EOS 352 Continuum Dynamics
Conservation laws

January 11, 2018

Overview

These notes cover the following

- Densities revisited
- Mass conservation in integral form
- Mass conservation in differential equation form
- Generalization to other scalar quantities
- Constitutive relations
- The heat equation

Densities

The 'high school' definition of density as

$$\rho = \frac{\text{mass of body}}{\text{volume}}$$

gives an average density for a body. Here we are interested in the density field $\rho(x, y, z, t)$, or the density at a point (x, y, z) at a time t. This can obviously be different from the definition above. Take a biological example. Measuring mass over volume for an animal will be different from making the same measurement for different body parts: bones and muscles will have a higher density, fat a lower density.

The density field is instead something you can think of as measuring the concentration of mass near a point (x, y, z) at time t. By saying 'near', what we really mean is taking the amount of mass δm in a small volume δV centered on (x, y, z), and defining

$$\rho(x, y, z, t) = \frac{\delta m}{\delta V},$$

or better still, taking the limit as $\delta V \to 0$.[1] At the heart of this definition is the idea that mass δm of an object should be proportional to its volume δV, provided that the volume is small: taking twice the volume should lead to an body that has twice the mass, provided that the properties of the material within it do not vary. This last requirement is ensured by making sure the volume is very small. Mathematically, this is where the limit comes in.

Density is then the 'constant of proportionality' that relates volume to mass, in the sense that

$$\delta m = \rho \delta V,$$

only that the constant of proportionality can now vary from place to place, and in time. The total mass of a body V can then be found by chopping it up into lots of little bits δV centered on different points (x, y, z), working out the individual masses of these δV's through $\delta m = \rho \delta V$ and summing. In the limit of small δV's, this becomes a volume integral, so the mass M of a body V is

$$M = \int_V \rho \, \mathrm{d}V.$$

The high school definition of density then gives

$$\bar{\rho} = \frac{\int_V \rho \, \mathrm{d}V}{V} = \frac{\int_V \rho \, \mathrm{d}V}{\int_V 1 \, \mathrm{d}V}$$

which we can see as an average of ρ over V (abstracting from averaging as adding up ρ's and dividing by the number of sample points to taking an integral and dividing by the volume of which we integrate instead).

Now, there is no reason why we should confine this concept of 'density' to mass. Densities can be associated with lots of other physical quantities. The simplest of

[1] You could get technical here and point out that you could have a very elongated shape centered on (x, y, z) that has a very small volume δV but encompasses points that are a long way from (x, y, z). What we mean of course is a volume that shrinks towards (x, y, z) from all directions, like a sphere of diminishing raidus. There is something of a physical caveat here: at very small scales, matter is made up of discrete objects, such as nuclei and electrons. The purpose of continuum mechanics is to understand behaviour at much larger scales, so by taking the limit $\delta V \to 0$, we assume that δV can be made very small compared with the scale we're interested in but much larger than the scale of an individual atom.

these that you will be familiar with is chemical concentration. In high school, this would be

$$c = \frac{\text{number of molecules or moles of a chemical}}{\text{volume of sample}}.$$

If we see concentration $c(x, y, z, t)$ as a field, then instead we would take the number of molecules or moles δn of a chemical contained in a small volume δV about (x, y, z) at time t and divide by δV:

$$c = \frac{\delta n}{\delta V}.$$

By the same argument as above, the total number of molecules or moles in a volume V is then given by

$$n = \int_V c \, \mathrm{d}V.$$

You can define densities analogously for all kinds of quantities, for instance a 'heat density' field $h(x, y, z, t)$ if you take the amount of heat[2] δE in a small volume δV around (x, y, z) and divide by δV

$$e = \frac{\delta E}{\delta V},$$

so that the total amount of heat in a volume V is

$$E = \int_V e \, \mathrm{d}V.$$

More generally still, you can define a density field $\phi(x, y, z, t)$ for some quantity Φ that is distributed in space as being the amount $\delta\phi$ in a small volume δV:

$$\phi = \frac{\delta \Phi}{\delta V}$$

and the total amount of Φ in a volume V is

$$\Phi = \int_V \phi \, \mathrm{d}V$$

Continuum dynamics is concerned with describing how different densities evolve in time, and how the associated physical quantities can be transported.

Exercise 1 *Let the velocity field of a continuum be $\mathbf{u} = (u, v, w)$. What is the density field associated with kinetic energy? Momentum in the x-direction? Do not rely on dimensional arguments.*

[2] internal energy, to be thermodynamically precise

Conservation of mass

There are different ways of arriving at the same answer here. Those of you who have taken EOSC 250 ('Fields and Fluxes') will have seen a slightly different version of the same thing. Here we consider a Langrangian volume $V(t)$: this is a volume that always includes the same 'particles' or 'bits of matter'. As a result, the volume will in general change shape over time, hence the notation $V(t)$. Because the volume includes the same particles at all times, its surface moves at a velocity given by the local velocity field $\mathbf{u}$ of the particles that sit on its surface.

Mass is a conserved quantity, and if $V(t)$ always contains the same bits of matter, then the mass $M(t)$ contained in it must not change. In other words,

$$\frac{\mathrm{d}M(t)}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \rho(x, y, z, t)\,\mathrm{d}V = 0. \tag{1}$$

Our aim is ultimately to turn this into a differential equation, because there are methods we can use to solve differential equations. Naturally, with ρ appearing in this expression, we expect to find a differential equation for ρ. But how to get there?

Note 1 *The first thing we have to do is manipulate the derivative. How do you differentiate an integral? The fundamental theorem of calculus is not of direct use here, as the derivative is not of the form* $\mathrm{d}/\mathrm{d}x \int^x f(y)\,\mathrm{d}y$. *But the analogue of a single rather than multiple integral is still instructive. Imagine a case where density depends only on* x *and* t*, and the volume* $V(t)$ *is a cuboid of base area* A *that extends from* $x_1(t)$ *to* $x_2(t)$*. Then*

$$M(t) = A\int_{x_1(t)}^{x_2(t)} \rho(x, t)\,\mathrm{d}x.$$

Differentiation with respect to t *gives*

$$\frac{\mathrm{d}M}{\mathrm{d}t} = A\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_1(t)}^{x_2(t)} \rho(x, t)\,\mathrm{d}x.$$

Now all you have to remember is what a derivative is:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \lim_{\delta t\to 0}\frac{f(t+\delta t) - f(t)}{\delta t}.$$

Put $f(t) = \int_{x_1(t)}^{x_2(t)} \rho(x, t)\,\mathrm{d}x$*. Then*

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \lim_{\delta t\to 0}\frac{\int_{x_1(t+\delta t)}^{x_2(t+\delta t)} \rho(x, t+\delta t)\,\mathrm{d}x - \int_{x_1(t)}^{x_2(t)} \rho(x, t)\,\mathrm{d}x}{\delta t}.$$

The hard part here is that the two integrals have different limits. But we can split the first integral up:

$$\int_{x_1(t+\delta t)}^{x_2(t+\delta t)} \rho(x,t+\delta t)\,\mathrm{d}x = \int_{x_1(t+\delta t)}^{x_1(t)} \rho(x,t+\delta t)\,\mathrm{d}x + \int_{x_1(t)}^{x_2(t)} \rho(x,t+\delta t)\,\mathrm{d}x + \int_{x_2(t)}^{x_2(t+\delta t)} \rho(x,t+\delta t)\,\mathrm{d}x.$$

Using this

$$\begin{aligned}
\frac{\mathrm{d}f}{\mathrm{d}t} &= \lim_{\delta t\to 0}\left[\frac{\int_{x_1(t+\delta t)}^{x_1(t)} \rho(x,t+\delta t)\,\mathrm{d}x + \int_{x_1(t)}^{x_2(t)} \rho(x,t+\delta t)\,\mathrm{d}x \int_{x_2(t)}^{x_2(t+\delta t)} \rho(x,t+\delta t)\,\mathrm{d}x}{\delta t}\right.\\
&\quad \left. -\frac{\int_{x_1(t)}^{x_2(t)} \rho(x,t)\,\mathrm{d}x}{\delta t}\right]\\
&= \lim_{\delta t\to 0}\int_{x_1(t)}^{x_2(t)} \frac{\rho(x,t+\delta t)\,\mathrm{d}x - \rho(x,t)}{\delta t}\,\mathrm{d}x + \lim_{\delta t\to 0}\frac{\int_{x_1(t+\delta t)}^{x_1(t)} \rho(x,t+\delta t)\,\mathrm{d}x}{\delta t}\\
&\quad + \lim_{\delta t\to 0}\frac{\int_{x_2(t)}^{x_2(t+\delta t)} \rho(x,t+\delta t)\,\mathrm{d}x}{\delta t}
\end{aligned}$$

There seem to be a lot of terms here, but just look at the three after the last equality sign. The first one has $[\rho(x,t+\delta t)-\rho(x,t)]/\delta t$ as the integrand, and this will turn into $\partial\rho/\partial t$ in the limit $\delta t\to 0$.[3] *This term clearly describes the effect of a changing density on the mass contained in $V(t)$. But if this term was the only one we had, then any change in density would lead to a change in mass, contrary to our assumption that mass must be conserved. Clearly the other two terms must account for this.*

To deal with the second and third terms, note that when δt is small, $x_1(t+\delta t)$ is very close to $x_1(t)$. But for an integral over a very small interval, the integrand is nearly constant, and so we can write

$$\int_{x_1(t+\delta t)}^{x_1(t)} \rho(x,t+\delta t)\,\mathrm{d}x \approx \rho(x_1(t+\delta t),t+\delta t)[x_1(t)-x_1(t+\delta t)]$$

and so

$$\begin{aligned}
\lim_{\delta t\to 0}\frac{\int_{x_1(t+\delta t)}^{x_1(t)} \rho(x,t+\delta t)\,\mathrm{d}x}{\delta t} &= \lim_{\delta t\to 0}\left[-\rho(x_1(t+\delta t),t+\delta t)\frac{x_1(t+\delta t)-x_1(t)}{\delta t}\right]\\
&= -\rho(x_1,t)\frac{\mathrm{d}x_1}{\mathrm{d}t}
\end{aligned}$$

Similarly, we can show that

$$\lim_{\delta t\to 0}\frac{\int_{x_2(t)}^{x_2(t+\delta t)} \rho(x,t+\delta t)\,\mathrm{d}x}{\delta t} = \rho(x_2,t)\frac{\mathrm{d}x_2}{\mathrm{d}t}.$$

[3] We ingore the complications introduced by interchanging the limit and the integral here.

The point about $V(t)$ being a Lagrangian volume is now that the faces $x = x_1(t)$ and $x = x_2(t)$ must move at the velocity of the material at those interfaces. In other words

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = u(x_1, t), \qquad \frac{\mathrm{d}x_2}{\mathrm{d}t} = u(x_2, t)$$

where u is the velocity of material in the x-direction. Therefore

$$\frac{\mathrm{d}M}{\mathrm{d}t} = A \int_{x_1}^{x_2} \frac{\partial \rho}{\partial t} \,\mathrm{d}x + \rho(x_2, t)u(x_2, t)A - \rho(x_1, t)u(x_1, t)A = 0.$$

As discussed, the first term signifies the rate of change of mass in V due to changes in density. The second and third describe the effect of deforming the boundaries of $V(t)$ at the velocity U of the material in it. Together, these effects must add up to zero: if $\partial\rho/\partial t < 0$, then the first term is negative (diminishing density tends to reduce mass). To offset this, we must have $u(x_2, t) > 0$ or $u(x_1, t) < 0$ or both: this implies the ends of the interval moving outwards, or the volume expanding to account for diminishing volume within it. (Think of a bicylce pump: density within it will drop if you pull the piston out while holding the valve shut. The drop in density is offset by an increase in volume, so the total mass can remain the same.)

From here, it is relatively straightforward to construct a differential equation. The first step is to recognize that the difference between the $\rho u A$ terms evaluated at (x_2, t) and (x_1, t) can be written as an integral by the fundamental theorem of calculus:

$$\rho(x_2, t)u(x_2, t)A - \rho(x_1, t)u(x_1, t)A = A \int_{x_1}^{x_2} \frac{\partial(\rho u)}{\partial x} \,\mathrm{d}x$$

so that $\mathrm{d}M/\,\mathrm{d}t$ can be written as a single integral

$$\frac{\mathrm{d}M}{\mathrm{d}t} = A \int_{x_1}^{x_2} \frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} \,\mathrm{d}x = 0.$$

The last step is the most subtle. The point is that, while $x_1(t)$ and $x_2(t)$ move at the local velocity u, their initial positions are completely arbitrary, so the integral

$$A \int_{x_1}^{x_2} \frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} \,\mathrm{d}x$$

must be zero no matter what its limits are. That, however, turns out to imply that the integrand is zero.

Because we are in one dimension, one can show this just by formally differentiating with respect to, say, x_2, keeping x_1 constant:

$$0 = \frac{\mathrm{d}}{\mathrm{d}x_1} \int_{x_1}^{x_2} \frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} \,\mathrm{d}x = \left(\frac{\partial \rho}{\partial t} \,\mathrm{d}x + \frac{\partial(\rho u)}{\partial x} \right)\Bigg|_{x=x_2}$$

and so

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0$$

at any arbitrary position x_2 (so the equation holds everywhere). Differentiation with respect to one of the end points of the range of integration however does not transfer well to higher dimensions, where there are no discrete end points, and we have instead a bounding surface. A slightly more subtle argument must be used. The simplest version of this is to look at x_2 close to x_1, so the range of integration is short, say $x_2 = x_1 + \delta x$. Then we can approximately treat the integrand as constant over the range of integration, and

$$\int_{x_1}^{x_2} \frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x}\,\mathrm{d}x \approx \left(\frac{\partial \rho}{\partial t}\,\mathrm{d}x + \frac{\partial(\rho u)}{\partial x}\right)\bigg|_{x=x_1} \delta x = 0$$

and again, with $\delta x \neq 0$, the integrand must vanish at an arbitrary position x_1

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} = 0.$$

A more sophisticated — and watertight — argument is given in note 2 below.

The next step is to generalize this argument to three dimensions.

We want to manipulate

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \rho(x,y,z,t)\,\mathrm{d}V$$

into more manageable form. To do so, use the basic definition of a derivative:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \lim_{\delta t \to 0} \frac{f(t+\delta t) - f(t)}{\delta t},$$

or, in the present case,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \rho(x,y,z,t)\,\mathrm{d}V = \lim_{\delta t \to 0} \frac{\int_{V(t+\delta t)} \rho(x,y,z,t+\delta t)\,\mathrm{d}V - \int_{V(t)} \rho(x,y,z,t)\,\mathrm{d}V}{\delta t}$$

To turn this into something that contains more recognizeable derivatives, we need to have integrals that have the same range of integration. Split the range of integration of the first integral as follows:

$$\int_{V(t+\delta t)} \rho(x,y,z,t+\delta t)\,\mathrm{d}V = \int_{V(t)} \rho(x,y,z,t+\delta t)\,\mathrm{d}V + \int_{V(t+\delta t)-V(t)} \rho(x,y,z,t+\delta t)\,\mathrm{d}V.$$

By $\int_{V(t+\delta t)-V(t)} \ldots \mathrm{d}V$ we mean an integral over the volume that is contained either in $V(t+\delta t)$ or in $V(t)$ but not in both, with the integral being treated as positive for any regions that are only $V(t+\delta)$ and not in $V(t)$, and as negative for any regions that are only in $V(t)$ but not in $V(t+\delta t)$ (see figure 1).

Using this,

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \rho(x,y,z,t)\,\mathrm{d}V &= \lim_{\delta t\to 0}\left[\frac{\int_{V(t)} \rho(x,y,z,t+\delta t)\,\mathrm{d}V - \int_{V(t)} \rho(x,y,z,t)\,\mathrm{d}V}{\delta t}\right. \\
&\quad \left. + \frac{\int_{V(t+\delta t)-V(t)} \rho(x,y,z,t+\delta t)\,\mathrm{d}V}{\delta t}\right] \\
&= \lim_{\delta t\to 0}\left[\int_{V(t)} \frac{\rho(x,y,z,t+\delta t)-\rho(x,y,z,t)}{\delta t}\,\mathrm{d}V\right] \\
&\quad + \lim_{\delta t\to 0}\frac{\int_{V(t+\delta t)-V(t)} \rho(x,y,z,t+\delta t)\,\mathrm{d}V}{\delta t}
\end{aligned}$$

The first term after the last equality is clearly $\int_V \partial\rho/\partial t\,\mathrm{d}V$. What about the second?

We need to look at the volume change in the volume, which we have denoted by '$V(t+\delta t)-V(t)$'. In the limit of small δt, we can expect this volume to be a thin layer pasted onto $v(t)$ (where $V(t+\delta t)$ is bigger than $V(t)$) or sliced off $V(t)$ (where $V(t+\delta t)$ is smaller than $V(t)$). How can we quantify this volume? The change in volume occurs because the particles at the surface of $V(t)$ move at the velocity field $\mathbf{u}$ of the continuum, and are therefore displaced by an amount $\mathbf{u}\delta t$ in δt.[4] We can split $V(t+\delta t)-V(t)$ into small subvolumes by taking the surface $S(t)$ of $V(t)$ and splitting it up into lots of small parts δS. Then we can identify where the particles that make up these δS's have moved to after a short time interval δt, and take the volume in between the old on new positions of δS as the volume elements δV that make up $V(t+\delta t)-V(t)$. To account for the sign of the integral (negative where the volume has shrunk in δt, positive where it has expanded), we can take δV as positive where δS has moved outwards, and as negative where it has moved inwards.

Each δV is then a prism shape with base area δS and side length $|\mathbf{u}\delta t|$. To get a volume δV, we need to take base times height, where height is the part of the side length that is actually perpendicular to δS, or $|\mathbf{u}\delta t|\cos(\theta)$ where θ is the angle between the side and the normal $\hat{\mathbf{n}}$ to δS:

$$\delta V = \delta S|\mathbf{u}\delta t|\cos(\theta).$$

[4]There is a second caveat to the applicability of continuum mechanics here, in addition to concerning ourselves only with scales much larger than the atomic scale: we also have to assume that at any given point (x,y,z) and instant in time t, there is a single velocity field $\mathbf{u}(x,y,z,t)$ that describes the motion of material there. This implies that nearby atoms interact with each other strongly enough that their net motion occurs at the same velocity. This is appropriate in many circumstances, especially in solids and liquids. For very low-density gases such as the outer parts of the earth's atmosphere or even outer space, particles are too widely spaced to collide often enough. Particles can therefore maintain unrelated velocities over distances that are "macroscopic" — comparable with scales at which we are trying to model the system — and continuum mechanics breaks down.

From the definition of a dot product, we have

$$\cos(\theta) = \frac{(\mathbf{u}\delta t) \cdot \hat{\mathbf{n}}}{|\mathbf{u}\delta t||\hat{\mathbf{n}}|}.$$

If we insist that $\hat{\mathbf{n}}$ is a unit normal so that $|\hat{\mathbf{n}}| = 1$, this gives

$$\delta V = \delta S \mathbf{u} \cdot \hat{\mathbf{n}} \delta t.$$

If we further insist that $\hat{\mathbf{n}}$ points to the outside of $V(t)$ (a so-called 'outward-pointing unit normal') then δV even has the correct sign: it is negative where $\mathbf{u}$ points into the volume $V(t)$ and where the volume has therefore shrunk in the time interval δt, while it is positive where $\mathbf{u}$ points out of the volume, meaning the volume has grown.

With these δV's in hand, we can write

$$\begin{aligned} \int_{V(t+\delta t)-V(t)} \rho(x, y, z, t+\delta t) \, \mathrm{d}V &= \sum \rho \delta V \\ &= \sum \rho \delta S \mathbf{u} \cdot \hat{\mathbf{n}} \delta t \\ &= \int_{S(t)} \rho \mathbf{u} \cdot \hat{\mathbf{n}} \, \mathrm{d}S \delta t \end{aligned}$$

Using this result, we find

$$\lim_{\delta t \to 0} \frac{\int_{V(t+\delta t)-V(t)} \rho(x, y, z, t+\delta t) \, \mathrm{d}V}{\delta t} = \int_{S(t)} \rho \mathbf{u} \cdot \hat{\mathbf{n}} \, \mathrm{d}S$$

and so

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{V(t)} \rho \, \mathrm{d}V = \int_{V(t)} \frac{\partial \rho}{\partial t} \, \mathrm{d}V + \int_{S(t)} \rho \mathbf{u} \cdot \hat{\mathbf{n}} \, \mathrm{d}S. \qquad (2)$$

This result is actually a general result that holds for *any* integrand ρ integrated over a Lagrangian volume $V(t)$. The equality (2) is known as *Reynolds' transport theorem*, and will occur repeatedly in studying conservation laws.

Now, from $\mathrm{d}M/\mathrm{d}t = 0$, we therefore have

$$\int_{V(t)} \frac{\partial \rho}{\partial t} \, \mathrm{d}V + \int_{S(t)} \rho \mathbf{u} \cdot \hat{\mathbf{n}} \, \mathrm{d}S = 0. \qquad (3)$$

Conservation of mass as a differential equation

As stated, we would like a differential equation that we can (in principle) apply solution techniques to that we may (or maybe may not yet...) know about. So far, what we have in (3) should be called an integro-differential equation. To make this a

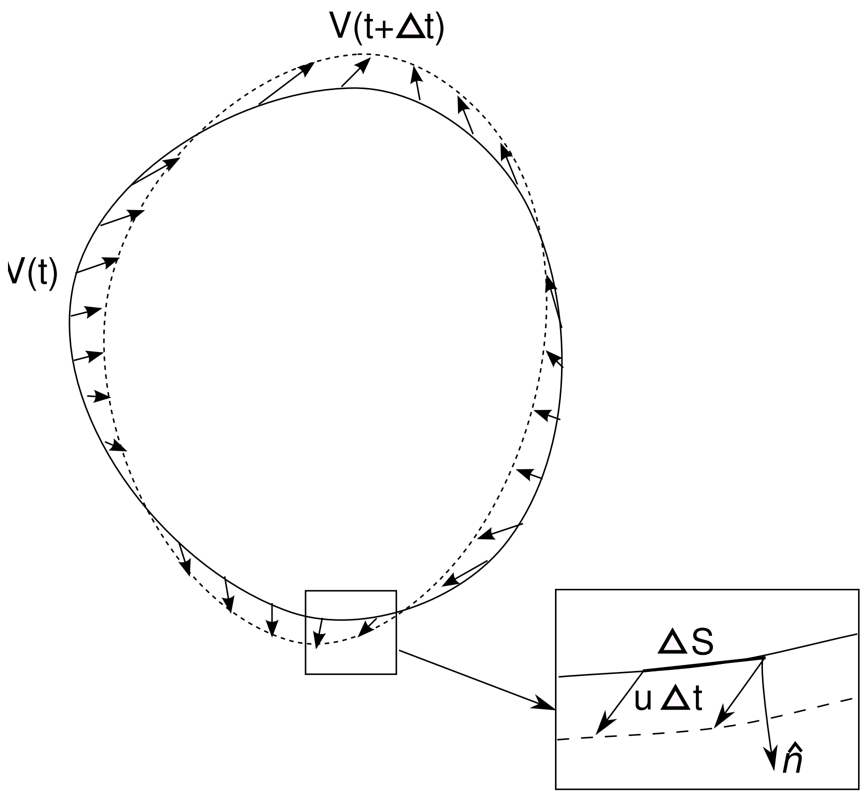

Figure 1: The deformation of a Lagrangian volume in time δt and integrating over the volume $V(t+\delta t) - V(t)$. The dashed surface bounds the volume $V(t+\delta t)$, the solid surface bounds the volume $V(t)$. The difference $V(t+\delta t) - V(t)$ is controlled by the movement of particles on the surface of the volume at velocity $\mathbf{u}(x, y, z, t)$.

little simpler, we can apply the divergence theorem, so that the integrals are at least taken over the same range of integration:

$$\int_{S(t)} \rho\mathbf{u}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = \int_{V(t)} \nabla\cdot(\rho\mathbf{u})\,\mathrm{d}V,$$

so that

$$\int_{V(t)} \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u})\,\mathrm{d}V = 0. \tag{4}$$

How do we take the next step? The point here is that $V(t)$ is an *arbitrary* volume: conservation of mass is something that must hold for any Lagrangian volume we choose, not just a specific one. In particular, $V(t)$ does not have to correspond to a something confined by a physically obvious boundary such as the membrane of a balloon or the walls of a bicycle pump. It can track the shape of any blob of matter. Hence we can make $V(t)$ very small indeed. For a small volume, the integrand $\partial\rho/\partial t + \nabla\cdot(\rho\mathbf{u})$ does not change by much over the volume, so that we can approximate

$$\int_{V(t)} \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u})\,\mathrm{d}V \approx \left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u})\right]|V(t)|$$

where $|V(t)|$ is the size of the volume $V(t)$. But $|V(t)|$ is not zero, and in order to have $\int_{V(t)} \partial\rho/\partial t + \nabla\cdot(\rho\mathbf{u})\,\mathrm{d}V = 0$, we can therefore demand that the integrand be zero:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) = 0. \tag{5}$$

Note 2 *To make this mathematically watertight requires a bit of extra thought, because the approximation above means that we cannot actually say that* $[\partial\rho/\partial t + \nabla\cdot(\rho\mathbf{u})]|V(t)| = 0$ *but only that it is close to zero. The relevant argument would then have to say* $[\partial\rho/\partial t+\nabla\cdot(\rho\mathbf{u})]|V(t)| = 0$ *has to be much closer to zero than* $|V(t)|$ *itself. This could be done by Taylor expanding about the centre of* $V(t)$*. There is a simpler argument that runs as follows: assume that the integrand in (4) is continuous. This means that variations in the integrand in a box around* (x, y, z) *approach zero as I make the size of the box smaller. A mathematical way of putting this is that, for any variation size* ϵ*, there is a distance* δ *from* (x, y, z) *so that variations in the integrand stay smaller than* ϵ*. The argument for why the integrand in (4) must be zero then runs as follows. Suppose that the integrand was not zero somewhere, say at a point* (x, y, z) *where* $D = \partial\rho/\partial t+\nabla\cdot(\rho\mathbf{u}) \neq 0$*. To make things, easier, suppose that* $D > 0$*. Picking* $\epsilon = D$ *in the definition of continuity above. It would then be possible to draw a sphere of some small radius* δ *around* (x, y, z) *so that the integrand varies by less than* D *within that sphere, and thefore is positive in the sphere. Picking* $V(t)$ *to be that sphere, we would find the integral* $\int_{V(t)} \partial\rho/\partial t + \nabla\cdot(\rho\mathbf{u})\,\mathrm{d}V > 0$*, which disagrees with (4). This allows us to say that such a point* (x, y, z) *cannot exist, and that the* *integrand has to be zero everywhere. Mathematically, this is known as a proof by contradiction: You assume that the thing you want to prove is false, and show that this necessarily leads to a statement that is not true, leaving only the possibiity that the thing you wanted to prove in the first place was actually correct. No need to memorize this argument, however.*

Note 3 *For a vector field* $\mathbf{q} = q_x(x, y, z, t)\mathbf{i} + q_y(x, y, z, t)\mathbf{j} + q_z(x, y, z, t)\mathbf{k}$*, the divergence* $\nabla \cdot \mathbf{q}$ *is defined through*

$$\nabla \cdot \mathbf{q} = \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z}.$$

The divergence theorem meanwhile says that

$$\int_V \nabla \cdot \mathbf{q} \, \mathrm{d}V = \int_S \mathbf{q} \cdot \hat{\mathbf{n}} \, \mathrm{d}S$$

where S *is the surface of the volume* V *and* $\hat{\mathbf{n}}$ *is an outward-pointing unit normal vector. You can use this to develop some intuition about divergences: take a small volume* V*, centered on some point* (x, y, z)*. For such a small volume, the volume integral can be approximated by the value of the integrand at* (x, y, z) *times the size of the volume* V *(because the integrand is approximately constant over the volume). This allows us to motivate the following alternative definiton of* $\nabla \cdot \mathbf{q}$*:*

$$\nabla \cdot \mathbf{q} = \lim_{V \to 0} \frac{1}{V} \int_S \mathbf{q} \cdot \hat{\mathbf{n}} \, \mathrm{d}S.$$

In other words, $\nabla \cdot \mathbf{q}$ *is a measure of how much the flow described by the vector field* $\mathbf{q}$ *is oriented away from* (x, y, z)*: recall that* $\hat{\mathbf{n}}$ *is an outward-pointing unit normal, and therefore, in the mean, points away from the centre of the volume* (x, y, z)*.* $\mathbf{q} \cdot \hat{\mathbf{n}}$ *is then a measure of whether* $\mathbf{q}$ *points towards or away from* (x, y, z)*.*

Note 4 *It should be obvious that, if (5) holds everywhere, then (3) must also automatically hold, simply by working the steps leading up to (5) in reverse. In particular, we can integrate over the Lagrangian volume* $V(t)$ *at any point in time to find (4),*

$$\int_{V(t)} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \, \mathrm{d}V = 0.$$

Applying the divergence theorem, this becomes (3),

$$\int_{V(t)} \frac{\partial \rho}{\partial t} \, \mathrm{d}V + \int_{S(t)} \rho \mathbf{u} \cdot \mathbf{n} \, \mathrm{d}V = 0.$$

From here, we only have to use Reynolds' transport theorem to retrieve the original statement (1),

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{V(t)} \rho \, \mathrm{d}V = 0, .$$

Conservation of other quantities

Mass is, in some ways, a rather special quantity to conserve. It cannot be exchanged between different bits of matter. Other conserved quantities like energy, momentum, charge etc. are different. Momentum is exchanged through forces, energy through heat flow, charge through electrical currents. Therefore, if we take the amount of energy, momentum, charge etc. (denoted by the generic symbol Φ with associated density ϕ below) in a Lagrangian volume $V(t)$ we can no longer so that this amount stays constant in time. Instead, we have to quantify the rate at which the quantity in question is exchanged with surrounding matter. In continuum mechanics, this process is called *conduction*, and is quantified through a flux field $\mathbf{q}$.

We assume that, at any point (x, y, z) and time t, there is a definite direction in which the quantity in question moves from one bit of matter to another, and we use that direction as the direction of the flux field $\mathbf{q}(x, y, z, t)$. The magnitude of the flux is then calculated (or measured as follows): take a small surface δS that is perpendicular to the direction of flow and measure the amount $\delta\Phi$ of whatever conserved quantity you're concerned with that is carried by conduction through δS in a short time δt. Then

$$|\mathbf{q}| = \frac{\delta\Phi}{\delta S \delta t}.$$

Hence, for δS perpendicular to $\mathbf{q}$, the amount of Φ that passes through in δS is

$$\delta\Phi = |\mathbf{q}|\delta S \delta t. \tag{6}$$

Again, at the heart of this is the idea that the amount $\delta\Phi$ transferred through a surface δS (of whatever the symbol Φ represents) should be proportional to the size of that surface and to the time δt elapsed, so long as 'conditions' do not change across the surface or in time — hence once more the insistence on small δS and δt (and δS is flat). $|\mathbf{q}|$ is the constant of proportionality, which can depend on where the surface δS is located, and on time (just as the 'constant of proportionality' ρ that relates volume δV to mass δm can depend on position and time).

In order to compute the rate at which Φ flows out of the surface of a Lagrangian volume, we then have to figure out what happens if part δS of the surface $S(t)$ is not perpendicular to the direction of the flux field $\mathbf{q}$. The answer is that the amount that passes through δS is now given by $|\mathbf{q}|$ times the size of the projection $\delta S'$ of δS onto a plane perpendicular to $\mathbf{q}$ times δt (see figure 2). The size of this projection is $\delta S' = \delta S \cos(\theta)$ where θ is the angle between the normal to δS and $\mathbf{q}$. From this,

$$\delta\Phi = \mathbf{q} \cdot \hat{\mathbf{n}} \delta S \delta t.$$

Exercise 2 *The argument we have just constructed is actually a bit of a sleight of hand. Why? See the appendix to these notes for more details.*

To figure out the rate at which Φ passes through a larger surface S, all we need to do is split that surface into lots of small elements δS, compute $\delta\phi$ for them, and sum. The rate (amount of Φ that passes through in δt, divided by δt) is then

$$\int_S \mathbf{q}\cdot\hat{\mathbf{n}}\,\mathrm{d}S.$$

At this point, we might be tempted to say the following: Let $\Phi(t)$ be the amount of Φ in a Lagrangian volume $V(t)$ at time t. The rate at which Φ changes (or more precisely, increases) is minus the rate at which Φ flows out, so

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)}\phi\,\mathrm{d}V = -\int_{S(t)}\mathbf{q}\cdot\hat{\mathbf{n}}\,\mathrm{d}S.$$

The snag is that it may also be possible to 'supply' Φ. Suppose we take Φ to be heat. There are other forms of energy (potential, kinetic) that can be converted to heat at rates that can be computed from basic physics. For instance, if you know the concentration of a radionuclide, the energy released per decay event and the half-life of the radionuclide, you can compute how much heat is being produced in the volume $V(t)$ due to radioactive decay. The way this is usually dealt with is to define a supply rate per unit volume a as a scalar field through

$$a(x,y,z,t) = \frac{\text{amount of }\Phi\text{ produced in small volume }\delta V\text{ around}(x,y,z)\text{ in time }\delta t}{\delta V\,\delta t}.$$

Note that, defined in this way, a looks very much like a density: it measures how concentrated heat production is near (x,y,z). The total rate of heat production in $V(t)$ is then naturally

$$\int_{V(t)} a\,\mathrm{d}V.$$

Instead, we therefore get the general conservation law

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)}\phi\,\mathrm{d}V = -\int_{S(t)}\mathbf{q}\cdot\hat{\mathbf{n}}\,\mathrm{d}S + \int_{V(t)} a\,\mathrm{d}V. \tag{7}$$

The term on the left can be dealt with using Reynolds' transport theorem:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)}\phi\,\mathrm{d}V = \int_{V(t)}\frac{\partial\phi}{\partial t}\,\mathrm{d}V + \int_{S(t)}\phi\mathbf{u}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$$

so that

$$\int_{V(t)}\frac{\partial\phi}{\partial t}\,\mathrm{d}V + \int_{S(t)}\phi\mathbf{u}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = -\int_{S(t)}\mathbf{q}\cdot\hat{\mathbf{n}}\,\mathrm{d}S + \int_{V(t)} a\,\mathrm{d}V. \tag{8}$$

Once more we can apply the divergence theorem and rearrange this equation to find

$$\int_{V(t)}\frac{\partial\phi}{\partial t} + \nabla\cdot(\phi\mathbf{u}) + \nabla\cdot\mathbf{q} - a\,\mathrm{d}V = 0 \tag{9}$$

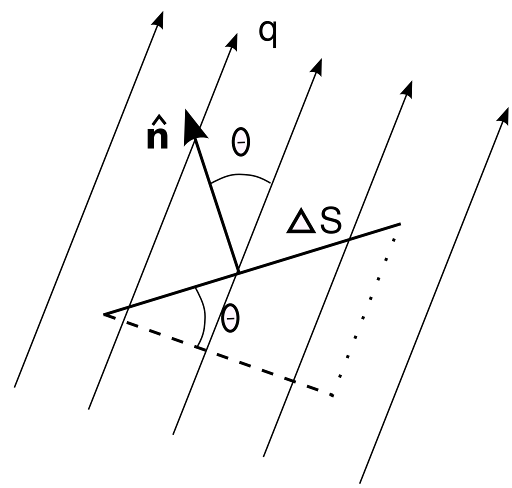

Figure 2: Conduction through a surface element δS that is at an oblique angle to the flux field $\mathbf{q}$. The rate of transfer through δS (shown as a thick solid line) is the same as the rate of transfer through the projection of δS onto a plane perpendicular to $\mathbf{q}$ (dahsed line). The size of this projection is $\delta S \cos(\theta)$ where θ is the angle between δS and the plane perpendicular to $\mathbf{q}$, so the rate of transfer per unit time is $|\mathbf{q}|\delta S \cos(\theta)$. θ is also the angle between $\mathbf{q}$ and the unit normal $\hat{\mathbf{n}}$ to δS, so the rate of transfer is equal to $\mathbf{q} \cdot \hat{\mathbf{n}}\delta S$.

and by the same arguments as we used for conservation of mass, we can show that the integrand must in fact be zero because the volume $V(t)$ is arbitrary:

$$\frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \mathbf{u}) + \nabla \cdot \mathbf{q} - a = 0. \tag{10}$$

This is the generic form of a conservation law. To understand the terms physically, we can rewrite it in the form

$$\frac{\partial \phi}{\partial t} = a - \nabla \cdot (\phi \mathbf{u}) - \nabla \cdot \mathbf{q}.$$

The density ϕ increases due to supply at rate a and decreases due to flow away from (x, y, z) due to the movement of material (or *advection*) at a rate $-\nabla \cdot (\phi \mathbf{u})$ and due to conduction away from (x, y, z) at a rate $-\nabla \cdot \mathbf{q}$.

Note 5 *One could question if (9) really does mean that the quantity Φ is 'conserved'. The volume $V(t)$ clearly experiences a loss $\int_{S(t)} \mathbf{q} \cdot \mathbf{n} \, \mathrm{d}S \delta t$ through its surface in time δt. In addition, an amount $\int_{V(t)} a \, \mathrm{d}V \delta t$ is added.*

The short answer is of course that the advection term $\int_{S(t)} \mathbf{q} \cdot \mathbf{n} \, \mathrm{d}S \delta t$ accounts for a loss from the volume $V(t)$ that is simultaneously added to another neighbouring *volume, so the total amount of* Φ *contained in both remains the same. Similarly, the amount of* $\int_{V(t)} a\,\mathrm{d}V\delta t$ *of* Φ *produced must be accounted for by an equal simultaneous loss of some other related quantity.*

To be specific, take the 'production' term $\int_{V(t)} a\,\mathrm{d}V\delta t$ *when* Φ *is 'heat', and suppose also for simplicity that there is no conduction. Let* ϕ_1 *be 'heat density'. In the absence of conduction, (7) is then*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \phi_1\,\mathrm{d}V = \int_{V(t)} a\,\mathrm{d}V.$$

But $\int_{V(t)} a\,\mathrm{d}V\delta t$ *is the amount of other forms of energy (kinetic, potential) converted to heat in the volume* $V(t)$ *during time* δt*. This is the same as saying that* $-\int_{V(t)} a\,\mathrm{d}V\delta t$ *is the amount of non-heat energy gained by the volume* $V(t)$*. Let* ϕ_2 *denote the density of 'other energy', and suppose again for simplicity that there is no conductive transport associated with these other forms of energy. Then (7) for 'other energy' becomes*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \phi_2\,\mathrm{d}V = -\int_{V(t)} a\,\mathrm{d}V.$$

But total energy content is the sum of heat and non-heat energy, $\int_V \phi_1+\phi_2\,\mathrm{d}V$*. Adding the two equations above gives*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \phi_1\,\mathrm{d}V + \frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \phi_2\,\mathrm{d}V = \int_{V(t)} a\,\mathrm{d}V - \int_{V(t)} a\,\mathrm{d}V.$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \phi_1 + \phi_2\,\mathrm{d}V = 0.$$

the production term $\int_{V(t)} a\,\mathrm{d}V\delta t$ *therefore does not lead to a gain or loss in total energy content — it only accounts for conversion of energy from one form to another,*

Similarly, to understand the effect of conduction better, consider the case of two adjoining Lagrangian volumes $V_1(t)$ *and* $V_2(t)$ *(see figure 3). Suppose that* $V_1(t)$ *and* $V_2(t)$ *can exchange* Φ *(think of it as heat if you will) through their common surface* S_i *by conduction (see figure 3), but that no heat is lost through the outer surface* S_0 *where* V_1 *and* V_2 *do not touch. For simplicity, ignore the production term* $\int_{V(t)} a\,\mathrm{d}V$*. Then (7) for the two reads*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_1(t)} \phi\,\mathrm{d}V = -\int_{S_i(t)} \mathbf{q}\cdot\mathbf{n}_1\,\mathrm{d}S$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_2(t)} \phi\,\mathrm{d}V = -\int_{S_i(t)} \mathbf{q}\cdot\mathbf{n}_2\,\mathrm{d}S,$$

and adding the two gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \phi\,\mathrm{d}V = -\int_{S_i(t)} \mathbf{q}\cdot(\mathbf{n}_1+\mathbf{n}_2)\,\mathrm{d}S$$

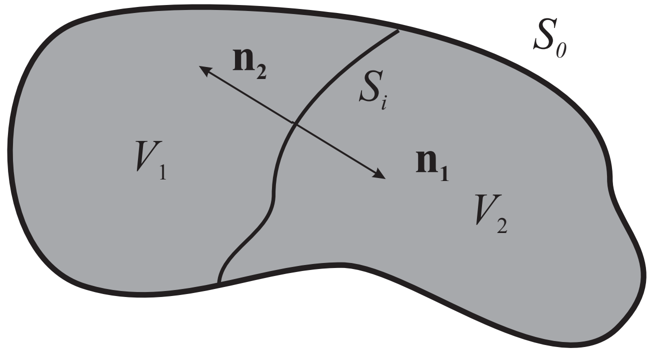

Figure 3: Conservation with conduction only between two neighbouring Lagrangian volumes.

where $V(t) = V_1(t) \cup V_2(t)$ is the total volume. But $\mathbf{n}_1$ and $\mathbf{n}_2$ are normals to the same surface, only they point in opposite directions. So $\mathbf{n}_2 = -\mathbf{n}_2$ and the integral on the right-hand side has zero integrand. Hence

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{V(t)} \phi \, \mathrm{d}V = 0$$

and $\Phi = \int_{V(t)} \phi \, \mathrm{d}V$ for the whole volume is conserved, though the sub-volumes V_1 and V_2 can exchange Φ with each other.

Heat

The conservation law (10) is completely generic, and can describe anything from conservation of mass ($\phi = \rho$, $\mathbf{q} = \mathbf{0}$, $a = 0$) to conservation of energy, charge etc. As such we cannot expect to find a general solution to this equation without making some more specific choices about some of the quantities involved. Mathematically, this is reflected in the fact that we have one equation in seven unknowns (ϕ and three components apiece for $\mathbf{u}$ and $\mathbf{q}$). Phyiscally, this is the result of not having stated anything about the properties of the materials in question.

While the conservation law (10) states completely fundamental physics, specifying material properties is generally something that is done empirically as it is usually impossible to use detailed microscopic physics at the scale of individual atoms to derive material behaviour relevant to the much larger scale of a continuum. The relevant mathematical relationships are then only models that work well as descriptions of the real world, but they do not have the same fundamental status as a conservation law. Such mathematical relationships are generally called *constitutive relations.*

We illustrate this here for the case of conservation of energy, where Φ is what

I will call heat. Then $\phi = e$ is a 'heat density',[5] and we expect this to depend on temperature as well as the material in question. Heat capacity (units J kg^{-1} K^{-1}) is a measure of how much the heat content per unit mass of some material increases given a small temperature increase.[6] If we treat c as constant, then we get simplest possible model for ϕ in this case, namely

$$\phi = \rho c T. \tag{11}$$

In addition, we expect heat conduction to depend on the temperature field as well. In particular, we expect heat to flow from hot to cold, and heat flux to be greatest where hot on cold are closest together, i.e. where the temperature gradient is biggest. The simplest empirical relationship that encapsulates these ideas is *Fourier's law*,

$$\mathbf{q} = -k\nabla T. \tag{12}$$

k is the thermal conductivity of the material in question, with units of W m^{-1} K^{-1}.

Note 6 *The gradient*

$$\nabla T = \frac{\partial T}{\partial x}\mathbf{i} + \frac{\partial T}{\partial y}\mathbf{j} + \frac{\partial T}{\partial z}\mathbf{k}$$

of a scalar field is perpendicular to contours (or 'isosurfaces') of T. Its direction is also the direction in which T increases most rapidly, and its magnitude gives the the relevant rate of increase. This is easiest to understand if we consider a curve C and ask how temperature changes along the curve. A curve can be parameterized in the form $(x(s), y(s), z(s))$ so that x, y and z are functions of a parameter s, which we can choose to be distance along the curve (also called 'arc length') from a fixed starting point. An example would be the unit circle. Distance along the unit circle can be measured through the angle traversed, and the circle can be parameterized as

$$(x(\theta), y(\theta), z(\theta)) = (\cos(\theta), \sin(\theta), 0)$$

Given a scalar field $T(x, y, z)$, the rate at which T changes with distance s is then given by

$$\frac{\mathrm{d}T}{\mathrm{d}s} = \nabla T \cdot \hat{\mathbf{t}} \tag{13}$$

where

$$\hat{\mathbf{t}} = \frac{\mathrm{d}x}{\mathrm{d}s}\mathbf{i} + \frac{\mathrm{d}y}{\mathrm{d}s}\mathbf{j} + \frac{\mathrm{d}z}{\mathrm{d}s}\mathbf{k}$$

[5] If you are familiar with thermodynamics, you may recognize that this should be 'internal energy density' rather than 'heat density'. and Φ should be referred to as 'internal energy', not heat.

[6] To be thermodynamically precise, Φ should not be referred to as 'heat' but as 'internal energy', and correspondingly, c would then be heat capacity measured at constant volume, so a change in temperature does not lead to a finite amount of work done in expanding the material in question.

is the unit tangent to the curve. Using the chain rule, it is easy to see that (13) holds, as we have

$$\frac{\mathrm{d}T}{\mathrm{d}s} = \frac{\partial T}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}s} + \frac{\partial T}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}s} + \frac{\partial T}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}s} = \nabla T \cdot \hat{\mathbf{t}}.$$

We'd still like to be sure that the interpretation of $\hat{\mathbf{t}}$ *as a unit tangent to the curve holds. To see this consider going a short distance from* $(x(s), y(s), z(s))$ *to* $(x(s+\delta s), y(s+\delta s), z(s+\delta s))$*. The displacement in doing so is*

$$\begin{aligned}\delta x\mathbf{i} + \delta y\mathbf{j} + \delta z\mathbf{k} &= [(x(s+\delta s) - x(s)]\,\mathbf{i} + [(y(s+\delta s) - y(s)]\,\mathbf{j} + [(z(s+\delta s) - z(s)]\,\mathbf{k} \\ &= \frac{\mathrm{d}x}{\mathrm{d}s}\delta s\mathbf{i} + \frac{\mathrm{d}y}{\mathrm{d}s}\delta s\mathbf{j} + \frac{\mathrm{d}z}{\mathrm{d}s}\delta s\mathbf{k} \\ &= \hat{\mathbf{t}}\delta s\end{aligned}$$

and this vector is clearly tangential to the curve (i.e., along *the curve) for small displacements* δs*. So* $\hat{\mathbf{t}}$ *is a tangent vector. It is also a unit vector if we insist that* δs *is the length of the displacement undergone in going from* s *to* δs*, i.e. if*

$$\delta s = \sqrt{(\delta x)^2 + (\delta y)^2 + (\delta z)^2}.$$

Rearranging

$$1 = \sqrt{\left(\frac{\delta x}{\delta s}\right)^2 + \left(\frac{\delta y}{\delta s}\right)^2 + \left(\frac{\delta z}{\delta s}\right)^2}$$

Or, in the limit,

$$\sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}s}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}s}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}s}\right)^2} = 1.$$

But the left-hand side is simply $|\hat{\mathbf{t}}|$*, so* $|\hat{\mathbf{t}}| = 1$ *and* $\hat{\mathbf{t}}$ *is a unit vector.*

Now it is easy to see that, if C *is a contour of the scalar field* T*, then* $\mathrm{d}T/\,\mathrm{d}s = 0$ *along* C*. Hence*

$$\nabla T \cdot \hat{\mathbf{t}} = 0$$

if $\hat{\mathbf{t}}$ *is tangent to contours of* T*. In other words,* ∇T *is perpendicular to temperature contours.*

Moreover, because $\hat{\mathbf{t}}$ *is a unit vector, we have*

$$\frac{\mathrm{d}T}{\mathrm{d}s} = \nabla T \cdot \hat{\mathbf{t}} = |\nabla T|\cos(\theta)$$

where θ *is the angle between* ∇T *and* $\hat{\mathbf{n}}$*. Now, as* $\cos(\theta) \leq 1$*, the last expression is biggest for a given temperature gradient if* $\cos(\theta) = 1$*, i.e. if* $\theta = 0$ *the curve is aligned with the gradient of* T*. Hence* ∇T *gives the direction of steepest increase in* T*. With* $\cos(\theta) = 1$*, we also have*

$$\frac{\mathrm{d}T}{\mathrm{d}s} = |\nabla T|$$

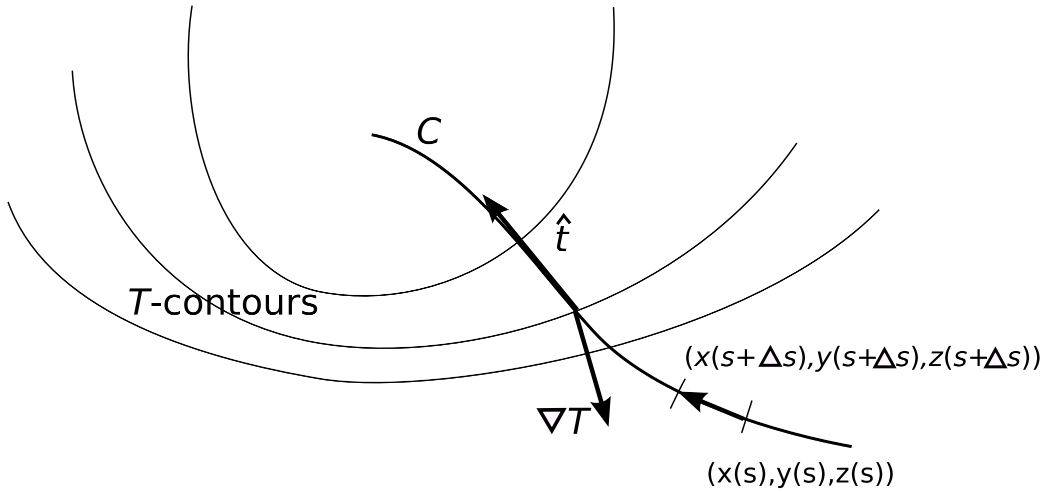

Figure 4: The relationship between the gradient ∇T, contours of T, and derivatives along a curve.

so the magnitude of ∇T also gives the rate of increase in T the direction where this increase is steepest.

Fourier's law therefore ensures that, at any point, conductive heat flux is in the direction in which temperature decreases *most rapidly (this is what the minus sign in (12) is for), and that conductive heat flux is proportional to the rate of decrease of temperature in that direction. Graphically, heat flux is perpendicular to temperature contours (from hot to cold) and inversely proportional to the distance between temperature contours.*

With (11) and (12), the conservation law (10) becomes

$$\frac{\partial(\rho c T)}{\partial t} + \nabla \cdot (\rho c T \mathbf{u}) - \nabla \cdot (k \nabla T) = a. \tag{14}$$

This is usually re-written slightly. Note that the product rule can be used to show that (recall that c is assumed to be constant)

$$\frac{\partial(\rho c T)}{\partial t} = cT \frac{\partial \rho}{\partial t} + \rho c \frac{\partial T}{\partial t}$$

and

$$\nabla \cdot (\rho c T \mathbf{u}) = cT \nabla \cdot (\rho \mathbf{u}) + \rho c \mathbf{u} \cdot \nabla T.$$

Substituting these in (14) gives

$$
\begin{aligned}
cT\frac{\partial\rho}{\partial t} + \rho c\frac{\partial T}{\partial t} + cT\nabla\cdot(\rho\mathbf{u}) + \rho c\mathbf{u}\cdot\nabla T - \nabla\cdot(k\nabla T) &= \\
cT\left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u})\right] + \rho c\frac{\partial T}{\partial t} + \rho c\mathbf{u}\cdot\nabla T - \nabla\cdot(k\nabla T) &= \\
\rho c\frac{\partial T}{\partial t} + \rho c\mathbf{u}\cdot\nabla T - \nabla\cdot(k\nabla T) &= a, \qquad (15)
\end{aligned}
$$

where we have used the mass conservation equation (5). The last line of (15) is a general form of the so-called *heat equation.* To get the heat equation that partial differential equations courses usually present, you have to assume that there is no advection ($\mathbf{u} = \mathbf{0}$) and that thermal conductivity k is spatially uniform ($k =$ constant). Then the (15) becomes

$$\rho c\frac{\partial T}{\partial t} - k\nabla^2 T = a$$

where

$$\nabla^2 T = \nabla\cdot\nabla T = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial x^2}$$

is called the *Laplacian* of the scalar field T.

Note 7 *We used a procedure in simplifying (14) into (15) above that is very common in continuum mechanics. This is to write a density ϕ in the form $\phi = \rho\psi$, where ψ now has dimensions of 'amount of conserved quantity Φ per unit mass'. Then the first two terms in (10) become, using the product rule,*

$$
\begin{aligned}
\frac{\partial\phi}{\partial t} + \nabla\cdot(\phi\mathbf{u}) &= \frac{\partial(\rho\psi)}{\partial t} + \nabla\cdot(\rho\psi\mathbf{u}) \\
&= \frac{\partial\rho}{\partial t}\psi + \rho\frac{\partial\psi}{\partial t} + \nabla\cdot(\rho\mathbf{u})\psi + \rho\mathbf{u}\cdot\nabla\psi \\
&= \psi\left[\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\mathbf{u})\right] + \rho\frac{\partial\psi}{\partial t} + \rho\mathbf{u}\cdot\nabla\psi \\
&= \rho\left[\frac{\partial\psi}{\partial t} + \mathbf{u}\cdot\nabla\psi\right]
\end{aligned}
$$

The combination of terms

$$\frac{D\psi}{Dt} = \frac{\partial\psi}{\partial t} + \mathbf{u}\cdot\nabla\psi$$

is often called a material derivative. *There is a simple reason for this name. Consider a point $\mathbf{x}(t) = (x(t), y(t), z(t))$ that moves at the velocity $\mathbf{u}(\mathbf{x}(t), t)$ of the material. Imagine following this point and measuring some quantity ψ as you follow that point,* *so that you are looking at $\psi(\mathbf{x}(t), t)$. By analogy with the derivation of (13), we can use the chain rule to write*

$$\begin{aligned}\frac{\mathrm{d}\psi(\mathbf{x}(t), t)}{\mathrm{d}t} &= \frac{\partial \psi}{\partial t} + (\nabla\psi) \cdot \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} \\ &= \frac{\partial \psi}{\partial t} + (\nabla\psi) \cdot \mathbf{u}\end{aligned}$$

as velocity is the rate of change of position, $\mathbf{u} = \mathrm{d}\mathbf{x}/\,\mathrm{d}t$.

Appendix: Why conductive flux is a vector field

We defined the magnitude of conductive flux, $|\mathbf{q}|$, through equation (6), which really states that the amount of 'stuff' $\delta\Phi$ that passes through a small surface δS in time δt is proportional to both δS and δt. The problem is that, as long as δS is flat, proportionality should hold regardless of how the surface δS is oriented. It is not really obvious that 'stuff' flows in a definite direction relative to which we can orient δS. What we will do here is show that such a direction really can be found, and that a conductive flux can be defined as a vector field $\mathbf{q}(x, y, z, t)$ such that

$$\delta\Phi = \mathbf{q} \cdot \hat{\mathbf{n}}\delta S\delta t \tag{16}$$

as we assumed previously. The derivation below is not entirely trivial (especially the later parts) and is not essential in understanding the remaining notes in this course, so long as you are happy to take (16) at face value.

The starting point for the derivation has to be only that

$$\delta\Phi = \tilde{q}\delta S\delta t \tag{17}$$

where the constant of proportionality $\tilde{q}$ depends not only on (x, y, z, t) but also on the orientation of δS, meaning we can write

$$\tilde{q} = \tilde{q}(x, y, z, t, \hat{\mathbf{n}}). \tag{18}$$

$\hat{\mathbf{n}}$ is the unit normal to δS, pointing towards the side of δS to which $\delta\Phi$ is transferred. We need to show that

$$\tilde{q}_(x, y, z, t, \hat{\mathbf{n}}) = \mathbf{q}(x, y, z, t) \cdot \hat{\mathbf{n}}$$

for a vector field $\mathbf{q}$ — which we can then call the 'conductive flux'.

We start by making one obvious observation: if I change sign of $\hat{\mathbf{n}}$, the magnitude of $\tilde{q}$ remains the same, but its sign changes. More simply, let Φ represent heat. Suppose I take joule of heat from one side of δS (call it side A) to the other (call it side B). I have added that joule to side B and removed it from A, which is the same as saying I have added minus one joule to side A. This means that

$$q(x, y, z, t, -\hat{\mathbf{n}}) = -q(x, y, z, t, \hat{\mathbf{n}}).$$

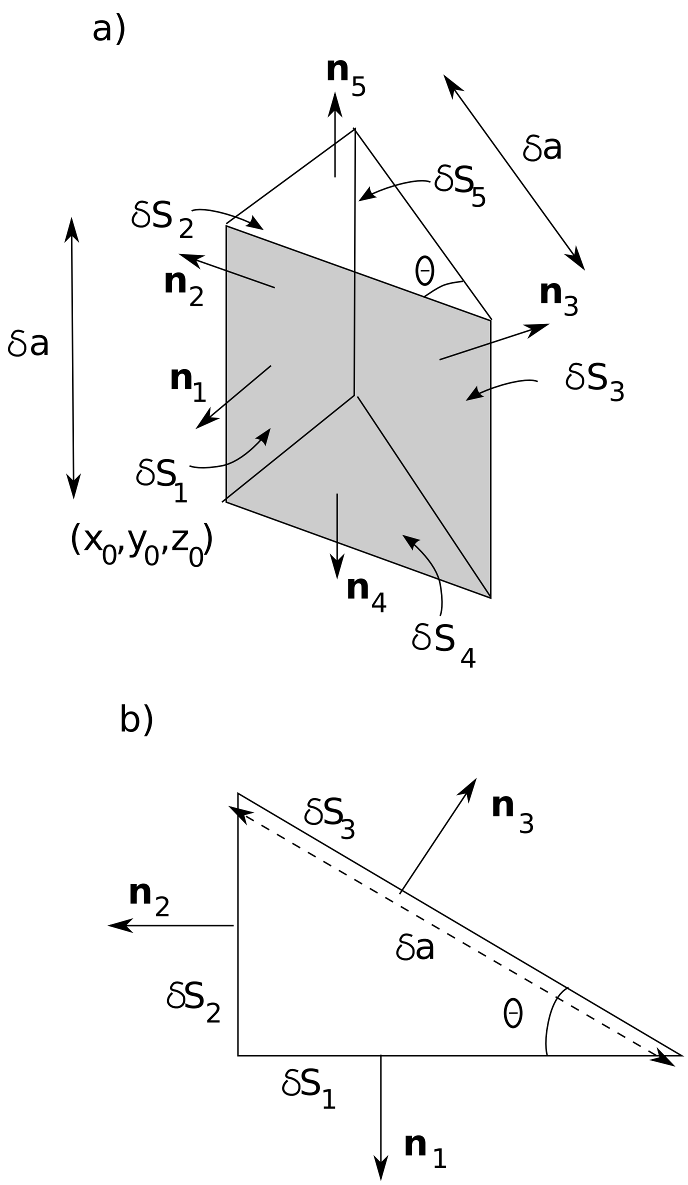

Figure 5: The prism shape of V: a) perspective view and b) top-down view.

Next, look at the prism in figure 5. Assume to start with that there is no production of 'Φ', no advection, and that the prism is in steady state. That means that the total transfer Φ through all the faces of the prism taken together must be zero. Assume that the prism is small enough so that we can use (17) for each face of the prism with $(x, y, z) = (x_0, y_0, z_0)$, the position of one of its vertices. Then

$$\begin{aligned}\tilde{q}(x_0, y_0, z_0, t, \hat{\mathbf{n}}_1)\delta S_1\delta t + \tilde{q}(x_0, y_0, z_0, t, \hat{\mathbf{n}}_2)\delta S_2\delta t + \tilde{q}(x_0, y_0, z_0, t, \hat{\mathbf{n}}_3)\delta S_3\delta t \\ +\tilde{q}(x_0, y_0, z_0, t, \hat{\mathbf{n}}_4)\delta S_4\delta t + \tilde{q}(x_0, y_0, z_0, t, \hat{\mathbf{n}}_5)\delta S_5\delta t &= 0,\end{aligned}$$

the normal $\hat{\mathbf{n}}$ in each case being outward-pointing as shown in the figure. But $\hat{\mathbf{n}}_4 = -\hat{\mathbf{n}}_5$ and $\delta S_4 = \delta S_5$, so the last two terms cancel. Put another way, what flows into the bottom of the prism flows out at the top. This leaves

$$\tilde{q}(x_0, y_0, z_0, t, \hat{\mathbf{n}}_1)\delta S_1 + \tilde{q}(x_0, y_0, z_0, t, \hat{\mathbf{n}}_2)\delta S_2 + \tilde{q}(x_0, y_0, z_0, t, \hat{\mathbf{n}}_3)\delta S_3 = 0.$$

Here the top-down view in panel b of the figure helps: clearly, we can write $\delta S_1 = \delta S_3 \cos(\theta)$, $\delta S_2 = \delta S_3 \sin(\theta)$. If we also put

$$q_1 = \tilde{q}(x_0, y_0, z_0, t, \hat{\mathbf{n}}_1), \qquad q_2 = \tilde{q}(x_0, y_0, z_0, t, \hat{\mathbf{n}}_2), \qquad q_3 = \tilde{q}(x_0, y_0, z_0, t, \hat{\mathbf{n}}_3),$$

we have

$$q_3 = -q_2 \sin(\theta) - q_1 \cos(\theta) = 0. \tag{19}$$

Next, we can play with the orientation of the prism, and with the internal angle θ. Suppose first I change the orientation of the prism until I have maximized transfer across the surface δS_1. With the orientation of the surface δS_1, suppose I change the size of the internal angle next. The orientation not only of δS_1 but also of δS_2 is then fixed before the internal angle is θ is changed, and is not affected by changes in θ. This means we can treat q_1 and q_2 as fixed, but q_3 must be treated as a function of θ when the internal angle is changed.

The fact that the face δS_1 was oriented to maximize flow across it means that q_3 should also be maximized when we change the internal angle so that δS_3 becomes aligned with δS_1. In other words,

$$\frac{\mathrm{d}q_3}{\mathrm{d}\theta} = 0 \qquad \text{when } \theta = 0.$$

Differentiating both sides of (19) with respect to θ, we get

$$\frac{\mathrm{d}q_3}{\mathrm{d}\theta} = -q_2 \cos(\theta) + q_1 \sin(\theta) = 0 \qquad \text{when } \theta = 0,$$

Since $\cos(0) = 1$, $\sin(0) = 0$, this implies that $q_2 = 0$. Then (19) says that

$$q_3 = -q_1 \cos(\theta) = q_1 \hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_3,$$

using the basic definition of dot products.

In other words, using the definitions of q_3 and q_1,

$$q_3 = \tilde{q}(x_0, y_0, z_0, t, \hat{\mathbf{n}}_3) = \tilde{q}(x_0, y_0, z_0, t, \hat{\mathbf{n}}_1)\hat{\mathbf{n}}_1 \cdot \hat{\mathbf{n}}_3. \tag{20}$$

$\hat{\mathbf{n}}_1$ was chosen specifically to make q_1 as large as possible given the position (x_0, y_0, z_0) of the prism. With that motivation, we can define conductive heat flux as a vector field

$$\mathbf{q}(x_0, y_0, z_0, t) = \tilde{q}(x_0, y_0, z_0, t, \hat{\mathbf{n}}_1)\hat{\mathbf{n}}_1,$$

and (20) becomes

$$\tilde{q}(x_0, y_0, z_0, t, \hat{\mathbf{n}}_3) = \mathbf{q}(x_0, y_0, z_0, t) \cdot \hat{\mathbf{n}}_3.$$

But there was nothing special about choosing the point (x_0, y_0, z_0), or the direction defined by $\hat{\mathbf{n}}_3$, so we can replace them by (x, y, z) and $\hat{\mathbf{n}}$. and (20) turns into

$$\tilde{q}(x, y, z, t, \hat{\mathbf{n}}) = \mathbf{q}(x, y, z, t) \cdot \hat{\mathbf{n}},$$

which is the desired equation (18). Notice that we have not only shown that this holds, but also that the conductive flux $\mathbf{q}$ can be understood as having a magnitude $|\mathbf{q}(x_0, y_0, z_0, t)| = q(x_0, y_0, z_0, t, \hat{\mathbf{n}}_1)$ that is the maximum rate of transfer across a given surface size, that is, maximized with respect to the orientation of the surface. Likewise, the direction of the conductive flux $\mathbf{q}$ is in the direction in which transfer is fastest.

The main weak spot in our derivation is that we have assumed that the prism is in steady state, with no advection or production. In fact, we have also cheated a bit by saying that we assume the prism to be small enough so that (17) can be applied to all of its sides. Presumably, (17) means that for a finite-sized body V with surface S, conduction transfers 'stuff' out of V at a rate

$$\int_S \tilde{q}(x, y, z, t, \hat{\mathbf{n}})\,\mathrm{d}S.$$

In the more general case, where we do not assume steady state or an absence of advection or production, the equivalent of (7) is then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \phi\,\mathrm{d}V + \int_S \tilde{q}(x, y, z, t, \hat{\mathbf{n}})\,\mathrm{d}S - \int_V a\,\mathrm{d}V = 0. \tag{21}$$

If you know the basics of Taylor expansions and assume that $\tilde{q}$ is sufficiently smooth[7], then we can write

$$\begin{aligned}\int_S \tilde{q}(x, y, z, t, \hat{\mathbf{n}})\,\mathrm{d}S \quad = \quad & \tilde{q}(x_0, y_0, z_0, t, \hat{\mathbf{n}}_1)\delta S_1 + \tilde{q}(x_0, y_0, z_0, t, \hat{\mathbf{n}}_2)\delta S_2 + \tilde{q}(x_0, y_0, z_0, t, \hat{\mathbf{n}}_3)\delta S_3 \\ & + \tilde{q}(x_0, y_0, z_0, t, \hat{\mathbf{n}}_4)\delta S_4 + \tilde{q}(x_0, y_0, z_0, t, \hat{\mathbf{n}}_5)\delta S_5 \\ & + \text{a correction that scales as } (\delta a)^3\end{aligned}$$

[7]We generally try to avoid getting into technicalities of this kind in these notes, but they are necessary at this point.

where δa is the edge length of the face δS_1 (see figure 5), so that all the faces have sizes that are proportional to $(\delta a)^2$ (meaning all the $\delta S's$ have size $(\delta a)^2$ times a combination of sines and cosines). The correction comes from the fact that $\tilde{q}$ actually varies by a small amount over each face — the variation in each case being approximately proportional to δa.

The two volume integrals in (21) also have sizes comparable to δa^3, because that is how the volume V depends on the edge length δa. In other words, we get

$$
\begin{aligned}
0 \;=\; & \tilde{q}(x_0, y_0, z_0, t, \hat{\mathbf{n}}_1)\delta S_1 + \tilde{q}(x_0, y_0, z_0, t, \hat{\mathbf{n}}_2)\delta S_2 + \tilde{q}(x_0, y_0, z_0, t, \hat{\mathbf{n}}_3)\delta S_3 \\
& + \tilde{q}(x_0, y_0, z_0, t, \hat{\mathbf{n}}_4)\delta S_4 + \tilde{q}(x_0, y_0, z_0, t, \hat{\mathbf{n}}_5)\delta S_5 + \text{a correction that scales as } (\delta a)^3
\end{aligned}
$$

By 'scales as δa^3', we mean of course that, if you change δa, the correction term is approximately proportional to δa^3. But using the fact that $\delta S_3 = (\delta a)^2$ and our previous results and notation, this means we get on dividing by δS_3 that

$$q_3 + q_2 \sin(\theta) + q_1 \cos(\theta) + \text{a correction that scales as } \delta a = 0.$$

What the correction term means is that if the sum of the conductive transfer terms $q_3 + q_2 \sin(\theta) + q_1 \cos(\theta)$ is not zero, this will to lead to violation of 'Φ-conservation' that is much bigger than the remaining terms (the 'correction that scales as δ') can make up for; more concisely, in the limit of $\delta a \to 0$, we are back to (19).