EOS 352 Continuum Dynamics
Fluid dynamics

March 28, 2020

Overview

These notes cover the following

- Constitutive relations for fluids
- inviscid fluids
- viscous fluids
- incompressible flow: the Navier-Stokes equations
- flow in a pipe

A constitutive relation for stress

Conservation of mass takes the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_i)}{\partial x_i} = 0. \tag{1a}$$

Conservation of momentum can be written as

$$\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = \frac{\partial \sigma_{ij}}{\partial x_j} + f_i \tag{1b}$$

while conservation of angular momentum demands that

$$\sigma_{ij} = \sigma_{ji} \tag{1c}$$

The third of these equalities implies that there are six components of the stress tensor to specify (σ_{11}, σ_{22}, σ_{33} as well as σ_{12}, σ_{13} and σ_{23}), from which the remaining ones can then be determined ($\sigma_{21} = \sigma_{12}$, $\sigma_{13} = \sigma_{31}$, $\sigma_{32} = \sigma_{23}$).

However, in (1a) and (1b) we still have only four equations (one in (1a) and one for each index $i = 1, 2, 3$ in (1b)), but ten unknowns (the six independent components of σ_{ij} as well as three velocity components u_i and density ρ). So more information is required. In fact, (1) must hold for any continuum, and therefore contains no information about the material in question, be it solid, liquid or gas. The missing information must therefore encapsulate the physics of how the material in question deforms. This information usually cannot be derived purely from first principles except in special circumstances (for instance, in an ideal gas), but must be constrained empirically. A simple example of a constitutive relation would be Fourier's law linking heat flux q_i to temperature gradient,

$$q_i = -k \frac{\partial T}{\partial x_i} \tag{2}$$

where the thermal conductivity k depends on the material and possibly other factors (such as temperature itself). In general k, cannot be derived from first principles (at least not in any straightforward way) but must be determined empirically. In fact, the validity of (2) must be tested empirically: is there a coefficient $-k$ that does not depend on the temperature gradient, such that heat flux equals that coefficient times the temperature gradient?[1]

The choice of constitutive relations becomes considerably more complicated when we have to figure out a model for how σ_{ij} depends on the state of deformation of a material. Along with σ_{ij}, we also in general have to figure out a constitutive relation for density ρ. While Fourier's law applies widely and, by changing k, can be used to describe the behaviour of many materials, even the functional form of constitutive relations for stress and density (rather than just the choice of numerical value for a coefficient like k) depends on the type of material. Here, we will focus on the behaviour of fluids (liquids and gases, but mostly liquids, and even some solids when subjected to forces for long enough).

[1] There are some additional constraints on possible constitutive relations that we will not consider in detail in this course. Some of these come out of considerations of *invariance* — the resulting equations must not be specific to a particular coordinate frame but must 'look' the same in all possible Cartesian coordinate systems. This is described briefly in the notes on invariance. Other constraints come out of thermodynamics, and relate to the fact that entropy cannot be destroyed. For instance, one can show (using something called the Clausius-Duhem inequality, which is a continuum version of the second law of thermodynamics) that k in Fourier's law must be positive, so heat cannot flow from cold to hot.

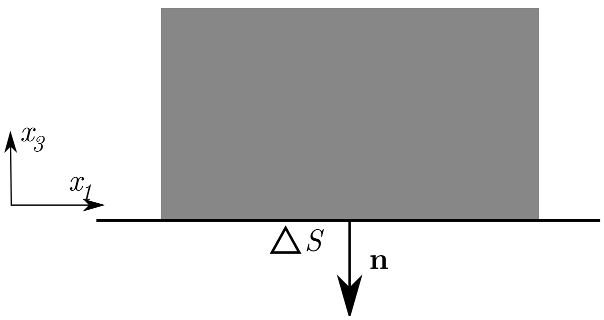

Figure 1: The force exerted by a fluid (in grey) above the x_1x_2-plane on a part δS of the plane is $p\delta S$ in the negative x_3-direction, or $(0, 0, -p\delta S) = p\delta S\mathbf{n}$ as a vector. The force exerted on the fluid by δS is therefore in the positive x_3-direction, given by $(0, 0, p\delta S) = -p\delta S\mathbf{n}$ in vector notation. In subscript notation, $\delta F_i = -pn_i\delta S$ for the force exerted on the fluid, and, from the definition of the stress tensor, this should be $\sigma_{ij}n_j\delta S$

Inviscid fluids

Recall that stresses σ_{ij} define surface forces. The most common understanding of how a fluid generates surface forces is that there is a pressure p in the fluid, defined as force per unit area. But pressure is a scalar, while forces are vectors and stresses are tensors. So how can we build a stress tensor from the idea of a pressure p?

The key is that the magnitude of the surface force generated by a pressure p in the fluid on a small surface element δS is always the same at $p\delta S$ regardless of the orientation of δS, *and* that the orientation of this force is perpendicular to the surface element δS, pointing *out* of the fluid. Forces being equal and opposite, this implies that the surface element generates a force on the fluid of magnitude $p\delta S$ pointing perpendicularly *into* the fluid. If n_i is an outward-pointing unit normal, this implies that the force on the fluid is (see figure 1)

$$\delta F_i = -pn_i\delta S.$$

Recall that the surface force exerted on an element of surface δS by a stress field σ_{ij} is

$$\delta F_i = \sigma_{ij}n_j\delta S$$

where n_i points *out* of the material that the force is being exerted on. Equating this with δF_i above gives

$$-pn_i\delta S = \sigma_{ij}n_j\delta S$$

or

$$-pn_i = \sigma_{ij}n_j.$$

This must hold regardless of the orientation of the surface element δS, so must hold for any choice of unit vector (n_1, n_2, n_3). Picking the combinations $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ in turn leaves us with

$$\begin{array}{ccc} \sigma_{11} = -p & \sigma_{12} = 0 & \sigma_{13} = 0 \\ \sigma_{21} = 0 & \sigma_{22} = -p & \sigma_{23} = 0 \\ \sigma_{31} = 0 & \sigma_{32} = 0 & \sigma_{33} = -p \end{array}$$

or

$$\sigma_{ij} = -p\delta_{ij} \tag{3}$$

for short. Note that this stress tensor immediately satisfies (1c). But (3) implies

$$\frac{\partial \sigma_{ij}}{\partial x_j} = -\frac{\partial (p\delta_{ij})}{\partial x_j} = -\frac{\partial p}{\partial x_i},$$

which we can recognize as the negative gradient of pressure p. Conservation of momentum (1b) then becomes

$$\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + f_i. \tag{4}$$

Together with (1a), this is known as the Euler equations.

A fluid that only exerts forces normal to a surface as described by (3) is known as an *inviscid fluid*. We can *close* the set of equations (1) (meaning, we can end up with as many variables as we have equations) if we also specify a constitutive relation for density ρ. The simplest is an incompressible fluid, for which ρ = constant, so (1a) becomes

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{5}$$

and in (4) and (5) we have four equations for the unknowns p, u_1, u_2 and u_3.

Alternatively, we may have a compressible gas, and ρ will typically depend on pressure as well as on temperature T. For instance, for a so-called *ideal gas*, we have

$$\rho = \frac{mp}{RT},$$

where m is the mass of one mole of gas and $R = 8.314$ J K^{-1} mol^{-1} is the ideal gas constant. A constitutive law (or 'equation of state') of this form would force us to consider conservation of energy (the heat equation) in addition to (4) and (1a). We defer this until later, where we consider the heat equation in the presence of mechanical work being done.

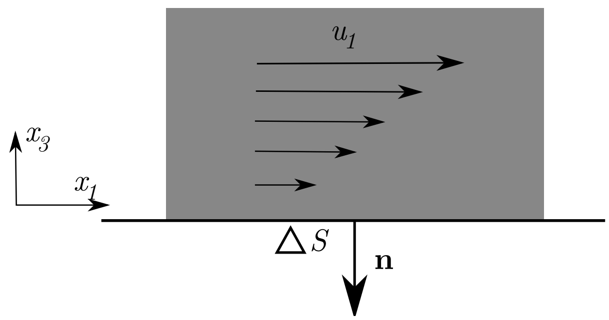

Figure 2: The force exerted by a fluid (in grey) above the x_1x_2-plane on a part δS of the plane is $p\delta S$; here the fluid has a velocity u_1 parallel to the x_1-axis that increases with height x_3 above the plane.

Viscous fluids

The point about an inviscid fluid is that the forces generated by the stress tensor σ_{ij} are always perpendicular to the surface they act on, so there are never any tangential forces, which we might associate with friction or drag on the surface. For this reason, (3) is often not an appropriate constitutive relation, as many fluids do exert tangential forces. Take honey as an example of a 'sticky' fluid that clearly adheres to surfaces and exerts a tangential force. (If it didn't it would flow off a spoon as easily as water, which it clearly doesn't).

To get an idea of how to build a plausible model for a fluid that does generate tangential forces, consider the fluid flowing over a flat surface. Assume that the flat surface is the x_1x_2-plane, and that the flow is purely in the x_1-direction, so that $\mathbf{u} = (u_1, 0, 0)$. Assume also that the flow is uniform. This means the velocity cannot depend on x_1 because the fluid would otherwise pile up somewhere or thin out somewhere else (the divergence $\nabla\cdot\mathbf{u} = \partial u_1/\partial x_1$ would be non-zero). By symmetry, we also expect the velocity not to depend on the transverse direction x_2. We do however, expect that the velocity might depend on distance above the plane: in particular, common experience would suggest that the flow velocity is faster the further the fluid is from the plate. Hence we suppose that $u_1 = u_1(x_3)$ (see also figure 2)

Now consider the forces exerted by the underlying plane on the fluid. From $\delta F_i = \sigma_{ij}n_j\delta S$, we have, for a surface element δS in the x_1x_2-plane with normal pointing out of the fluid (i.e., down, with $n_1 = n_2 = 0$, $n_3 = -1$),

$$\delta F_1 = -\sigma_{13}\delta S, \qquad \delta F_2 = -\sigma_{23}\delta S, \qquad \delta F_3 = -\sigma_{33}\delta S.$$

Now we expect a tangential force that opposes the motion of the fluid, so $\delta F_1 < 0$ if $u_1 > 0$ and $\delta F_2 = 0$ by symmetry.

If $\delta F_1 < 0$, we expect σ_{13} to be positive. But what should it relate to? We may expect that the force gets larger the faster the fluid flows, so somehow σ_{13} should increase with velocity u_1. However, if the fluid sticks to the surface (the x_1x_2-plane at $x_3 = 0$), the velocity at the surface is probably zero. In any case, we do note really expect the force to depend on velocity as such: if the surface itself were moving, we would expect the force to depend on some measure of the difference between velocity in the fluid and the velocity of the surface. But what measure of this velocity difference should we use? If the fluid velocity depends on x_3, then there is presumably a velocity gradient, with the fluid flowing faster the further away from the surface we go, and the size of the stress σ_{13} should relate to this. Just as Fourier's law models heat flux as depending on temperature gradients as a measure of how close hot and cold material are to each other, an obvious model would relate the stress σ_{13} to the velocity gradient, which is a measure of how close fast- and slow-moving fluid are to each other and how strongly they might therefore interact. A plausible model would therefore take the form

$$\sigma_{13} = \mu \frac{\partial u_1}{\partial x_3}, \tag{6}$$

where μ is a constant of proportionality analogous to thermal conductivity k in Fourier's law. This constant of proportionality is known as *viscosity.*

Exercise 1 *What are the units of* σ_{13} *and* $\partial u_1/\partial x_3$*? What are the units of viscosity?*

Assuming that (6) holds for the simple geometry we have used above, we still need to generalize this equation to give us the whole stress tensor σ_{ij} for a more general flow, in which all the velocity components u_i can potentially be non-zero, and where they can depend on all the x_i's. The most tempting generalization would appear to be

$$\sigma_{ij} = \mu \frac{\partial u_i}{\partial x_j} \tag{7}$$

as this is obviously consistent with (6).

There are however two problems with (7). The first is that there is no reason why it should satisfy (1c), as there is no reason why $\partial u_i/\partial x_j = \partial u_j/\partial x_i$. The second is that it predicts non-zero forces only when there is movement with finite velocity gradients in the fluid — but from our discussion of inviscid fluids, we expect that there can be normal surface forces due to pressure in the fluid even when the fluid is at rest.

In order to satisfy (1c), we can make the stress tensor symmetric by writing

$$\sigma_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \tag{8}$$

This still satisfies (6) for the simple geometry for which we derived (6), as we have in that case

$$\sigma_{13} = \mu \left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \right) = \mu \frac{\partial u_1}{\partial x_3},$$

u_3 being zero for the geometry under consideration. In addition, we also have $\sigma_{23} = \mu(\partial u_2/\partial x_3 + \partial u_3/\partial x_2) = 0$ as $u_2 = u_3 = 0$ for the geometry we considered above.

Equation (8) does satisfy the angular momentum conservation law $\sigma_{ij} = \sigma_{ji}$, but it will still only generate stresses if there is flow. To get around this, we somehow need to introduce a pressure variable that can account for normal forces (perpendicular to the surface) when the fluid is at rest (i.e., when $\partial u_i/\partial x_j = 0$). The usual way to introduce this in continuum mechanics is to *define* pressure as

$$p = -\sigma_{ii}/3, \tag{9}$$

which is consistent with an inviscid fluid, where $\sigma_{ij} = -p\delta_{ij}$, so that $\sigma_{ii} = -p\delta_{ii} = -p(\delta_{11} + \delta_{22} + \delta_{33}) = -3p$.

Note 1 *Note that equation (9) is $p = ((-\sigma_{11}) + (-\sigma_{22}) + (-\sigma_{33}))/3$ when written out. It should therefore be understood as a mean over the stress components σ_{11}, σ_{22} and σ_{33}. Each of these corresponds to a normal force: imagine a small surface δS, and orient it parallel to the x_1x_2-plane. The component of force normal to it being exerted by the material below on the material above is then $-\sigma_{33}\delta S$ (recall that the force generated by a stress tensor is generally $\delta F_i = \sigma_{ij}n_j$, where n_i is the normal pointing* away *from the material the force is being exerted on). We can similarly compute $-\sigma_{22}\delta S$ as the normal force generated if the surface δS is rotated into the x_1x_3-plane, and $-\sigma_{11}\delta S$ as the normal force generated if the surface δS is rotated into the x_2x_3-plane. Pressure is simply the mean over these normal forces divided by the area they act on.*

What we would now like is something that looks like (8) and produces (6) for the simple flow geometry considered above, but reduces to $\sigma_{ij} = -p\delta_{ij}$ when the fluid is at rest, when we expect the discussion in the preceding section on inviscid fluids to hold. Note that with (8), we would have $\sigma_{kk} = \mu(\partial u_k/\partial x_k + \partial u_k/\partial x_k) = 2\mu\partial u_k/\partial x_k$. To obtain something that satisfies (9), we should therefore subtract $[2\mu(\partial u_k/\partial x_k)/3+p]\delta_{ij}$ from the right-hand side of (8),

$$\sigma_{ij} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij}\right) - p\delta_{ij}. \tag{10}$$

This is the general form of a viscous constitutive relation for σ_{ij}. We can easily check that, with this prescription of σ_{ij}, we have

$$\begin{aligned}
\sigma_{ii} &= \mu\left(\frac{\partial u_i}{\partial x_i} + \frac{\partial u_i}{\partial x_i} - \frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ii}\right) - p\delta_{ii} \\
&= \mu\left(\frac{\partial u_i}{\partial x_i} + \frac{\partial u_i}{\partial x_i} - \frac{2}{3}\frac{\partial u_k}{\partial x_k} \times 3\right) - 3p \\
&= -3p
\end{aligned}$$

as required by (9). In addition, when there are no velocity gradients, we recover the inviscid stress (3). For the simple flow geometry we considered leading up to (6), we also still obtain $\sigma_{13} = \mu \partial u_1/\partial x_3$, $\sigma_{23} = 0$.

Whether (10) is correct for a particular material can generally only be established by experiment. A wide variety of fluids can be described using this constitutive relation. For instance, (10) holds for water with $\mu = 1.8 \times 10^{-3}$ Pa s^{-1} at its melting point.[2] A fluid for which μ is a constant is called a *Newtonian fluid*. An inviscid fluid can be obtained simply by putting $\mu = 0$.

Exercise 2 *Let $(u_1, u_2, u_3) = (-\omega x_2, \omega x_1, 0)$ and $p = \rho g(h - x_3)$, where ω, ρ, g and h are constants. Compute σ_{ij} as a function of (x_1, x_2, x_3) from (10). Also compute the components of the vector $\partial \sigma_{ij}/\partial x_j$.*

Exercise 3 *Let $(u_1, u_2, u_3) = (u_0(1 - x_3^2/h^2), 0, 0)$ and $p = -Cx_1$, where u_0, h and C are constants. Compute σ_{ij} as a function of (x_1, x_2, x_3) from (10). Also compute the components of the vector $\partial \sigma_{ij}/\partial x_j$. If there are no body forces and μ is constant, what does C have to be in order for the fluid to be in steady state?* Hint: If the fluid is in steady state, then $\partial u_i/\partial t = 0$ in (1b). *Sketch the velocity field in the x_1x_3-plane.*

Note 2 *The combination of velocity gradients*

$$\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}$$

occurs frequently enough to be given its own symbol. Specifically, the strain rate tensor D_{ij} *is defined through*

$$D_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

(10) can then be written as

$$\sigma_{ij} = 2\mu\left(D_{ij} - \frac{2}{3}D_{kk}\delta_{ij}\right) - p\delta_{ij}.$$

Often, the first term on the right-hand side is referred to as the deviatoric stress τ_{ij} *defined through*

$$\tau_{ij} = \sigma_{ij} + p\delta_{ij} \tag{11}$$

with p given by (9), so that

$$\tau_{ij} = 2\mu\left(D_{ij} - \frac{1}{3}D_{kk}\delta_{ij}\right).$$

[2] As with many other fluids, the viscosity of water is temperature-dependent, and decreases somewhat with increasing temperature. This is even more pronounced with supersaturated sugar solutions like honey or corn syrup: these are very sticky at low temperatures, with high viscosity, and become runnier, with lower μ, at higher temperatures.

Note 3 *The mean normal force generated by the deviatoric stress is zero. From (11), it follows that by setting $i = j$ and summing,*

$$\tau_{ii} = \sigma_{ii} + p\delta_{ii}$$

But $\delta_{11} = \delta_{22} = \delta_{33} = 1$, so $\delta_{ii} = 3$ and

$$\tau_{ii} = \sigma_{ii} + 3p$$

But $p = -\sigma_{ii}/3$ by definition, so

$$\tau_{ii} = \tau_{11} + \tau_{22} + \tau_{33} = 0.$$

(10) is the general form of stress for a viscous fluid. The definition of pressure is however not unique: it was originally intended to ensure that there is a stress field of the form $\sigma_{ij} = -p\delta_{ij}$ when there are no velocity gradients. We could easily define a new pressure variable

$$\bar{p} = p + \mu_b \frac{\partial u_k}{\partial x_k} \tag{12}$$

for some coefficient μ_b, and obtain from (10) that

$$\sigma_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij} \right) + \mu_b \frac{\partial u_k}{\partial x_k}\delta_{ij} - \bar{p}\delta_{ij} \tag{13}$$

The stress tensor then still reduces to $\sigma_{ij} = -\bar{p}\delta_{ij}$ when the fluid is at rest (in which case $p = \bar{p}$), and also predicts (6) for the geometry considered there. However, $\bar{p}$ does not satisfy the usual continuum mechanical definition of pressure in (9).

There are situations in which a pressure $\bar{p}$ is more useful than p defined in (9). In compressible gases, for instance, density is usually taken to be a function of pressure. However, the pressure that appears in the constitutive relation for ρ is not the continuum mechanical pressure defined by (9) but the thermodynamic pressure. This is generally related to the continuum mechanical pressure p through a relation of the form (12), with $\bar{p}$ denoting the thermodynamic pressure and μ_b known as the *bulk viscosity*. To offset the original viscosity μ from μ_b, μ is sometimes referred to as the *dynamic viscosity*.

When $\bar{p}$ is thermodynamic pressure, the term

$$\tau^e_{ij} = \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\frac{\partial u_k}{\partial x_k}\delta_{ij} \right) + \mu_b \frac{\partial u_k}{\partial x_k}\delta_{ij}$$

in (13) is usually referred to as the *extra stress*. This is similar to deviatoric stress, but unlike the latter, we generally have $\tau^e_{ii} \neq 0$. In terms of τ^e_{ij} and $\bar{p}$ we have

$$\sigma_{ij} = \tau^e_{ij} - \bar{p}\delta_{ij}.$$

Incompressible Fluids and the Navier-Stokes equations

A frequently encountered type of viscous fluid is an incompressible fluid: most liquids, for instance, are nearly incompressible. This implies that

$$\rho = \rho_0 = \text{constant}.$$

Conservation of mass (1a) can then be reduced to

$$\frac{\partial u_i}{\partial x_i} = 0 \tag{14}$$

With this in hand, the viscous stress prescription (10) can be written as $\sigma_{ij} = \mu(\partial u_i/\partial x_j + \partial u_j/\partial x_i) - p\delta_{ij}$, and (1b) becomes

$$\rho_0 \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) = \frac{\partial}{\partial x_j} \left[\mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] - \frac{\partial p}{\partial x_i} + f_i \tag{15}$$

Further simplification of (15) is possible if μ is a constant (i.e., if the fluid is Newtonian). μ can then be taken outside the derivative:

$$\rho_0 \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) = \mu \left(\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{\partial^2 u_j}{\partial x_j \partial x_i} \right) - \frac{\partial p}{\partial x_i} + f_i. \tag{16}$$

But

$$\frac{\partial^2 u_j}{\partial x_j \partial x_i} = \frac{\partial^2 u_j}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i} \left(\frac{\partial u_j}{\partial x_j} \right) = 0$$

as $\partial u_j/\partial x_j = 0$. This allows us to simplify (16) to

$$\rho_0 \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) = \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} - \frac{\partial p}{\partial x_i} + f_i. \tag{17}$$

(14) and (17) together are usually referred to as the *Navier-Stokes equations*, and are widely used to describe the behaviour of simple incompressible liquids like water.

It turns out that the Navier-Stokes equations as stated above can be written in standard vector notation, as the divergence of the stress tensor on the right-hand side of (17) involves only the Laplacian of velocity u_i and the gradient of pressure p. In standard vector notation, we obtain

$$\begin{aligned} \nabla \cdot \mathbf{u} &= 0, \\ \rho_0 \left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} \right) &= \mu \nabla^2 \mathbf{u} - \nabla p + \mathbf{f}. \end{aligned}$$

We persist with the index notation version below, but it is important that you can recognize this form of the Navier-Stokes equations as being the same as (14) and (17), since you may encounter the vector notation version in other pieces of writing about fluid dynamics. Note that the brackets around $\mathbf{u} \cdot \nabla$ are also often omitted in the second vector-notation equation above.

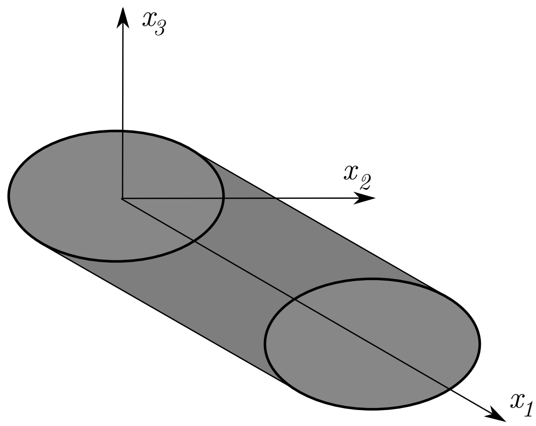

Figure 3: Geometry for a unidirectional flow in a pipe along the x_1-axis.

Unidirectional flow

Having elaborately constructed a theory for viscous flow, this is a good time to actually calculate a solution. Consider an incompressible, Newtonian fluid whose flow is purely in the x_1-direction, for instance through a pipe of uniform cross-section that is parallel to the x_1-axis (see figure 3). To make things easier, suppose that the body force f_i is a constant, and assume a steady state so the flow does not depend on time. Then

$$u_1 = u_1(x_1, x_2, x_3), \qquad u_2 = u_3 = 0.$$

The mass conservation equation (14) now becomes

$$\frac{\partial u_1}{\partial x_1} = 0$$

so $u_1 = u_1(x_2, x_3)$ does not depend on the along-flow direction x_1.

Exercise 4 *Suppose that the pipe is inclined at an angle θ to the horizontal, pointing downwards in the positive x_1-direction, and let x_2 be transverse to the slope (so the x_2-axis is horizontal). If ρ_0 is fluid density and g is acceleration due to gravity, and body forces are purely gravitational, what are f_1, f_2 and f_3?*

Next, we consider the momentum conservation equation (17). Pick $i = 2$ first. With $u_2 = 0$, this simply becomes

$$\frac{\partial p}{\partial x_2} = f_2,$$

where f_2 is, by assumption, constant. Similarly, with $i = 3$, we have

$$\frac{\partial p}{\partial x_3} = f_3.$$

If we define a slightly altered pressure variable $P = p - f_2 x_2 - f_3 x_3$, we have

$$\frac{\partial P}{\partial x_2} = \frac{\partial P}{\partial x_3} = 0,$$

and $P = P(x_1)$ depends only on the along-flow direction.

Next, take $i = 1$ in (17), and substitute $p = P + f_2 x_2 + f_3 x_3$. We get, with $u_2 = u_3 = 0$ and $\partial u_1/\partial x_1 = 0$,

$$0 = \mu \frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2} - \frac{\partial P}{\partial x_1} + f_1 \tag{18}$$

Exercise 5 *Show explicitly that (18) holds, by demonstrating that the terms on the left-hand side of (17) vanish here for $i = 1$.*

We can now also show that P must depend linearly on position x_1. Differentiate (18) with respect to x_1:

$$0 = \mu \frac{\partial^3 u_1}{\partial x_1 \partial x_2^2} + \frac{\partial^2 u_1}{\partial x_1 \partial x_3^2} - \frac{\partial^2 P}{\partial x_1^2}$$

as f_1 is assumed to be constant. But

$$\frac{\partial^3 u_1}{\partial x_1 \partial x_2^2} = \frac{\partial^3 u_1}{\partial x_2^2 \partial x_1} = \frac{\partial^2}{\partial x_2^2}\left(\frac{\partial u_1}{\partial x_1}\right) = 0$$

on account of (14). Similarly,

$$\frac{\partial^3 u_1}{\partial x_1 \partial x_3^2} = 0,$$

leaving

$$\frac{\partial^2 P}{\partial x_1^2} = 0.$$

But we already known that $P = P(x_1)$, so the only possibility now is that $P = P_0 - C x_1$ with P_0 and C constant. We put a minus sign because we expect fluid to flow from high to low pressure, so pressure decreasing along the x_1-axis should facilitate fluid flowing in the positive x_1-direction; the opposite case is of course also covered if we simply make C negative.

Exercise 6 *With the same setting as (4), what form does pressure $p(x_1, x_2, x_3)$ take? Can you interpret any part of this as a 'hydrostatic pressure'?*

Note that

$$C = -\frac{\partial P}{\partial x_1} = -\frac{\partial p}{\partial x_1}$$

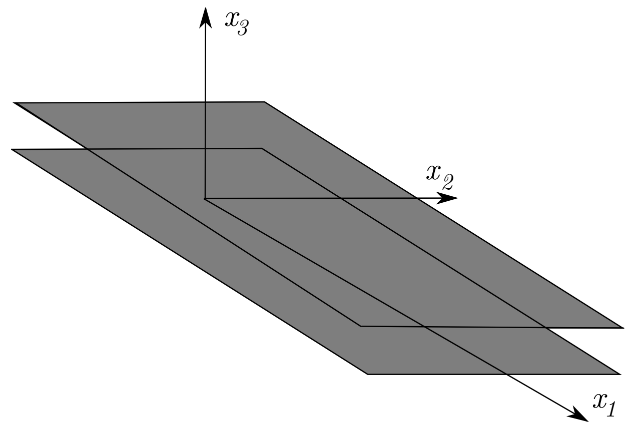

Figure 4: Geometry for flow between parallel plates x_1-axis.

is the negative pressure gradient. Substituting this back in (18) gives

$$0 = \mu \left(\frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2} \right) + C + f_1$$

or, if we combine the component of body force in the x_1-direction and the pressure gradient into one symbol

$$f = C + f_1,$$

we get

$$-\mu \left(\frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2} \right) = f \tag{19}$$

with f constant.

The term on the left is the two-dimensional Laplacian of $u_1(x_2, x_3)$; in more traditional (x, y, z)-coordinates and with $u = u_1$, we would have written

$$-\mu \left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = f. \tag{20}$$

This problem is relatively easy to solve in simple geometries, provided we also provide appropriate boundary conditions on u_1. Take for instance flow between parallel plates (figure 4). If the plates are at $x_3 = 0$ and $x_3 = h$, then by symmetry we expect $u_1 = u_1(x_3)$. If we also assume that the fluid cannot slip over the plates, then we have $u_1(0) = u_1(h) = 0$, and therefore

$$-\mu \frac{\mathrm{d}^2 u_1}{\mathrm{d} x_3^2} = f, \qquad u_1(0) = u_1(h) = 0 \tag{21}$$

with f constant. Simple integration and appliation of the boundary conditions gives

$$u_1(x) = -\frac{f}{2\mu} x^2 + ax + b, \qquad u_1(0) = b = 0, \qquad u_1(h) = -\frac{f}{2\mu} h^2 + ah = 0 \tag{22}$$

so $a = fh/(2\mu)$ and

$$u_1(x) = -\frac{f}{2\mu}x^2 + \frac{fh}{2\mu}x = \frac{fx(h-x)}{2\mu} \tag{23}$$

The flow velocity half-way between the pipes is therefore $u_1(h/2) = fh^2/(8\mu)$, which increases quadratically with the spacing between the plates, and decreases with the viscosity (or stickiness) of the fluid.

Exercise 7 *We can also compute the flow in a circular pipe. Suppose that the pipe is centered on the x_1 axis, and the velocity depends only on distance r from the centre of the pipe. It is then appropriate to switch to cylindrical polar coordinates with $x_2 = r\cos(\theta)$ and $x_3 = r\sin(\theta)$. We have $u_1 = u_1(r)$ and the Laplacian of u_1 can be written in the form*

$$\frac{\partial^2 u_1}{\partial x_2^2} + \frac{\partial^2 u_1}{\partial x_3^2} = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u_1}{\mathrm{d}r}\right) \tag{24}$$

As above, suppose that the fluid is stuck at the walls of the pipe, so $u_1 = 0$ there, If the radius of the pipe is R, solve (18) to derive

$$u_1(r) = \frac{f(R^2 - r^2)}{4\mu}$$

The rate Q at which water comes out of the pipe, measured in mass per unit time, can be computed from the surface integral

$$Q \int_S \rho u_i n_i \, \mathrm{d}S$$

over a cross-section across the pipe. Find Q in terms of R, f and μ. If you double the width of a fire hose and keep its length and the water pressure in the water mains the same, how much will the rate at which water comes out of the hose be increased by if the calculation above is correct? If 10^{-3} Pa s, $R = 5$ cm and the flow is driven purely by a pressure gradient along a 100 m long hose, with a mains pressure of 350 kPa, what water flow rate in kilograms per second and cubic metres per second would you calculate?

Note that with typical water flow, the assumption of a unidirectional flow in which $u_2 = u_3 = 0$ actually breaks down; the flow will spontaneously develop so-called eddies.

Boundary conditions

The Navier-Stokes equations (14) and (15) require as many scalar boundary conditions at each boundary as there are velocity components — so three if we are in three dimensions.

There are some boundary conditions that occur frequently, For instance, when a fluid is in contact with a rigid wall, one often assumes that the fluid is stuck to the wall, in which case

$$u_i = 0.$$

Where a fluid is in contact with a vacuum (or an inviscid fluid at negligible pressure), there is then no force at that interface, so $\delta F_i = \sigma_{ij} n_j \delta S = 0$ and hence[3]

$$\sigma_{ij} n_j = 0. \tag{25}$$

There are some cases in which there may be contact with a rigid wall, but slip between the wall and the fluid is possible. In that case, we have no fluid penetrating into the wall, so

$$u_i n_i = 0. \tag{26}$$

But this is only one scalar boundary condition, and we cannot say that the tangential component of velocity is zero if there is slip. Instead, we may assume that the tangential component of velocity is related in some way to the tangential component of stress at the boundary.

We need to make more precise what we mean by this. The force on a small element of boundary is $\delta F_i = \sigma_{ij} n_j \delta S$. The tangential component of a vector can be computed by subtracting the normal component from the vector. The *length* of the normal component of a vector is $a_j n_j$, and its direction is n_i, so the written as a *vector*, the normal component is $a_j n_j n_i$. The tangential component of $\mathbf{a}$ is therefore $a_i - a_j n_j n_i = a_j(\delta_{ij} - n_i n_j)$.[4] The tangential component of δF_i is therefore $\delta F_j(\delta_{ij} - n_i n_j)$. Taking care not to repeat indices more than is permitted, this becomes

$$\delta F_j(\delta_{ij} - n_i n_j) = \sigma_{jk} n_k (\delta_{ij} - n_i n_j) \delta S.$$

We can then define the shear stress at the surface as being the tangential component of force divided by the small element of surface δS it acts on, so

$$\tau_{s,i} = \sigma_{jk} n_k (\delta_{ij} - n_i n_j).$$

This is the shear stress that acts *on the fluid*, with the usual outward-pointing sign convention for n_i.

For a boundary where there is slip, we may expect that this shear stress opposes the motion, and so is directed in the direction of $-\mathbf{u}$. The unit vector in that direction is $u_i/|\mathbf{u}|$. The magnitude of the shear stress depends on the physics of the contact between fluid and rigid wall. A typical assumption is that it depends on the magnitude of velocity, and is an increasing function $f(|\mathbf{u}|)$ of $|\mathbf{u}|$. In that case, we would have

$$\sigma_{jk} n_k (\delta_{ij} - n_i n_j) = f(|\mathbf{u}|) u_i / |\mathbf{u}|. \tag{27}$$

[3]There are reasons why this may not hold; for instance, if surface tension is significant, the right-hand side of (25) is not zero but given by the curvature of the fluid surface.

[4]The tensor $\delta_{ij} - n_i n_j$ can therefore be thought of as a tensor that *projects* vectors onto a surface.

This provides the remaining two scalar boundary condition. It may look like this is a vector equation, and so should have three components. However, we can show that (by construction) the vectors on both sides have zero component parallel to $\mathbf{n}$, and so effectively only have two independent components. Specifically, we have for the left-hand side

$$\sigma_{jk}n_k(\delta_{ij} - n_i n_j)n_i = \sigma_{jk}n_k(n_j - n_i n_i n_j) = \sigma_{jk}n_k(n_j - n_j) = 0$$

as $n_i n_i = 1$, and so the tangential component of $\sigma_{ij}n_j$ really does have zero component parallel to n_i. For the right-hand side of (27)

$$f(|\mathbf{u}|)u_i/|\mathbf{u}|n_i = [f(|\mathbf{u}|)/|\mathbf{u}|]u_i n_i = 0$$

as $u_i n_i = 0$ from (26).

Note 4 *Note that the shear stress $\sigma_{jk}n_k(\delta_{ij} - n_i n_j)$ can actually be computed from the deviatoric part τ_{ij} of the stress tensor σ_{ij} alone: we have $\sigma_{ij} = \tau_{ij} - p\delta_{ij}$ and hence*

$$\begin{aligned}\sigma_{jk}n_k(\delta_{ij} - n_i n_j) &= (\tau_{jk} - p\delta_{jk})n_k(\delta_{ij} - n_i n_j) \\ &= \tau_{jk}n_k(\delta_{ij} - n_i n_j) - p n_j(\delta_{ij} - n_i n_j) \\ &= \tau_{jk}n_k(\delta_{ij} - n_i n_j) - p(n_i - n_i n_j n_j) \\ &= \tau_{jk}n_k(\delta_{ij} - n_i n_j)\end{aligned}$$

since $n_j n_j = 1$.

Note 5 *The nature of the boundary conditions imposed actually turns out to be just as important as the Navier-Stokes equations themselves in determining a solution. As usual, these boundary conditions contain information about physics at the boundary that is not contained in the partial differential equations (14) and (15) themselves. For instance, if we had tried to insist on zero tangential force at the wall in the parallel-plate example above, we would have found the boundary conditions $\mathrm{d}u_1/\mathrm{d}x_3 = 0$ on $x_3 = 0$, h. But we would not have been able to solve the problem in that case. Can you show this, and think of a reason why? What assumption that we made previously must fail?* Hint: Think of this as a mechanics problem: in steady state, all components of force must balance. Which ones can we not balance with zero tangential stress boundary conditions?

When the fluid is contact with a vacuum and (25) applies, it is generally the case that the surface of the fluid also evolves in time. This is a common occurrence, with obvious examples being waves on the ocean, or the surface of a mountain stream. A free surface that can evolve requires an equation that describes that evolution, known as a *kinematic boundary condition*. We can derive such a condition simply. A surface is generally parameterized by $F(x_1, x_2, x_3) = 0$ for some function F (the surface is then the 0 isosurface or zero contour of F). If the surface can evolve in time, this should be written as $F(x_1, x_2, x_3, t) = 0$. A simple example for a surface that can be written as a height above the x_1x_2-plane through $x_3 = h(x_1, x_2, t)$ would be $F(x_1, x_2, x_3, t) = h(x, y, t) - z$.

The surface of the fluid is also a material surface, meaning that it moves at the velocity $\mathbf{u}$ of the material there. Take a material point $(x(t), y(t), z(t))$ on the surface. For this point, we have

$$F(x_1(t), x_2(t), x_3(t), t) = 0$$

for all times t, and therefore

$$\frac{\mathrm{d}}{\mathrm{d}t} F(x_1(t), x_2(t), x_3(t), t) = 0.$$

Applying the chain rule, we get

$$\frac{\partial F}{\partial t} + \nabla F \cdot \left(\frac{\mathrm{d}x_1}{\mathrm{d}t}, \frac{\mathrm{d}x_2}{\mathrm{d}t}, \frac{\partial x_3}{\partial t} \right) = 0.$$

But the time derivative of $(x_1(t), x_2(t), x_3(t))$ is simply the velocity $\mathbf{u}$, so

$$\frac{\partial F}{\partial t} + \nabla F \cdot \mathbf{u} = 0.$$

For the surface given by $F(x_1, x_2, x_3, t) = h(x_1, x_2, t) - x_3$, this becomes

$$\frac{\partial h}{\partial t} + u_1 \frac{\partial h}{\partial x_1} + u_2 \frac{\partial h}{\partial x_2} = u_3.$$