \times \mathbf{F}_{PP'} = \sum_P \sum_{P'} \mathbf{r}_{P'} \times \mathbf{F}_{P'P} = -\sum_P \sum_{P'} \mathbf{r}_{P'} \times \mathbf{F}_{PP'}$$

because the forces $\mathbf{F}_{PP'}$ and $\mathbf{F}_{P'P}$ are equal and opposite. Now, using this, we can write

$$\begin{aligned}\sum_P \sum_{P'} \mathbf{r}_P \times \mathbf{F}_{PP'} &= \frac{1}{2}\left(\sum_P \sum_{P'} \mathbf{r}_P \times \mathbf{F}_{PP'} + \sum_P \sum_{P'} \mathbf{r}_P \times \mathbf{F}_{PP'}\right) \\ &= \frac{1}{2}\left(\sum_P \sum_{P'} \mathbf{r}_P \times \mathbf{F}_{PP'} - \sum_P \sum_{P'} \mathbf{r}_{P'} \times \mathbf{F}_{PP'}\right) \\ &= \frac{1}{2}\sum_P \sum_{P'} (\mathbf{r}_P - \mathbf{r}_{P'}) \times \mathbf{F}_{PP'}.\end{aligned}$$

But $\mathbf{r}_P - \mathbf{r}_{P'}$ is the vector that joins particle P' to particle P. If the force $\mathbf{F}_{PP'}$ is parallel to the direction from P to P', it follows that $\mathbf{r}_P - \mathbf{r}_{P'}$ and $\mathbf{F}_{PP'}$ are parallel and hence their cross product is zero. it follows that

$$\frac{1}{2}\sum_P \sum_{P'} (\mathbf{r}_P - \mathbf{r}_{P'}) \times \mathbf{F}_{PP'} = \mathbf{0}$$

and therefore that

$$\frac{\mathrm{d}\mathbf{L}_{\text{tot}}}{\mathrm{d}t} = \mathbf{0}$$

so that angular momentum is conserved.

All the basic forces in nature (gravitational, electrostatic, ...) are such that forces act along the line connecting two particles, and consequently angular momentum is conserved.

Cross products and angular momentum in subscript notation

Angular momentum is defined as a cross product of two vectors. Taking a cross product is not a straightforward to write in subscript notation as most other vector operations. Specifically, if $\mathbf{r} = (x_1, x_2, x_3)$ and $\mathbf{u} = (u_1, u_2, u_3)$, then $\mathbf{L} = (L_1, L_2, L_3) = \mathbf{r} \times m\mathbf{u}$ is of the form

$$L_1 = m(x_2u_3 - x_3u_2) \qquad L_2 = m(x_3, u_1 - x_1u_3) \qquad L_3 = m(x_1u_2 - x_2u_1), \tag{5}$$

and there does not appear to be a simple formula for the vector component L_i.

The reason why cross products are difficult to write in succinct subscript notation is that they actually are slightly unusual constructs in vector algebra: they require a right-hand rule to define their direction: in a left-handed world, the would point in the opposite direction. Nature of course does not really discriminate between left- and right-handed rotations in this way, and in particular, does not care that the x_1-, x_2- and x_3-axes by convention form a right-handed coordinate system. Cross-products are sometimes called pseudovectors because they change direction in going from left-handed to right-handed coordinate systems.

In subscript notation, it is actually much easier to write them as so-called antisymmetric tensors with two indices. Suppose that we were to define a tensor $\mathcal{L}_{ij}$ through

$$\mathcal{L}_{ij} = m(x_iu_j - x_ju_i). \tag{6}$$

Then the components of $\mathcal{L}_{ij}$ could be written out explicitly in the form

$$\mathcal{L}_{11} = \mathcal{L}_{22} = \mathcal{L}_{33} = 0, \qquad \mathcal{L}_{12} = m(x_1u_2 - x_2u_1) = -\mathcal{L}_{21}, \qquad \mathcal{L}_{13} = m(x_1u_3 - x_3u_1) = -\mathcal{L}_{31},$$

$$\mathcal{L}_{23} = m(x_2u_3 - x_3u_2) = -\mathcal{L}_{32}, \tag{7}$$

or in matrix form

$$\boldsymbol{\mathcal{L}} = \begin{pmatrix} 0 & m(x_1u_2 - x_2u_1) & m(x_1u_3 - x_3u_1) \\ -m(x_1u_2 - x_2u_1) & 0 & m(x_2u_3 - x_3u_2) \\ -m(x_1u_3 - x_3u_1) & -m(x_2u_3 - x_3u_2 & 0 \end{pmatrix}$$

It should be clear that the matrix $\boldsymbol{\mathcal{L}}$ has precisely three independent entries. Further, comparing (7) with (5), it should be clear that, if we know the components (L_1, L_2, L_3) of the vector $\mathbf{L}$, then we immediately know all the components $\mathcal{L}_{ij}$ of the matrix $\boldsymbol{\mathcal{L}}$, as we have

$$L_1 = \mathcal{L}_{23} = -\mathcal{L}_{32}, \qquad L_2 = -\mathcal{L}_{13} = \mathcal{L}_{31}, \qquad L_3 = \mathcal{L}_{12} = -\mathcal{L}_{21}, \tag{8}$$

and $\mathcal{L}_{11} = \mathcal{L}_{22} = \mathcal{L}_{33} = 0$. In other words, there is no information contained in $\mathbf{L}$ that is not also contained in $\boldsymbol{\mathcal{L}}$ or vice versa: if we know one of these objects, we can reconstruct the other.

So instead of using the cross product $\mathbf{L} = \mathbf{r} \times m\mathbf{u}$ as the definition of angular momentum, we can equally use the tensor $\mathcal{L}_{ij}$. Similarly, we can define torque as an antisymmetric tensor

$$\mathcal{T}_{ij} = x_i F_j - x_j F_i, \tag{9}$$

and just as we did above in (8), we can show that the cross product vector $\mathbf{T} = (T_1, T_2, T_3)$ relates to the matrix $\boldsymbol{\mathcal{T}}$ through

$$T_1 = \mathcal{T}_{23} = -\mathcal{T}_{32}, \qquad T_2 = -\mathcal{T}_{13} = \mathcal{T}_{31}, \qquad T_3 = \mathcal{T}_{12} = -\mathcal{T}_{21}, \tag{10}$$

But (4) in component form reads $\mathrm{d}L_i/\,\mathrm{d}t = T_i$. With (8) and (10), this becomes

$$\frac{\mathrm{d}\mathcal{L}_{ij}}{\mathrm{d}t} = \mathcal{T}_{ij}. \tag{11}$$

or

$$\frac{\mathrm{d}(m(x_i u_j - x_j u_i))}{\mathrm{d}t} = x_i F_j - x_j F_i.$$

Our next task is to re-write this for a continuum.

Conservation of angular momentum for a continuum

In the tensor notation above, the angular momentum contained in a small volume δV around position (x_1, x_2, x_3) and moving with velocity (u_1, u_2, u_3) is

$$\delta\mathcal{L}_{ij} = \delta m(x_i u_j - x_j u_i) = \rho(x_i u_j - x_j u_i)\delta V$$

and the total angular momentum in a Lagrangian volume $V(t)$ can therefore be found by summing

$$\mathcal{L}_{ij} = \int_{V(t)} \rho(x_i u_j - x_j u_i)\,\mathrm{d}V$$

The fundamental assumption made in continuum mechanics in general is that torques on the volume arise entirely because of surface and body forces. Recall that the surface force on a surface element δS can be written as

$$\delta F_i = \sigma_{ik} n_k \delta S,$$

where we have chosen k as the dummy index to avoid confusion with the fixed indices i and j that appear above. The associated torque is therefore

$$\delta\mathcal{T}_{ij} = x_i \delta F_j - x_j \delta F_i = (x_i \sigma_{jk} n_k - x_j \sigma_{ik} n_k)\delta S.$$

Similarly, the net force on a volume element δV due to a body force f_i is

$$\delta F_i = f_i \delta V$$

with associated torque

$$\delta\mathcal{T}_{ij} = x_i \delta F_j - x_j \delta F_i = (x_i f_j - x_j f_i)\delta V.$$

The total torque can therefore be found by summing

$$\mathcal{T}_{ij} = \int_{S(t)} (x_i \sigma_{jk} - x_j \sigma_{ik}) n_k \, \mathrm{d}S + \int_{V(t)} (x_i f_j - x_j f_i) \, \mathrm{d}V$$

Substituting these into (11) gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{V(t)} \rho (x_i u_j - x_j u_i) \, \mathrm{d}V = \int_{S(t)} (x_i \sigma_{jk} - x_j \sigma_{ik}) n_k \, \mathrm{d}S + \int_{V(t)} (x_i f_j - x_j f_i) \, \mathrm{d}V$$

We can now follow the same procedure as before to turn this into a differential equation, using Reynolds' transport theorem and the divergence theorem. The only difference is that the conserved quantity $\mathcal{L}_{ij}$ and hence the associated density $\rho(x_i u_j - x_j u_i)$ now have two indices i and j, rather than one index i for momentum p_i and momentum density ρu_i. All this requires is somewhat more careful handling of the repeated indices that appear in the dot products in Reynolds' transport theorem and the divergence theorem — these must now not be i or j, so we will use k below.

Applying Reynolds' transport theorem to the time derivative,

$$\int_{V(t)} \frac{\partial}{\partial t} \left[\rho (x_i u_j - x_j u_i)\right] \, \mathrm{d}V + \int_{S(t)} \left[\rho (x_i u_j - x_j u_i)\right] u_k n_k \, \mathrm{d}S =$$

$$\int_{S(t)} (x_i \sigma_{jk} - x_j \sigma_{ik}) n_k \, \mathrm{d}S + \int_{V(t)} (x_i f_j - x_j f_i) \, \mathrm{d}V$$

Next, using the divergence theorem,

$$\int_{V(t)} \left[\frac{\partial}{\partial t} \left[\rho (x_i u_j - x_j u_i)\right] + \frac{\partial}{\partial x_k} \left[\rho (x_i u_j - x_j u_i) u_k\right] - \frac{\partial}{\partial x_k} \left[x_i \sigma_{jk} - x_j \sigma_{ik}\right] - (x_i f_j - x_j f_i) \right] \mathrm{d}V = 0$$

And the usual argument about $V(t)$ being arbitrary allows us to conclude that the integrand must be zero,

$$\frac{\partial}{\partial t} \left[\rho (x_i u_j - x_j u_i)\right] + \frac{\partial}{\partial x_k} \left[\rho (x_i u_j - x_j u_i) u_k\right] - \frac{\partial}{\partial x_k} \left[x_i \sigma_{jk} - x_j \sigma_{ik}\right] - (x_i f_j - x_j f_i) = 0. \tag{12}$$

Reduction to a symmetric stress tensor

Equation (12) looks exceedingly complicated, but we can in fact reduce it to something very simply by using the momentum conservation equation

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_k)}{\partial x_k} = \frac{\partial \sigma_{ik}}{\partial x_k} + f_i \tag{13}$$

All we need to do is apply the product rule to the derivatives in (12) and recognize a few facts about differentiating the terms x_i and x_j that appear in some of the derivatives.

We will apply the product rule to each term in (12) in turn. Take the first term,

$$\begin{aligned}\frac{\partial}{\partial t}\left[\rho(x_i u_j - x_j u_i)\right] &= \frac{\partial(\rho u_j x_i)}{\partial t} - \frac{\partial(\rho u_i x_j)}{\partial t} \\ &= \frac{\partial(\rho u_j)}{\partial t} x_i + \rho u_j \frac{\partial x_i}{\partial t} - \frac{\partial(\rho u_i)}{\partial t} x_j - \rho u_i \frac{\partial x_j}{\partial t}\end{aligned}$$

But now you have to remember what the partial derivative $\partial/\partial t$ stands for: differentiating with respect to t while holding x_1, x_2 and x_3 constant. It follows that

$$\frac{\partial x_i}{\partial t} = \frac{\partial x_j}{\partial t} = 0$$

as this amounts to differentiating constants. Hence

$$\frac{\partial}{\partial t}\left[\rho(x_i u_j - x_j u_i)\right] = \frac{\partial(\rho u_j)}{\partial t} x_i - \frac{\partial(\rho u_i)}{\partial t} x_j \tag{14}$$

Next, take the second term in (12),

$$\begin{aligned}\frac{\partial}{\partial x_k}\left[\rho(x_i u_j u_k - x_j u_i u_k)\right] &= \frac{\partial(\rho u_j u_k x_i)}{\partial x_k} - \frac{\partial(\rho u_i u_k x_j)}{\partial x_k} \\ &= \frac{\partial(\rho u_j u_k)}{\partial x_k} x_i + \rho u_j u_k \frac{\partial x_i}{\partial x_k} - \frac{\partial(\rho u_i u_k)}{\partial x_k} x_j - \rho u_i u_k \frac{\partial x_j}{\partial x_k}\end{aligned} \tag{15}$$

Again we have to take a closer look at the partial derivatives $\partial x_i/\partial x_k$ and $\partial x_j/\partial x_k$. The partial derivative with respect to x_k is a derivative taken with all the *other* x_i's (i.e., with $i \neq k$) held constant, as well as with t held constant. The derivative of x_i with respect to x_k is therefore zero if $i \neq k$, and one if $i = k$. To see this, we can write the derivatives out explicitly for all the possible combinations of i and k,

$$\frac{\partial x_1}{\partial x_1} = 1, \qquad \frac{\partial x_2}{\partial x_1} = 0, \qquad \frac{\partial x_3}{\partial x_1} = 0,$$

$$\frac{\partial x_1}{\partial x_2} = 0, \qquad \frac{\partial x_2}{\partial x_2} = 1, \qquad \frac{\partial x_3}{\partial x_2} = 0,$$

$$\frac{\partial x_1}{\partial x_3} = 0, \qquad \frac{\partial x_2}{\partial x_3} = 0, \qquad \frac{\partial x_3}{\partial x_3} = 1.$$

In matrix form, this becomes the identity matrix

$$\begin{pmatrix} \frac{\partial x_1}{\partial x_1} & \frac{\partial x_2}{\partial x_1} & \frac{\partial x_3}{\partial x_1} \\ \frac{\partial x_1}{\partial x_2} & \frac{\partial x_2}{\partial x_2} & \frac{\partial x_3}{\partial x_2} \\ \frac{\partial x_1}{\partial x_3} & \frac{\partial x_2}{\partial x_3} & \frac{\partial x_1}{\partial x_3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

To facilitate writing this in subscript notation, we define a special tensor known as the *Kronecker delta* by

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise,} \end{cases} \tag{16}$$

so that

$$\frac{\partial x_i}{\partial x_k} = \delta_{ik}, \qquad \frac{\partial x_j}{\partial x_k} = \delta_{jk} \tag{17}$$

Now consider a sum over a repeated index involving the Kronecker delta, say

$$\delta_{ij} a_j = \sum_{j=1}^{3} \delta_{ij} a_j = \delta_{i1} a_1 + \delta_{i2} a_2 + \delta_{i3} a_3.$$

Only one of the three terms on the right can be non-zero, because the Kronecker delta is zero except if both its indices are the same. Take $i = 1$. Then $\delta_{i1} a_1 = a_1$ but $\delta_{i2} a_2 = \delta_{i3} a_3 = 0$, and $\delta_{ij} a_j = a_1$. Similarly, if $i = 2$, we get $\delta_{ij} a_j = a_2$, and if $i = 3$, we get $\delta_{ij} a_j = a_3$, This can be summarized by saying

$$\delta_{ij} a_j = a_i.$$

We can do the same thing for tensors with more indices. For instance, $\delta_{ij}\sigma_{jk} = \sigma_{ik}$ by the same argument as above: $\delta_{ij}\sigma_{jk} = \sum_{j=1}^{3} \delta_{ij}\sigma_{jk}$, and the only term in the sum that is not zero is that for which $j = i$.

Exercise 1 *Recall that a repeated index indicates summation. In standard matrix notation, how would you write $a_i = \delta_{ij} a_j$? How would you write $\sigma_{ik} = \delta_{ij}\sigma_{ij}$?*

With these properties of the Kronecker delta in hand, (15) becomes

$$\begin{aligned} \frac{\partial}{\partial x_k}\left[\rho(x_i u_j u_k - x_j u_i u_k)\right] &= \frac{\partial(\rho u_j u_k)}{\partial x_k} x_i + \rho u_j u_k \delta_{ik} - \frac{\partial(\rho u_i u_k)}{\partial x_k} x_j - \rho u_i u_k \delta_{jk} \\ &= \frac{\partial(\rho u_j u_k)}{\partial x_k} x_i + \rho u_j u_i - \frac{\partial(\rho u_i u_k)}{\partial x_k} x_j - \rho u_i u_j \\ &= \frac{\partial(\rho u_j u_k)}{\partial x_k} x_i - \frac{\partial(\rho u_i u_k)}{\partial x_k} x_j \end{aligned} \tag{18}$$

as $\rho u_j u_i - \rho u_i u_j = 0$.

Similarly, we can deal with the third term in (12):

$$\begin{aligned}
\frac{\partial}{\partial x_k}\left[x_i\sigma_{jk} - x_j\sigma_{ik}\right] &= \frac{\partial \sigma_{jk}}{\partial x_k}x_i + \sigma_{jk}\frac{\partial x_i}{\partial x_k} - \frac{\partial \sigma_{ik}}{\partial x_k}x_j - \sigma_{ik}\frac{\partial x_j}{\partial x_k} \\
&= \frac{\partial \sigma_{jk}}{\partial x_k}x_i + \sigma_{jk}\delta_{ik} - \frac{\partial \sigma_{ik}}{\partial x_k}x_j = \sigma_{ik}\delta_{jk} \\
&= \frac{\partial \sigma_{jk}}{\partial x_k}x_i + \sigma_{ji} - \frac{\partial \sigma_{ik}}{\partial x_k}x_j - \sigma_{ij} \qquad (19)
\end{aligned}$$

Substituting (14), (18) and (19) in (12), we get

$$\frac{\partial(\rho u_j)}{\partial t}x_i - \frac{\partial(\rho u_i)}{\partial t}x_j + \frac{\partial(\rho u_j u_k)}{\partial x_k}x_i - \frac{\partial(\rho u_i u_k)}{\partial x_k}x_j - \frac{\partial \sigma_{jk}}{\partial x_k}x_i - \sigma_{ji} + \frac{\partial \sigma_{ik}}{\partial x_k}x_j + \sigma_{ij} - (x_i f_j - x_j f_i) = 0 \qquad (20)$$

Now this does not look any simpler than the original, but recall that we wanted to use the momentum conservation equation (13). Hence we group terms we can recognize from (13) together. This allows us to rewrite (20) as

$$\left[\frac{\partial(\rho u_j)}{\partial t} + \frac{\partial(\rho u_j u_k)}{\partial x_k} - \frac{\partial \sigma_{jk}}{\partial x_k} - f_j\right]x_i - \left[\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_k)}{\partial x_k} - \frac{\partial \sigma_{ik}}{\partial x_k} - f_i\right]x_j - \sigma_{ji} + \sigma_{ij} = 0 \qquad (21)$$

But if we replace all the i's by j's in (13), we can recognize that the first expression in square brackets is zero. Similarly, the second expression in square brackets is zero, and we are left rather prosaically with

$$-\sigma_{ji} + \sigma_{ij} = 0$$

as an alternative form of (12). In other words, conservation of angular momentum boils down to the stress tensor σ_{ij} being *symmetric*,

$$\sigma_{ij} = \sigma_{ji}.$$