EOS 352 Continuum Dynamics
Complex variables

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Overview

These notes provide the basic knowledge of complex variables you need to solve some partial differential equations with them:

- Basic definitions and concepts
- Algebra with complex numbers
- Taylor series
- Complex exponentials

Basic definitions and concepts

All ordinary numbers that you will be familiar with from high school (natural numbers, integers, rational and irrational numbers, together making up the 'real' numbers) have positive squares: if x is a real number, then $x^2 \geq 0$. To formally solve equations like $x^2 = -1$, imaginary numbers were invented, and these turn out to have a lot of uses. The 'imaginary unit' i is defined as

$$i = \sqrt{-1}.$$

An imaginary number is i times a real number, for instance $4i$ or $-\pi i$ are imaginary numbers.

A complex number is the sum of a real and a complex number, for instance

$$z = 4 + 7i.$$

More generally, a complex number takes the form

$$z = a + ib$$

where a and b are real. Obviously any real number is also a complex number (put $b = 0$) and any imaginary number is likewise a complex number (put $a = 0$). z is a commonly used symbol for a complex number, even though this can cause confusion with the vertical coordinate z in many Cartesian coordinate systems (so it is important to be explicit). When written in this form, a is called the 'real' part of z and b is called the imaginary part, often denoted by

$$a = \mathrm{Re}(z), \qquad b = \mathrm{Im}(z).$$

Two complex numbers can only be equal to one another if their real and imaginary parts are the same. Let $z_1 = a + ib$ and $z_2 = c + id$. In order to have $z_1 = z_2$ we must have $a = c$, $b = d$. To see this, put

$$a + ib = c + id$$

Rearranging,

$$a - c = i(d - b).$$

But a real number $a - c$ can only equal an imaginary number $i(d - b)$ if they are both equal to zero: taking the square on both sides,

$$(a - c)^2 = -(d - b)^2.$$

The left hand side is positive if $a \neq c$ while the right-hand side is negative if $b \neq d$, and so the only way both sides can be equal is if $a = c$ and $b = d$.

The *complex conjugate* $\bar{z}$ of a complex number $z = a + ib$ is defined as

$$\bar{z} = a - ib,$$

i.e. the same as z but with the sign of the imaginary part reversed. The modulus of a complex number $z = a + ib$ is defined as

$$|z| = \sqrt{a^2 + b^2}$$

Note that $|\bar{z}| = |z|$. For a real number $z = a$, we have $|z| = \sqrt{a^2} = a$ if $a > 0$ or $-a$ if $a < 0$, so $|z| = |a|$, where $|a|$ is the usual absolute value of a real number. This ensures that there is no ambiguity in using the notation $|\cdot|$, as it generalizes taking the absolute value to complex numbers.

More generally, the modulus of z can be seen as the length of a vector with x-component a and y-component b. In fact, complex numbers are often described geometrically in this way: namely, as points lying in a plane (the complex plane) with coordinate of the point relative to one axis (the 'imaginary axis') giving the imaginary part of the complex number, while the coordinate along the other ('real') axis giving the real part of the complex number. Note that

$$|z| = 0$$

implies that $z = 0$.

Exercise 1 *Find $\bar{z}$ and $|\bar{z}|$ for*

1. $z = 3 + 4i$
2. $z = -1 + i$
3. $z = \pi i$

Algebra

The usual rules of algebra still hold in complex algebra, like commutativity (you can switch the order of two terms in a sum or product), associativity (when multiplying or adding more than two numbers together, the order in which this is done is immaterial) and distributivity (expanding brackets when multiplying a sum by another number). In particular, if z_1, z_2 and z_3 are complex mumbers, then

$$\begin{aligned} z_1 + z_2 &= z_2 + z_1 \\ (z_1 + z_2) + z_3 &= z_1 + (z_2 + z_3) \\ z_1 \times z_2 &= z_2 \times z_1 \\ (z_1 \times z_2) \times z_3 &= z_1 \times (z_2 \times z_3) \\ z_1 \times (z_2 + z_3) &= z_1 \times z_2 + z_1 \times z_3. \end{aligned}$$

Addition and subtraction

To add two complex numbers, simply add their real and imaginary parts. So, if $z_1 = a + ib$ and $z_2 = c + id$, then

$$z_1 + z_2 = (a + c) + i(b + d).$$

Therefore

$$(4 + i) + (2 - 3i) = 6 - 2i.$$

To subtract two complex numbers, subtract their real and imaginary parts instead:

$$z_1 - z_2 = (a - c) + i(b - d).$$

Therefore

$$(4 + i) - (2 - 3i) = 2 + 4i.$$

As a result of the above, you can see that the real part of $z_1 + z_2$ $(= a + b)$ is the real part of z_1 $(= a)$ plus the real part of z_2 $(= b)$:

$$\mathrm{Re}(z_1 + z_2) = \mathrm{Re}(z_1) + \mathrm{Re}(z_2).$$

It is however *not* true in general that $|z_1 + z_2| = |z_1| + |z_2|$.

Note 1 *Note that the real and imaginary parts of a complex number z can be expressed in terms of z and $\bar{z}$: If $z = a + ib$ then $\bar{z} = a - ib$ and so*

$$\begin{aligned}\mathrm{Re}(z) = a &= \frac{z + \bar{z}}{2}, \\ \mathrm{Im}(z) = b &= \frac{z - \bar{z}}{2}.\end{aligned}$$

Exercise 2 *Calculate $|z_1|$, $|z_2|$ and $|z_1 + z_2|$ for $z_1 = 4 - 3i$, $z_2 = 1 + i$.*

Multiplication

To multiply imaginary numbers with each other, you only have to remember that $i^2 = -1$ and hence $i^3 = -i$, $i^4 = 1$, $i^5 = i$ and so forth. To multiply two complex numbers, simply expand the product and use the fact that $i^2 = -1$. If $z_1 = a + ib$ and $z_2 = c + id$, then

$$\begin{aligned}z_1 z_2 &= (a + ib)(c + id) \\ &= ac + iad + ibc + i^2 bd \\ &= ac - bd + i(ad + bc)\end{aligned}$$

I don't recommend memorizing this as a formula, instead, just expand whenever you need to do a product:

$$\begin{aligned}(4 + i) \times (2 - 3i) &= 4 \times 2 - i \times 4 \times 3 + i \times 2 - i^2 \times 3 && (1) \\ &= 8 - 12i + 2i + 3 && (2) \\ &= 11 - 10i. && (3)\end{aligned}$$

There is a use to the general formula above though: first, we can show that the modulus of $z_1 z_2$ is the same as the modulus of z_1 times the modulus of z_2. We have

$$\begin{aligned}|z_1||z_2| &= \sqrt{a^2 + b^2}\sqrt{c^2 + d^2} \\ &= \sqrt{(a^2 + b^2)(c^2 + d^2)} \\ &= \sqrt{a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2}\end{aligned}$$

while

$$
\begin{aligned}
|z_1 z_2| &= \sqrt{(ac-bd)^2+(ad+bc)^2} \\
&= \sqrt{(a^2c^2-2acbd+b^2d^2)+(a^2d^2+2adbc+b^2c^2)} \\
&= \sqrt{a^2c^2+b^2d^2+a^2d^2+b^2c^2} \\
&= |z_1||z_2|
\end{aligned}
$$

It is generally *not true* however that $\mathrm{Re}(z_1 z_2)$ is the same as $\mathrm{Re}(z_1)$ times $\mathrm{Re}(z_2)$. From the general formula, $\mathrm{Re}(z_1 z_2) = ac - bd$, while $\mathrm{Re}(z_1)\mathrm{Re}(z_2) = ac$.

We can also show that the square of the modulus of a complex number is given multiplying the complex number by its conjugate:

$$
\begin{aligned}
z_1 \bar{z}_1 &= (a+ib)(a-ib) \\
&= a^2 - iab + iba - i^2 b^2 \\
&= a^2 + b^2 \\
&= |z_1|^2
\end{aligned}
$$

Division

Division is the only slightly difficult thing about complex algebra. What we are after is a way of writing

$$\frac{z_1}{z_2} = \frac{a+ib}{c+id}$$

in the form $e + if$, with e and f real. The trick is to multply by top and bottom of the fraction by the complex conjugate of z_2:

$$
\begin{aligned}
\frac{z_1}{z_2} &= \frac{z_1 \bar{z}_2}{z_2 \bar{z}_2} \qquad (4) \\
&= \frac{z_1 \bar{z}_2}{|z_2|^2}, \qquad (5)
\end{aligned}
$$

where the denominator $|z_2|^2 = c^2 + d^2$ is now a real number and all we need to know is how to multiply two complex numbers, z_1 and $\bar{z}_2$.

More explicitly,

$$
\begin{aligned}
\frac{a+ib}{c+id} &= \frac{(a+ib)(c-id)}{(c+id)(c-id)} \\
&= \frac{ac - iad + ibc - i^2 bd}{c^2+d^2} \\
&= \frac{ac + bd + i(bc - ad)}{c^2+d^2}
\end{aligned}
$$

Again, I don't recommend memorizing this formula, instead learn the method.

As an example, take

$$\begin{aligned}
\frac{1+i}{5-i} &= \frac{(1+i)(5+i)}{(5-i)(5+i)}\\
&= \frac{5+i+5i+i^2}{25+1}\\
&= \frac{4+6i}{26}
\end{aligned}$$

Note also that

$$\begin{aligned}
\left|\frac{1}{z}\right| &= \left|\frac{\bar{z}}{|z|^2}\right| = \frac{1}{|z|^2} \times |\bar{z}| = \frac{1}{|z|^2} \times |z|\\
&= \frac{1}{|z|}
\end{aligned}$$

and therefore that

$$\begin{aligned}
\left|\frac{z_1}{z_2}\right| &= \left|\frac{1}{z_2} \times z_1\right| = \left|\frac{1}{z_2}\right| |z_1|\\
&=\frac{|z_1|}{|z_2|}
\end{aligned}$$

Exercise 3 *Express the following in the form $a + ib$ where a and b are real:*

1. $z_1 + z_2$ *if* $z_1 = -3 + 2i$, $z_2 = 2 - 3i$
2. $z_1 - z_2$ *if* $z_1 = -3 + 2i$, $z_2 = 2 - 3i$
3. $(4 + 2i)(-3 + i)$
4. $(1 - 2i)/(4 + 3i)$
5. $(2 + i)(8 - 4i)$

Distributivity revisited

Distributivity is key when you are doing algebra: basically, how / when can you exchange the order of operations? Which complex algebra, there are some subtleties that can lead you into mistakes. For standard algebraic operations, multiplication and division are distributive over addition and subtraction, so

$$z_1 \times (z_2 + z_3) = z_1 \times z_2 + z_1 \times z_3, \qquad \frac{z_2 + z_3}{z_1} = \frac{z_2}{z_1} + \frac{z_3}{z_1}$$

but not vice versa:

$$z_1 + z_2 \times z_3 \not\equiv (z_1 + z_2) \times (z_1 + z_3), \qquad z_1 + z_2/z_3 \not\equiv (z_1 + z_2)/(z_1 + z_3),$$

and — getting this being a common beginner mistake —

$$\frac{z_1}{z_2 + z_3} \not\equiv \frac{z_1}{z_2} + \frac{z_1}{z_3}.$$

(note that $\not\equiv$ means that the two sides are not necessarily equal, in "most cases" won't be, but you can construct special cases in which equality does hold, usually involving one of the complex numbers involved being zero. '$\equiv$' is a symbol for equality used to indicate an *identity*, a statement that always holds, while '$=$' may be an equation.)

Complex algebra introduces additional operations: taking moduli, complex conjugation and taking real and imaginary parts. Here it is easy to go wrong, so tread carefully. Below is a collection of a few relevant observations, most of which repeat what we have established already

$$
\begin{aligned}
|z_1 z_2| &\equiv |z_1||z_2| && \text{modulus is distributive over multiplication}\\
|z_1/z_2| &\equiv |z_1|/|z_2| && \text{ditto for division}\\
|z_1 + z_2| &\not\equiv |z_1| + |z_2| && \text{but not addition}\\
|z_1 - z_2| &\not\equiv |z_1| - |z_2| && \text{or subtraction}\\
\overline{z_1 + z_2} &\equiv \bar{z}_1 + \bar{z}_2 && \text{complex conjugation is distributive over addition}\\
\overline{z_1 - z_2} &\equiv \bar{z}_1 - \bar{z}_2 && \text{and subtraction}\\
\overline{z_1 \times z_2} &\equiv \bar{z}_1 \times \bar{z}_2 && \text{and multiplication}\\
\overline{z_1/z_2} &\equiv \bar{z}_1/\bar{z}_2 && \text{and divison}\\
\mathrm{Re}(z_1 + z_2) &\equiv \mathrm{Re}(z_1) + \mathrm{Re}(z_2) && \text{Re is distributive over addition}\\
\mathrm{Re}(z_1 - z_2) &\equiv \mathrm{Re}(z_1) - \mathrm{Re}(z_2) && \text{ditto subtraction}\\
\mathrm{Re}(z_1 \times z_2) &\not\equiv \mathrm{Re}(z_1) \times \mathrm{Re}(z_2) && \text{but not multiplication}\\
\mathrm{Re}(z_1/z_2) &\not\equiv \mathrm{Re}(z_1)/\mathrm{Re}(z_2) && \text{or division}
\end{aligned}
$$

'Im' satisfies the same distributivity rules as 'Re'. One important exception to distributivity over multiplication is that Re and Im are distributive over multiplication if one factor is real, so

$$\mathrm{Re}(az_1) = \mathrm{Re}(a)\mathrm{Re}(z_1) = a\mathrm{Re}(z_1),$$

if a is real, while

$$\mathrm{Im}(az_1) = a\mathrm{Im}(z_1)$$

if a is again real.

As usual, you should not necessarily *memorize* these rules: you should know the definitions of $|\cdot|$, $\bar{\cdot}$ and $\mathrm{Re}(\cdot)$, $\mathrm{Im}(\cdot)$ and be able to use them to derive the relevant result when necessary.

Taylor series

We eventually want to make sense of functions of complex variables rather than real variables. While functions like cos and sin have a geometrical interpretation for real arguments, this is not possible for complex arguments: we only have the rules of algebra above. What is done instead is typically to define functions in terms of infinite series: sums that involve powers of the variable, so each term can be understood in terms of the algebraic operations above.

The basic idea of a Taylor series is to approximate a function by its tangent. This means that, close to a point x_0, the change in f can be related to the change in x by the derivative of f,

$$f'(x_0) \approx \frac{f(x) - f(x_0)}{x - x_0},$$

or

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0).$$

Of course, this approximation becomes poor when the point x is a longer distance from x_0. To wit, if we take $f(x) = \cos(x)$ and $x_0 = 0$, we have $f(x) = 1$ and $f'(x_0) = 0$. The formula above would then suggest that $f(x) = 1$, which is a poor approximation at larger x.

A more systematic way to developing an approximation is by using the fundamental theorem of calculus

$$f(x) - f(x_0) = \int_{x_0}^{x} f'(x_1)\, \mathrm{d}x_1.$$

Beware that x_1 here is just a *dummy variable*: it doesn't matter what we use as the integration variable, so long as it is a variable that can represent a sum going from x_0 to x. This means for instance that we cannot use x as a variable of integration: the expression $\int_{x_0}^{x} f(x)\, \mathrm{d}x$ doesn't make sense because x simultaneously has to be fixed (the limit) and vary (the integration variable).

From this we also have

$$f(x) = f(x_0) + \int_{x_0}^{x} f'(x_1)\, \mathrm{d}x_1.$$

But we can deal with the integral by writing

$$f'(x_1) = f'(x_0) + \int_{x_0}^{x_1} f'(x_2)\, \mathrm{d}x_2,$$

where x_2 is another dummy variable. So

$$\begin{aligned}
\int_{x_0}^{x} f'(x_1)\,\mathrm{d}x_1 &= \int_{x_0}^{x} f'(x_0) + \left[\int_{x_0}^{x_1} f''(x_2)\,\mathrm{d}x_2\right] \mathrm{d}x_1 \\
&= \int_{x_0}^{x} f'(x_0)\,\mathrm{d}x_1 + \int_{x_0}^{x} \left[\int_{x_0}^{x_1} f''(x_2)\,\mathrm{d}x_2\right] \mathrm{d}x_1 \\
&= f'(x_0)(x-x_0) + \int_{x_0}^{x} \left[\int_{x_0}^{x_1} f''(x_2)\,\mathrm{d}x_2\right] \mathrm{d}x_1
\end{aligned}$$

This gives

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \int_{x_0}^{x} \left[\int_{x_0}^{x_1} f''(x_2)\,\mathrm{d}x_2\right] \mathrm{d}x_1. \tag{6}$$

The right-hand side gives the first two terms of the Taylor series, plus a correction (the integral). The correction is small if the first derivative remains close to $f'(x_0)$ throughout the interval from x_0 to x.

To get more terms in the Taylor series, we can apply the same approach again to $f''(x_2)$,

$$f''(x_2) = f''(x_0) + \int_{x_0}^{x_2} f'''(x_3)\,\mathrm{d}x_3$$

where x_3 is another dummy variable. Substituting in (6) gives

$$f(x) = f(x_0)+f'(x_0)(x-x_0)+\int_{x_0}^{x}\int_{x_0}^{x_1} f''(x_0)\,\mathrm{d}x_2\,\mathrm{d}x_1+\int_{x_0}^{x}\left[\int_{x_0}^{x_1}\left[\int_{x_0}^{x_2} f'''(x_3)\,\mathrm{d}x_3\right]\mathrm{d}x_2\right]\mathrm{d}x_1.$$

We can work out the first integral on the right-hand side exactly

$$\begin{aligned}
\int_{x_0}^{x}\int_{x_0}^{x_1} f''(x_0)\,\mathrm{d}x_2\,\mathrm{d}x_1 &= f''(x_0)\int_{x_0}^{x_1}(x_1-x_0)\,\mathrm{d}x_1 \\
&= f''(x_0)\left[(x_1-x_0)^2/2\right]_{x_0}^{x} \\
&= f''(x_0)\frac{(x-x_0)^2}{2}
\end{aligned}$$

and substituting into (6) gives us the first three terms of the Taylor series plus another correction term:

$$f(x) = f(x_0)+f'(x_0)(x-x_0)+f''(x_0)\frac{(x-x_0)^2}{2}+\int_{x_0}^{x}\left[\int_{x_0}^{x_1}\left[\int_{x_0}^{x_2} f'''(x_3)\,\mathrm{d}x_3\right]\mathrm{d}x_2\right]\mathrm{d}x_1.$$

We can continue like this to get any number of terms in the Taylor series (assuming the function is smooth enough to keep differnetiating). Let $f^{(n)}(x_0) = \left.\mathrm{d}^n f/\,\mathrm{d}x^n\right|_{x=x_0}$

be the nth derivative of f at x_0 (this is easier to write than putting n primes on f). The general form of the correction term after n terms is

$$\begin{aligned}
&\int_{x_0}^{x}\left[\int_{x_0}^{x_1}\dots\left[\int_{x_0}^{x_{n-1}} f^{(n)}(x_n)\,\mathrm{d}x_n\right]\dots\mathrm{d}x_2\right]\mathrm{d}x_1 = \\
&\int_{x_0}^{x}\left[\int_{x_0}^{x_1}\dots\left[\int_{x_0}^{x_{n-1}}\left[f^{(n)}(x_0)+\int_{x_0}^{x_n} f^{(n+1)}(x_{n+1})\,\mathrm{d}x_{n+1}\right]\mathrm{d}x_n\right]\dots\mathrm{d}x_2\right]\mathrm{d}x_1 = \\
&\int_{x_0}^{x}\left[\int_{x_0}^{x_1}\dots\left[\int_{x_0}^{x_{n-1}} f^{(n)}(x_0)\,\mathrm{d}x_n\right]\dots\mathrm{d}x_2\right]\mathrm{d}x_1 \\
&+\int_{x_0}^{x}\left[\int_{x_0}^{x_1}\dots\left[\int_{x_0}^{x_n} f^{(n+1)}(x_{n+1})\,\mathrm{d}x_{n+1}\right]\dots\mathrm{d}x_2\right]\mathrm{d}x_1
\end{aligned}$$

The term on the last line represents another correction, while $f^{(n)}(x_0)$ being a constant allows us to compute the integral immediately after the last equality directly:

$$\begin{aligned}
&\int_{x_0}^{x}\left[\int_{x_0}^{x_1}\dots\left[\int_{x_0}^{x_{n-1}} f^{(n)}(x_0)\,\mathrm{d}x_n\right]\dots\mathrm{d}x_2\right]\mathrm{d}x_1 = \\
&f^{(n)}(x_0)\int_{x_0}^{x}\left[\int_{x_0}^{x_1}\dots\left[\int_{x_0}^{x_{n-2}}\left[\int_{x_0}^{x_{n-1}} 1\,\mathrm{d}x_n\right]\mathrm{d}x_{n-1}\right]\dots\mathrm{d}x_2\right]\mathrm{d}x_1 = \\
&f^{(n)}(x_0)\int_{x_0}^{x}\left[\int_{x_0}^{x_1}\dots\left[\int_{x_0}^{x_{n-3}}\left[\int_{x_0}^{x_{n-2}}(x_{n-1}-x_0)\,\mathrm{d}x_{n-1}\right]\mathrm{d}x_{n-2}\right]\dots\mathrm{d}x_2\right]\mathrm{d}x_1 = \\
&f^{(n)}(x_0)\int_{x_0}^{x}\left[\int_{x_0}^{x_1}\dots\left[\int_{x_0}^{x_{n-3}}\frac{(x_{n-2}-x_0)^2}{2}\,\mathrm{d}x_{n-2}\right]\dots\mathrm{d}x_2\right]\mathrm{d}x_1
\end{aligned}$$

Every time we compute one of the integrals, we increase the power of $(x_{n-i}-x_0)$ by one and divide by that power. As there are n nested integrals in total, we get

$$\int_{x_0}^{x}\left[\int_{x_0}^{x_1}\dots\left[\int_{x_0}^{x_{n-1}} f^{(n)}(x_0)\,\mathrm{d}x_n\right]\dots\mathrm{d}x_2\right]\mathrm{d}x_1 = f^{(n)}(x_0)(x-x_0)^n/n!.$$

Hence we can write

$$\begin{aligned}
f(x) &= \sum_{n=0}^{N} f^{(n)}(x_0)(x-x_0)^n/n! \\
&\quad+\int_{x_0}^{x}\left[\int_{x_0}^{x_1}\dots\left[\int_{x_0}^{x_N} f^{(N+1)}(x_{N+1})\,\mathrm{d}x_{N+1}\right]\dots\mathrm{d}x_2\right]\mathrm{d}x_1
\end{aligned}$$

An infinite series can then be constructed if the last term (the integral) becomes small for large $N\to\infty$, in which case

$$f(x)=\sum_{n=0}^{\infty} f^{(n)}(x_0)(x-x_0)^n/n!.$$

This does not always work, but for the functions we're interested in, it does work.

Exponentials and trigonometric functions

The exponential function $f(x) = \exp(x)$ is particularly simple to expand in a Taylor series. For any n, we have

$$\frac{\mathrm{d}^n f}{\mathrm{d}x^n} = \exp(x).$$

Picking $x_0 = 0$, we have $f^{(n)}(x_0) = 1$, and so

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Note 2 *As an aside, we could equally have picked an arbitrary x_0 to find*

$$\begin{aligned}
\exp(x) &= \sum_{n=0}^{\infty} \exp(x_0) \frac{(x - x_0)^n}{n!} \\
&= \exp(x_0) \sum_{n=0}^{\infty} \frac{(x - x_0)^n}{n!} \\
&= \exp(x_0) \exp(x - x_0),
\end{aligned}$$

which is nothing more than the statement that $\exp(a + b) = \exp(a)\exp(b)$ *if we pick* $a = x_0$ *and* $b = x - x_0$.

The trigonometric functions $\cos(x)$ and $\sin(x)$ are similarly easy to expand. If $f(x) = \sin(x)$, then $f'(x) = \cos(x)$, $f''(x) = -\sin(x)$, $f'''(x) = -\cos(x)$, $f''''(x) = \sin(x)$. Because $f''''(x) = f(x)$, it follows that $f^{(i)}(x) = f^{(i+4)}(x) = f^{(i+4n)}(x)$ if n is any integer. Hence

$$\begin{aligned}
f^{(4n)}(x) &= \sin(x) \\
f^{(4n+1)}(x) &= \cos(x) \\
f^{(4n+2)}(x) &= -\sin(x) \\
f^{(4n+3)}(x) &= -\cos(x).
\end{aligned}$$

In particular, if we put $x_0 = 0$, we have

$$\begin{aligned}
f^{(4n)}(x_0) &= 0 \\
f^{(4n+1)}(x) &= 1 \\
f^{(4n+2)}(x) &= 0 \\
f^{(4n+3)}(x) &= -1.
\end{aligned}$$

With this in hand, we can write down the Taylor series for $\sin(x)$, expanding about $x_0 = 0$. From our calculations of derivatives, it follows that all the even terms in the series are zero, and the odd terms alternatve sign, so we get

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots$$

The same procedure can also be applied to $\cos(x)$, in which case all the odd terms vanish, and the even terms change sign, to give

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

Exercise 4 *Find the Taylor expansion for* $\log(1+x)$*, expanding about* $x_0 = 0$*. Will this always work as an infinite series? If not, give a reason.*

Complex exponentials

To make sense of $\exp(z)$ for a complex argument z, we *define* the exponential function by its Taylor series,

$$\exp(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \frac{z^4}{4!} + \dots$$

Now, if we choose $z = ix$, this gives

$$\exp(ix) = 1 + ix + \frac{(ix)^2}{2} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \dots$$

We can expand all the powers to find that $(ix)^2 = -x^2$, $(ix)^3 = -ix^3$, $(ix)^4 = x^4$ $(ix)^5 = ix^5$ and so on. Then

$$\exp(ix) = 1 + ix - \frac{x^2}{2} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} + \dots$$

Next, group all the terms that have a factor i and those that do not:

$$\begin{aligned} \exp(ix) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \\ &\quad + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right) \end{aligned}$$

But the two series, the one with a factor i and the one without, can be recognized as the Taylor expansions of $\sin(x)$ and $\cos(x)$, respectively, so that

$$\exp(ix) = \cos(x) + i\sin(x).$$

This is known as *Euler's formula*, and plays an important role in the use of complex variables in physics-based applications. A simple consequence is that

$$\cos(x) = \mathrm{Re}(\exp(ix)) = \frac{\exp(ix) + \exp(-ix)}{2}.$$

For more general *complex* exponentials, we can use

$$\exp(x + iy) = \exp(x)\exp(iy) = \exp(x)\left[\cos(y) + i\sin(y)\right]$$

So

$$\mathrm{Re}(\exp(x + iy)) = \exp(x)\cos(y).$$

Exercise 5 *If you are familiar with hyperbolic sine and cosine functions, you will know that these are*

$$\cosh(x) = \frac{\exp(x) + \exp(-x)}{2}, \qquad \sinh(x) = \frac{\exp(x) - \exp(-x)}{2}.$$

From Euler's formula, deduce that

$$\cosh(ix) = \cos(x), \qquad \sinh(ix) = i\sin(x)$$

How would you demonstrate that

$$\cos(ix) = \cosh(x), \qquad \sin(ix) = i\sinh(x)?$$

Exercise 6 *Express the following in the form $a + ib$, where u, v, x and y are real:*

1. $(u + iv)\exp(ix)$
2. $\frac{\exp(u+iv)}{x+iy}$
3. $|\exp(x + iy)|$
4. $\bar{z}$ *if* $z = \exp(x + iy)$. *Can you also express this as* $\exp(u + iv)$ *for some choice of real u and v?*
5. $\mathrm{Re}\left((u + iv)\exp(-\exp(ix))\right)$

Exercise 7 *Euler's formula makes it easier to derive trignometric formulas involving sums and multiples of an angle. Take for instance* $\cos(A+B)$ *and* $\sin(A+B)$*, which you can expand as a sum of cosines and sines of A and B. To derive the relevant formulas, begin with*

$$\cos(A + B) + i\sin(A + B) = \exp[i(A + B)] = \exp(iA)\exp(iB)$$

Expand $\exp(iA)$ *and* $\exp(iB)$ *using Euler's formula, and manipulate to show that*

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B), \qquad \sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B).$$

The polar form of a complex number

With the result $\exp(i\theta) = \cos(\theta) + i\sin(\theta)$, we can represent any complex number in the *polar form*

$$a + ib = r\exp(i\theta) = r(\cos(\theta) + i\sin(\theta)).$$

To do so, note that this amounts to finding r and θ so that

$$a = r\cos(\theta), \qquad b = r\sin(\theta).$$

Exercise 8 *Show that*

$$r = \sqrt{a^2 + b^2}, \qquad \tan(\theta) = \frac{b}{a}. \tag{7}$$

The inverse of tan *is non-unique. More precisely, it is unique up to the addition of* $n\pi$*, where* n *is an integer: if* $\tan(\theta) = b/a$*, then we also have* $\tan(\theta + n\pi) = b/a$ *for any integer* n*. Show that the choice of inverse* tan *above is actually more restrictive: if* $a+ib = r\exp(i\theta)$ *with* r *and* θ *given by (7), then we also have* $a+ib = r\exp(i\theta+2n\pi)$ *but not* $a+ib = r\exp(i\theta+(2n+1)\pi)$*. What does* $r\exp(i\theta+(2n+1)\pi)$ *equal in terms of* a *and* b*? The way to choose which inverse* tan *to use is given by the fact that we have taken* r *to be positive, and so* $\cos(\theta)$ *has the same sign as* a*. If we restrict* θ *so that* $-pi < \theta \leq \pi$*, then only one of the possible inverse tangents satisfies this requirement.*

As exercise 8 shows, if $z = a + ib = r\exp(i\theta)$, then

$$r = |z|.$$

The angle θ is known as the *argument* of z, denoted by $\arg(z)$, which is usually restricted lie in the range $-\pi < \theta \leq \pi$. The polar form of complex numbers can make some operations much easier, because of the rules governing the multiplication and exponentiation of exponentials. Take for instance the multiplication of two complex numbers

$$z_1 = r_1 \exp(i\theta_1), \qquad z_2 = r_2 \exp(i\theta_2).$$

Then

$$z_1 z_2 = r_1 \exp(i\theta_1) \times r_2 \exp(i\theta_2) = r_1 r_2 \exp[i(\theta_1 + \theta_2)],$$

or

$$z_1 z_2 = |z_1||z_2| \exp\{i[\arg(z_1) + \arg(z_2)]\}$$

The modulus of the product is the product of moduli, and the argument of the product is the *sum* of arguments:

$$|z_1 z_2| = |z_1||z_2|, \qquad \arg(z_1 z_2) = \arg(z_1) + \arg(z_2).$$

Similarly, consider taking the square root of a complex number. This is apparently straightforward in polar form since we can write

$$z^{1/2} = [r\exp(i\theta)]^{1/2} = r^{1/2}\exp(i\theta/2) = |z|^{1/2}\exp[i\arg(z)/2].$$

Apparently, we take the square root of the modulus and halve the argument. There is a caveat, however: when writing $z = r\exp(i\theta)$, it *does* become important that θ is not unique, but we can add $2n\pi$ to θ for any integer n and obtain the same complex number, so $z = r\exp(i\theta) + r\exp[i(\theta + 2n\pi)]$, and hence we obtain

$$z^{1/2} = r^{1/2}\exp[i(\theta/2 + n\pi)] = r^{1/2}\exp(i\theta/2)\exp(in\pi).$$

But $\exp(in\pi) = \cos(n\pi) + i\sin(n\pi)$. If n is an integer, $\sin(n\pi) \equiv 0$ (since $\sin(0) = \sin(\pi) = sin(2\pi) = \ldots = 0$. Similarly, $\cos(n\pi) = 1$ if n is even (since $\cos(0) = \cos(2\pi) = \cos(4\pi) = \ldots = 1$ and $\cos(n\pi) = -1$ if n is odd (since $\cos(\pi) = cos(3\pi) = \cos(5\pi) = \ldots = -1$. Hence $\exp(in\pi) = \pm 1$, depending on the value of n chosen, and we find

$$z^{1/2} = \pm r^{1/2}\exp(i\theta/2) = \pm|z|^{1/2}\exp[i\arg(z)/2]$$

with $\arg(z)$ being the argument of z that lies between 0 and 2π. The sign ambiguity of the square root is explicit here: if z is not real, then we cannot fall back on the usual convention that $\sqrt{x}$ is equal to the *positive* number a satisfying $a^2 = x$.[1]

In fact, we can go further and similarly use the polar form of a complex number to write any power of a complex number (including fractional powers, that is, roots) as

$$z^\alpha = |z|^\alpha \exp\{i[\alpha\arg(z) + 2\alpha n\pi]\};$$

picking a particular value of n is usually referred to as picking the *branch* of the power of z.

Exercise 9 *Find both values of $\sqrt{i}$ in the form $a + ib$, starting with the polar form of i. Similarly, find all third roots $(-1)^{1/3}$ in the form $a + ib$.*

Note 3 *A situation you are likely to meet in applications of complex variables is finding the real part of $f(t) = A\exp(i\omega t)$, where A is a complex number, ω is a real number (often an 'angular frequency') and t is a real variable, often time. Using polar forms, this is simply*

$$\begin{aligned}\mathrm{Re}(f(t)) =& \mathrm{Re}\left(A\exp(i\omega t)\right)\\ =& \mathrm{Re}\left[|A|\exp(i\theta)\exp(i\omega t)\right]\\ =& \mathrm{Re}\left(|A|\exp[i(\omega t + \theta)]\right)\\ =& |A|\cos(\omega t + \theta)\end{aligned}$$

where $\theta = \arg(A)$ represents a phase shift *to the cosine function.*

Exercise 10 *Instead of having to take the real part of $f(t) = A\exp(i\omega t)$ as in note 3., you may simply have a function of the form*

$$g(t) = A\exp(i\omega t) + B\exp(-i\omega t)$$

Suppose you are told that $g(t)$ is real for all t. Show that this implies that

$$B = \bar{A}$$

and that

$$g(t) = 2|A|\cos(\omega t + \theta), \qquad \theta = \arg(A)$$

which is simply twice the real part of $f(t)$.

[1] If we accept that $\arg(z)$ lies between $-\pi$ and π, then we are guaranteed that the $|z|^{1/2}\exp(i\arg(z)/2)$ has a positive real part, which is the reason why the argument is usually defined to lie between $-\pi$ and π.

Differentiation

Calculus with complex variables is a vast subject with many applications. Here we will be concerned with only a small part thereof: how to differentiate a complex function with respect to a real argument. Let

$$f(t) = g(t) + ih(t)$$

where g and h are real, so that f is complex. We have

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\mathrm{d}g}{\mathrm{d}t} + \frac{\mathrm{d}h}{\mathrm{d}t}$$

so that

$$\mathrm{Re}\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right) = \frac{\mathrm{d}g}{\mathrm{d}t}.$$

But $g = \mathrm{Re}f$, so

$$\mathrm{Re}\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right) = \frac{\mathrm{d}(\mathrm{Re}(f(t)))}{\mathrm{d}t}.$$

In other words, the order of Re and $\mathrm{d}/\,\mathrm{d}t$ can be interchanged.

Example 1 *Let* $f(t) = \exp(it)$*, so that* $\mathrm{Re}f(t) = \cos(t)$ *and*

$$\frac{\mathrm{dRe}(f(t))}{\mathrm{d}t} = \frac{\mathrm{d}(\cos(t))}{\mathrm{d}t} = -\sin(t).$$

But

$$\frac{\mathrm{d}f}{\mathrm{d}t} = i\exp(it) = i(\cos(t) + i\sin(t)) = -\sin(t) + i\cos(t)$$

so that

$$\mathrm{Re}\left(\frac{\mathrm{d}f}{\mathrm{d}t}\right) = -\sin(t) = \frac{\mathrm{d}(\mathrm{Re}(f(t)))}{\mathrm{d}t}$$

as promised.