EOS 352 Continuum Dynamics
Conservation of momentum

April 30, 2014

Overview

These notes cover the following

- Conserving vector-valued quantities by conserving each component
- Fluxes for vector-valued quantities: tensors
- Momentum conservation: local form of Newton's second law
- Surface forces and stresses
- Body forces

Conservation laws for scalar quantities in subscript notation

Recall that, in order to write down conservation laws for scalar quantities, we used the following steps: If Φ is a conserved quantity and ϕ the associated density, then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \phi \,\mathrm{d}V = -\int_{S(t)} \mathbf{q}\cdot\mathbf{n}\,\mathrm{d}S + \int_{V(t)} a\,\mathrm{d}V,$$

where $V(t)$ is a Lagrangian volume with surface $S(t)$ and outward-pointing surface normal $\mathbf{n}$, while $\mathbf{q}$ is a conductive flux, and a is a supply rate density. Using Reynolds' transport theorem, the left-hand side becomes

$$\int_{V(t)} \frac{\partial \phi}{\partial t}\,\mathrm{d}V + \int_{S(t)} \phi\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S = -\int_{S(t)} \mathbf{q}\cdot\mathbf{n}\,\mathrm{d}S + \int_{V(t)} a\,\mathrm{d}V,$$

and with the divergence theorem and some rearrangement, we get

$$\int_{V(t)} \left[\frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \mathbf{u}) + \nabla \cdot \mathbf{q} - a \right] \mathrm{d}V = 0,$$

and a slightly subtle argument about V being arbitrary finally leads to the conclusion that this last integrand must be identically equal to zero (i.e., zero everywhere and at all times), and hence to the differential equation

$$\frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \mathbf{u}) + \nabla \cdot \mathbf{q} = a$$

Now, it may not be obvious immediately how to translate this to the case where Φ is itself vector-valued. As a first step, we can translate the derivation above into subscript notation, however. This is straightforward, simply replacing dot products and divergences with sums over repeated indices. The basic conservation law statement in integral form is then

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{V(t)} \phi \, \mathrm{d}V = - \int_{S(t)} q_j n_j \, \mathrm{d}S + \int_{V(t)} a \, \mathrm{d}V. \tag{1}$$

Note that we could equally have written $q_i n_i$ instead of $q_j n_j$ on the right because j is a dummy index; the choice of j is made because it will make our notation later a bit simpler. Here and in everything that follows, we apply the summation convention, so $q_j n_j$ really stands for $\sum_{j=1}^{3} q_j n_j$. Reynolds' transport theorem then renders (1) as

$$\int_{V(t)} \frac{\partial \phi}{\partial t} \mathrm{d}V + \int_{S(t)} \phi u_j n_j \, \mathrm{d}S = - \int_{S(t)} q_j n_j \, \mathrm{d}S + \int_{V(t)} a \, \mathrm{d}V, \tag{2}$$

and the divergence theorem as

$$\int_{V(t)} \left[\frac{\partial \phi}{\partial t} + \frac{\partial (\phi u_j)}{\partial x_j} + \frac{\partial q_j}{\partial x_j} - a \right] \mathrm{d}V = 0 \tag{3}$$

with the same argument about the arbitrariness of $V(t)$ leading to

$$\frac{\partial \phi}{\partial t} + \frac{\partial (\phi u_j)}{\partial x_j} + \frac{\partial q_j}{\partial x_j} = a \tag{4}$$

For conservation of mass, we have $\phi = \rho$ as the ordinary mass density, while conductive flux and supply are zero, $q_i = 0$ and $a = 0$. Hence

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho u_j)}{\partial x_j} = 0. \tag{5}$$

Exercise 1 *Write (5) out explicitly, i.e., expand the sums over j so that all indices explicitly are 1, 2 and 3 and no j's are left (this is analogous to writing out $a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3$).*

Conservation of vector-valued quantities: fluxes as tensors

Next, we would like to write down a similar set of steps to describe conservation of momentum. But how do we conserve a vector-valued quantity? The answer naturally is that we conserve each component of that quantity separately. In other words, we conserve x-, y- and z-components of momentum, except that we should now call them x_1-, x_2- and x_3-components of momentum. We can treat each of those just like we would treat a scalar quantity. That is, we can take Φ to be each of those components in turn.

A bit of care is required as we do this. Take Φ to be the x_1-component of momentum first. We need to begin by figuring out the associated density ϕ_1.

Note 1 *If you have made it to this point, you should hopefully have enough physics background to know that the momentum of a point particle (i.e., an object of finite mass but zero spatial extent) with mass m and velocity $\mathbf{u}$ is the vector-valued quantity $\mathbf{p} = m\mathbf{u}$, where $\mathbf{p}$ is the usual symbol for denoting momentum. In subscript notation, therefore, $p_i = mu_i$.*

If the momentum of a small mass δm travelling at velocity $\mathbf{u}$ is $(\delta m)\mathbf{u}$, then its x_1-component of momentum (or 'x_1-momentum' for short) is $(\delta m)u_1$. The density associated with x_1-momentum is therefore

$$\phi_1(x_1, x_2, x_3, t) = \frac{x_1\text{-momentum contained in a small volume } \delta V \text{ around } (x_1, x_2, x_3) \text{ at time } t}{\delta V}$$

More specifically,

$$\phi_1 = \frac{\delta m u_1}{\delta V} = \rho u_1$$

if we recognize that δm must be the mass contained in δV and therefore $\rho = \delta m/\delta V$ is the ordinary mass density.

Following the procedure for a generic conserved scalar quantity Φ above, we can therefore write that conservation of x_1-momentum should take the form

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \phi_1 \,\mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \rho u_1 \,\mathrm{d}V = -\int_{S(t)} \mathbf{q}_1 \cdot \mathbf{n}\,\mathrm{d}S + \int_{V(t)} a_1 \,\mathrm{d}V$$

where we have put an extra label '1' on the conductive flux $\mathbf{q}_1$ and the supply rate density a_1 to indicate that we are looking specifically x_1-momentum. But in subscript notation, this is simply

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \rho u_1 \,\mathrm{d}V = -\int_{S(t)} q_{1j} n_j \,\mathrm{d}S + \int_{V(t)} a_1 \,\mathrm{d}V \tag{6}$$

where the labels 1 still indicate that we are concerned in each case with the x_1-component of momentum, whereas the index j on q_{1j} and n_j indicate the jth component of the conductive flux $\mathbf{q}_1$ and of the normal vector $\mathbf{n}$, respectively. In other words, $\mathbf{q}_1 = (q_{11}, q_{12}, q_{13})$. Note that (6) is simply (1) with $\phi = \phi_1 = \rho u_1$ and extra labels '1' affixed to q_j and a.

Following the same program as above and applying the Reynolds' transport theorem again, we have

$$\int_{V(t)} \frac{\partial(\rho u_1)}{\partial t}\,\mathrm{d}V + \int_{S(t)} (\rho u_1) u_j n_j\,\mathrm{d}S = -\int_{S(t)} q_{1j} n_j\,\mathrm{d}S + \int_{V(t)} a_1\,\mathrm{d}V, \tag{7}$$

which is simply (2) with $\phi = \phi_1 = \rho u_1$ and labels '1' again attached to q_j and a. Note that the second term on the left-hand side, $\int_{S(t)}(\rho u_1)u_j n_j\,\mathrm{d}S$, really does have two occurences of the velocity field u_i: once as part of the momentum density ρu_1, and once from the normal velocity component $u_j n_j$ that is involved in the evolution of the Lagrangian volume $V(t)$ over time. We will discuss this in more detail later.

Subsequently using the divergence theorem, we get

$$\int_{V(t)} \left[\frac{\partial(\rho u_1)}{\partial t} + \frac{\partial(\rho u_1 u_j)}{\partial x_j} + \frac{\partial q_{1j}}{\partial x_j} - a_1\right]\mathrm{d}V = 0 \tag{8}$$

and finally, because V is arbitrary, we must have

$$\frac{\partial(\rho u_1)}{\partial t} + \frac{\partial(\rho u_1 u_j)}{\partial x_j} + \frac{\partial q_{1j}}{\partial x_j} = a_1. \tag{9a}$$

Exercise 2 *Write (9a) out explicitly, i.e., expand the sums over j so that all indices explicitly are 1, 2 and 3 and no j's are left (this is analogous to writing out $a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3$).*

We will consider this physically in more detail in a moment. First, we should recognize that we can follow exactly the same procedure for the other two components of momentum. For instance, the density associated with the x_2-component of momentum is ρu_2, and we can denote the conductive flux of x_2 momentum by $\mathbf{q}_2 = (q_{21}, q_{22}, q_{23})$ and to produce

$$\frac{\partial(\rho u_2)}{\partial t} + \frac{\partial(\rho u_2 u_j)}{\partial x_j} + \frac{\partial q_{2j}}{\partial x_j} = a_2, \tag{9b}$$

where it is important to recognize that there is no reason why the conductive flux of x_1-momentum should be the same as the conductive flux of x_2-momentum, so in general $\mathbf{q}_1 \not\equiv \mathbf{q}_2$, or $q_{1i} \not\equiv q_{2i}$ in subscript notation. The same goes for the supply rate density, where we generally expect $a_1 \not\equiv a_2$. We can proceed similarly for x_3-momentum, finding

$$\frac{\partial(\rho u_3)}{\partial t} + \frac{\partial(\rho u_3 u_j)}{\partial x_j} + \frac{\partial q_{3j}}{\partial x_j} = a_3, \tag{9c}$$

But looking at the three equations (9), they can generically be written as

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j} + \frac{\partial q_{ij}}{\partial x_j} = a_i \tag{10}$$

with $i = 1$, $i = 2$ and $i = 3$. Here we finally meet a new type of object, q_{ij}, which has two subscript indices. For a fixed i, q_{ij} is a vector whose components are labelled by j, and this vector physically is the conductive flux of i-momentum. In other words, q_{ij} is the conductive flux that carries x_i-momentum along the x_j-direction.[1] As there are three components of momentum, there are three such vectors, which in classical vector notation we called $\mathbf{q}_1$, $\mathbf{q}_2$ and $\mathbf{q}_3$, and q_{ij} combines these fluxes into a single object that you can think of as a flux of a vector-valued quantity. For a scalar-valued conserved quantity, flux is a single vector with three components. For a vector-valued conserved quantity like momentum, we now have nine flux components, and the notation with two subscripts already suggests (at least if you have some background in matrix algebra) that an object like q_{ij} can best be represented as a matrix rather than a vector,

$$\begin{pmatrix} q_{11} & q_{12} & q_{13} \\ q_{21} & q_{22} & q_{23} \\ q_{31} & q_{32} & q_{33} \end{pmatrix}$$

In this matrix, each row corresponds to one of the conductive flux vectors $\mathbf{q}_1$, $\mathbf{q}_2$ and $\mathbf{q}_3$. An object like this, with more than one subscript index, is generally called a *tensor*.

Conductive momentum flux: Newton's second law and surface forces

The statement that q_{ij} is the conductive flux of x_i-momentum in the x_j-direction is probably quite hard to grasp. All of the discussion above is very abstract, as we have not really developed a good understanding of what a 'conductive flux of momentum' and a 'supply of momentum' should mean. We will deal with this next, and show how different terms in (6) and (10) can be interpreted.

Note 2 *Recall that, in classical point particle dynamics, Newton's second law says that the net force on a particle equals its mass times its acceleration,*

$$\mathbf{F} = m\mathbf{a} = m\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t}$$

[1] The idea that x_i-momentum can be conducted in the x_j-direction tends to be hard to grasp at first. We will return to this very shortly, in the next section.

if we recognize that acceleration is defined as the rate at which the particle velocity changes over time. A better way to write Newton's second law (as the above assumes a fixed particle mass m) is that

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\mathbf{u}) = \mathbf{F} \tag{11}$$

where $m\mathbf{u} = \mathbf{p}$ is the momentum of the particle. In subscript notation, this would be

$$\frac{\mathrm{d}}{\mathrm{d}t}(mu_i) = F_i.$$

or

$$\frac{\mathrm{d}p_i}{\mathrm{d}t} = F_i.$$

If we have multiple particles of, say, mass m_A, m_B etc. and with velocities $\mathbf{u}_A$, $\mathbf{u}_B$ etc, then for each we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(m_P\mathbf{u}_P) = \mathbf{F}_P$$

where P is a generic label for the particle (so P can be A, B etc — which we use here to label particles instead of numbers so that we do net get confused over what labels particles and what labels vector components). The total momentum of all the particles is then $\mathbf{p}_{\mathrm{tot}} = \sum_P m_P\mathbf{u}_P$ and the total force is $\mathbf{F}_{\mathrm{tot}} = \sum_P \mathbf{F}_P$. But summing the last equation above then gives

$$\frac{\mathrm{d}\mathbf{p}_{\mathrm{tot}}}{\mathrm{d}t} = \mathbf{F}_{\mathrm{tot}}.$$

Recall equation (6), which we can generalize for conservation of momentum in the i-direction as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \rho u_i \,\mathrm{d}V = -\int_{S(t)} q_{ij}n_j \,\mathrm{d}S + \int_{V(t)} a_i \,\mathrm{d}V \tag{12}$$

But the left-hand side is simply the rate of change of momentum contained in the volume $V(t)$, written in component notation notation. As $V(t)$ is a Lagrangian volume, it always contains the same bits of matter, or, in the language of note 2, the same 'particles'. It therefore follows that the right-hand side of (6) is simply the total force acting on the volume $V(t)$ at time t, and (12) is a statement of Newton's second law for a continuum (where we do not actually have point particles, but mass and therefore momentum are spread out in space).

This total force has two parts, $-\int_{S(t)} q_{ij}n_j \,\mathrm{d}S$ and $\int_{V(t)} a_i \,\mathrm{d}V$. Remember that $-\int_{S(t)} q_{ij}n_j \,\mathrm{d}S$ is the effect of conduction of momentum through the boundary $S(t)$ of $V(t)$. If we were talking about heat rather than momentum, this would represent the transfer of heat into $V(t)$ due to conduction through the contact with surrounding material. In the case of momentum, this force must therefore be the result of momentum transfer due to contact with surrounding material across the surface S.

The simplest way to understand the term $-\int_{S(t)} q_{ij} n_j \,\mathrm{d}S$ is therefore as a contact force exerted on a surface: this could for instance be the force exerted by a table on a stationary cup of coffee on top of it, or the force exerted by the cup on the stationary coffee inside it. It could also be the force between a road and a car tire as the car is braking.

Recall that the integral $-\int_{S(t)} q_{ij} n_j \,\mathrm{d}S$ really comes from a sum $-\sum_{\delta S} q_{ij} n_j \delta S$ over small surface elements. In other words, if $-\sum_{\delta S} q_{ij} n_j \delta S$ is a total surface force, then each element of surface δS experiences a force

$$\delta F_i = -q_{ij} n_j \delta S, \tag{13}$$

exerted on the volume V.

Exercise 3 *Show that the components of the tensor q_{ij} have dimensions of force over area, and therefore units of Pascals.*

In equation (13), δF_i is the force exerted on the volume V, whereas the usual definition of a conductive flux $\mathbf{q}$ and of $\mathbf{n}$ as an outward-pointing unit normal makes $\mathbf{q} \cdot \mathbf{n} \delta S$ equal to the rate of conduction *out of* V. This explains why the force exerted on V, or equally, the the rate of conduction of momentum *into* V, is $-q_{ij} n_j \delta S$. In continuum mechanics, it is often useful not to have to deal with the negative sign involved. By convention, we therefore define a new quantity through

$$\sigma_{ij} = -q_{ij}$$

The symbol σ on the left is the Greek letter sigma, and σ_{ij} is called the *stress tensor*. In terms of the stress tensor, the surface force exerted on V through the a surface element δS is then

$$\delta F_i = \sigma_{ij} n_j \delta S \tag{14}$$

and the total surface force on the volume V is

$$F_i = \int_S \sigma_{ij} n_j \,\mathrm{d}S.$$

Exercise 4 *Calculate $\sigma_{ij} n_j$ if σ_{ij} can be represented by the matrix*

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

and n_i is normal to the triangle S with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$, with n_i pointing out of the volume V lying below that triangle. To do this, you can write out (14) explicitly for each $i = 1, 2, 3$. You will need to know the components n_i, which you will have to compute from knowing the shape of the triangle S.

Then compute the force (F_1, F_2, F_3) acting on the volume V through the triangle with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

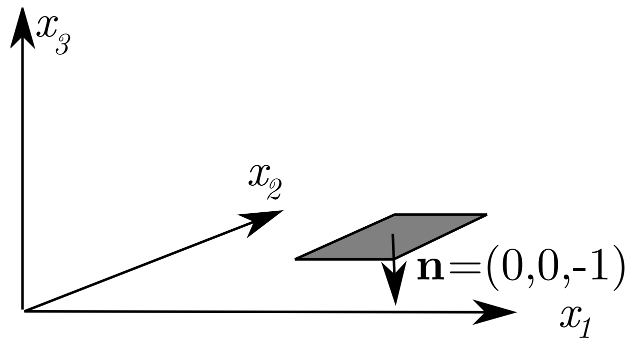

Figure 1: A surface element in the x_1x_2-plane with downward-pointing unit normal.

Note 3 *If you are familiar with matrix algebra, you will recognize that the vector* $\sigma_{ij}n_j$ *(which is equal to* $\sum_{j=1}^{3}\sigma_{ij}n_j$ *by the summation convention) is simply the product* $\boldsymbol{\sigma}\mathbf{n}$*, where* $\boldsymbol{\sigma}$ *is the matrix with entries* σ_{ij}*, and* $\mathbf{n}$ *is the column vector* $(n_1, n_2, n_3)^{\mathrm{T}}$.

Often one of the most difficult things to grasp when first looking at conservation of momentum is how x_i-momentum can be conducted in the x_j-direction: that is, why do we need to have q_{ij} components with $i \neq j$. Having translated conduction of momentum into surface forces, we can easily illustrate why this has to be the case. Take the difference between the force exerted by a coffee cup on the coffee within it (which we will assume to be at rest), and the force a road exerts on a braking car tire.

We may expect that the surface force δF_i exerted by the coffee cup on the liquid coffee at rest within it across a surface element δS should be perpendicular to the surface — there is no reason why there should be a tangential force. Consider an element of surface δS that lies in the x_1x_2-plane (i.e., the xy plane in non-subscript notation), with the coffee above the x_1x_2-plane. Then $n_1 = n_2 = 0$, $n_3 = -1$, as the normal n_i has to be outward-pointing to the coffee in order to calculate the force on the coffee. Then

$$\delta F_1 = \sum_{j=1}^{3} \sigma_{1j} n_j \delta S = -\sigma_{13}\delta S, \tag{15a}$$

and similarly

$$\delta F_2 = -\sigma_{23}\delta S, \qquad \delta F_3 = -\sigma_{33}\delta S \tag{15b}$$

But we have just said that we expect the force to be purely normal. The normal is parallel to the x_3-axis, and the force δF_i should be too. In other words, the tangential components δF_1 and δF_2 should be zero, so $\sigma_{13} = \sigma_{23} = 0$. In terms of the original conductive flux, this also means $q_{13} = q_{23} = 0$. To use the original language of fluxes, there is now no conductive flux of x_1- or x_2-momentum in the x_3-direction. Meanwhile σ_{33} must be negative if the cup exerts a positive force on the coffee above it.

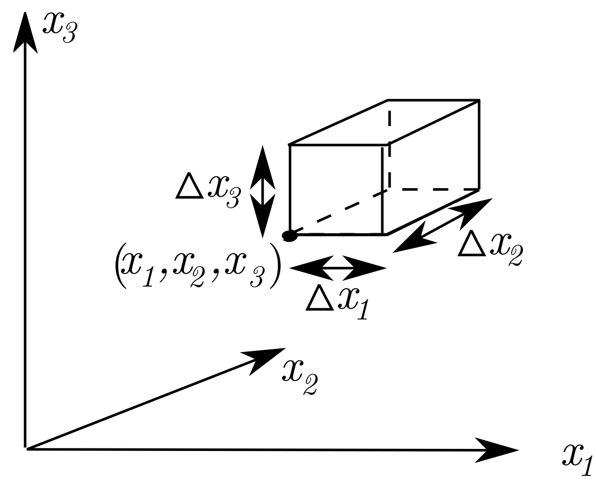

Figure 2: The small volume in exercise 5.

The reason why σ_{13} and σ_{23} were zero above was that there were no tangential forces. For a counterexample in which such forces occur, take the case of the car tire. Suppose the road lies in the x_1x_2-plane with the car above the plane moving in the x_1-direction. If the car is braking, the force on the tire therefore has a component in the negative x_1-direction. In addition, there may be a normal force on the tire in the positive x_3-direction, presumably supporting the weight of the car. But the calculations in (15) still hold, so we now have $\sigma_{13} > 0$, $\sigma_{23} = 0$ and $\sigma_{33} < 0$. The component σ_{13}, signifying a flux of x_1-momentum in the x_3-direction, is of course associated with the tangential force component $-\sigma_{13}\delta S$, which arises because there is friction between the tire and the road. We see that components σ_{ij} with $i \neq j$ are associated with tangential surface forces: $\sigma_{ij}\delta S$ is the x_i-component of force generated on a surface δS that is perpendicular to the x_j-axis.

It is important to understand that surface forces can be computed not only at surfaces S separating different materials like coffee and cup, or tire and road. They can be computed on any surface S, even if the surface lies within a larger body made of a single material. For instance, if we know the stress tensor σ_{ij} everywhere in a tire, we compute the force exerted by one part of the tire on another, even though both parts are made of the same material.

Exercise 5 *Consider a small cuboid of size $\delta x_1 \times \delta x_2 \times \delta x_3$ with one corner at (x_1, x_2, x_3) and the others at $(x_1 + \delta x_1, x_2, x_3)$, $(x_1, x_2 + \delta x_2, x_3)$, $(x_1, x_2, x_3 + \delta x_3)$ etc. in a general stress field σ_{ij} that can depend on position. The cuboid has six faces.*

Show that there is a face — all it face 1 —experiencing a net force δF_i^1 given by

$$\delta F_1^1 = -\int_{x_2}^{x_2+\delta x_2}\int_{x_3}^{x_3+\delta x_3} \sigma_{11}(x_1, x_2', x_3')\,\mathrm{d}x_3'\,\mathrm{d}x_2' \approx -\sigma_{11}(x_1, x_2, x_3)\delta x_2 \delta x_2$$
$$\delta F_2^1 = -\int_{x_2}^{x_2+\delta x_2}\int_{x_3}^{x_3+\delta x_3} \sigma_{21}(x_1, x_2', x_3')\,\mathrm{d}x_3'\,\mathrm{d}x_2' \approx -\sigma_{21}(x_1, x_2, x_3)\delta x_2 \delta x_3,$$
$$\delta F_3^1 = -\int_{x_2}^{x_2+\delta x_2}\int_{x_3}^{x_3+\delta x_3} \sigma_{31}(x_1, x_2', x_3')\,\mathrm{d}x_3'\,\mathrm{d}x_2' \approx -\sigma_{31}(x_1, x_2, x_3)\delta x_2 \delta x_3$$

and another face – call it face 2 — experiencing a net force δF_i^2 given by

$$\delta F_1^2 = \int_{x_2}^{x_2+\delta x_2}\int_{x_3}^{x_3+\delta x_3} \sigma_{11}(x_1+\delta x_1, x_2', x_3')\,\mathrm{d}x_3'\,\mathrm{d}x_2' \approx \sigma_{11}(x_1+\delta x_1, x_2, x_3)\delta x_2 \delta x_2$$
$$\delta F_2^2 = \int_{x_2}^{x_2+\delta x_2}\int_{x_3}^{x_3+\delta x_3} \sigma_{21}(x_1+\delta x_1, x_2', x_3')\,\mathrm{d}x_3'\,\mathrm{d}x_2' \approx \sigma_{21}(x_1+\delta x_1, x_2, x_3)\delta x_2 \delta x_3,$$
$$\delta F_3^2 = \int_{x_2}^{x_2+\delta x_2}\int_{x_3}^{x_3+\delta x_3} \sigma_{31}(x_1+\delta x_1, x_2', x_3')\,\mathrm{d}x_3'\,\mathrm{d}x_2' \approx \sigma_{31}(x_1+\delta x_1, x_2, x_3)\delta x_2 \delta x_3$$

Hence show that the net force $\delta F_i^1 + \delta F_i^2$ on the cuboid due to those two faces is approximately

$$\left(\frac{\partial \sigma_{11}}{\partial x_1}, \frac{\partial \sigma_{21}}{\partial x_1}, \frac{\partial \sigma_{31}}{\partial x_3}\right)\delta V$$

where $\delta V = \delta x_1 \delta x_2 \delta x_3$. Similarly compute the net force due to the remaining four faces to show that the cuboid experiences a total force approximately equal to

$$\left(\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3}, \frac{\partial \sigma_{21}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3}, \frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3}\right)\delta V$$

or, in subscript notation

$$F_i = \frac{\partial \sigma_{ij}}{\partial x_j}\delta V.$$

Note 4 *Note that*

$$\frac{\partial \sigma_{ij}}{\partial x_j} = \sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j}$$

can be thought of as the divergence of the stress tensor σ_{ij}, just as

$$\frac{\partial q_j}{\partial x_j} = \sum_{j=1}^{3} \frac{\partial q_j}{\partial x_j}$$

is the divergence of the flux vector q_i.

The divergence theorem still holds for a tensor σ_{ij} or q_{ij}, a fact we implicitly used in (8). We have

$$\int_S \sigma_{ij} n_j \, \mathrm{d}S = \int_V \frac{\partial \sigma_{ij}}{\partial x_j} \, \mathrm{d}V, \tag{16}$$

or, writing the sums implied by the summation convention explicitly

$$\int_S \sum_{j=1}^{3} \sigma_{ij} n_j \, \mathrm{d}S = \int_V \sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j} \, \mathrm{d}V. \tag{17}$$

This is trivial to understand if we fix the index i. For instance, if we take $i = 1$, σ_{1j} is a vector with components $(\sigma_{11}, \sigma_{12}, \sigma_{13})$, and for this vector, we know that the divergence theorem holds:

$$\int_S \sigma_{1j} n_j \, \mathrm{d}S = \int_V \frac{\partial \sigma_{1j}}{\partial x_j} \, \mathrm{d}V.$$

Now we can repeat the same with $i = 2$ and $i = 3$, and find that (16) holds for any $i = 1, 2, 3$.

Supply of momentum: body forces

In addition to the surface force term $-\int_{S(t)} q_{ij} n_j \, \mathrm{d}S = \int_{S(t)} \sigma_{ij} n_j \, \mathrm{d}S$, (12) contains another force term, the 'momentum supply rate' $\int_V a_i \, \mathrm{d}V$. This is clearly not a force exerted on the boundary of $V(t)$. Instead, it arises from a sum of forces $\delta F_i = a_i \delta V$ that are exerted directly on the small volume elements δV that make up the whole volume $V(t)$. Forces of this kind must be the result of long-range forces rather than of forces that result from direct contact with neighbouring material.

A good example would be the effect of gravity. This is the result of long-distance interactions between objects that have finite mass. Consider a gravitational field $\mathbf{g}$ due to some massive object acting on a continuum. In subscript notation, this would then be written as g_i. The force due to gravity on a small volume δV is then $\delta F_i = \delta m g_i = \rho g_i \delta V$ as the mass of the volume is $\delta m = \rho \delta V$. Comparing with $\delta F_i = a_i \delta V$ above, this suggests

$$a_i = \rho g_i.$$

for gravitational forces.

Just as we introduced a different symbol $\sigma_{ij} = -q_{ij}$ above to compute surface forces, the usual symbol for 'momentum supply density' a_i is not a_i but

$$f_i = a_i.$$

Also, f_i is usually referred to not as a supply density but as a *body force* (to distinguish it from a surface force).

Conservation of momentum in standard notation and simplification

Having introduced the new notation $\sigma_{ij} = -q_{ij}$ and $f_i = a_i$, the differential equation (10) describing conservation of momentum becomes

$$\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j} = \frac{\partial \sigma_{ij}}{\partial x_j} + f_i \tag{18}$$

This equation is a local form of Newton's second law (11). The two terms on the left-hand side are the change of momentum of a small Lagrangian volume. The first term describes how the concentration of x_i-momentum around a point changes in time, while the second describes how x_i-momentum is carried by the motion of the material itself (which allows material with high momentum to move from one place to another, thereby changing the spatial distribution of momentum over time). The right-hand side is the net force on the same Lagrangian volume, with the first term describing surface forces and the second describing long-range body forces.

As we did previously with the heat equation, we can use the mass conservation equation (5) to simplify (18). Apply the product rule to the left-hand side of (18) to find

$$\begin{aligned}
\frac{\partial(\rho u_i)}{\partial t} + \frac{\partial(\rho u_i u_j)}{\partial x_j} &= \frac{\partial \rho}{\partial t} + \rho\frac{\partial u_i}{\partial t} + \frac{\partial(\rho u_j)}{\partial x_j} u_i + \rho u_j \frac{\partial u_i}{\partial x_j} \\
&= \left(\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u_j)}{\partial x_j}\right) u_i + \rho\frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} \\
&= \rho\frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j}
\end{aligned}$$

from (5). Therefore (18) becomes

$$\rho\frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = \frac{\partial \sigma_{ij}}{\partial x_j} + f_i. \tag{19}$$

By itself, this equation cannot be solved — it contains thirteen unknowns (nine components of σ_{ij}, three components of u_i and ρ) but only three equalities (one for each i). Even taken together with (5), we therefore only have four equations for thirteen unknowns. Further information about the physics involved in the motion of a particular material is therefore required. (Note that this has an analogy in classical particle mechanics — we cannot solve Newton's second law (11) for velocity $\mathbf{u}$ unless we are given extra information about the forces $\mathbf{F}$ and mass m.) This information comes in part from the need to conserve not only momentum and mass (which we have already dealt with) but also angular momentum (which we have not). However, this is not enough, and the material nature of the continuum in question does matter —- we require further constitutive relations.

Newton's third law

We have talked about Newton's second law for a continumm being (12), or equally its reformulation in terms of σ_{ij} and f_i instead of q_{ij} and a_j, which is

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \rho u_i \,\mathrm{d}V = \int_{S(t)} \sigma_{ij} n_j \,\mathrm{d}S + \int_{V(t)} f_i \,\mathrm{d}V \tag{20}$$

You may wonder what happened to Newton's third law. In fact, it is Newton's third law that ensures that momentum is conserved in classical, point-particle mechanics.

Note 5 *To see that this last statement holds, consider a set of objects that we label A, B, C etc. (ordinarily, we might have labelled them 1, 2 etc,, but as we are already using number subscripts to indicate vector components in this text, we choose to use upper case letters to signify different point particles). Let the force exerted by object B on object A be $\mathbf{F}_{AB}$. Newton's third law says that*

$$\mathbf{F}_{AB} = -\mathbf{F}_{BA}.$$

If there are no external forces (so that the force acting on object A is purely composed of forced exerted on it by the other objects), then we have

$$m_A \frac{\mathrm{d}\mathbf{u}_A}{\mathrm{d}t} = \mathbf{F}_{AB} + \mathbf{F}_{AC} + \ldots,$$

and similarly for objects B, C etc. Here m_A is the mass of object A and $\mathbf{u}_A$ is its velocity.

To make our notation more compact, let P be the label for some object, so P could be A, B, C etc. Then

$$m_P \frac{\mathrm{d}\mathbf{u}_P}{\mathrm{d}t} = \sum_{P'} \mathbf{F}_{PP'}$$

where the sum over P' can be taken to be the sum over all the particles present, and we can put $\mathbf{F}_{PP} = \mathbf{0}$; an object does not exert a force on itself.

The total momentum in the system of particles is the sum over indiividual particle momenta,

$$\mathbf{p} = \sum_P m_P \mathbf{u}_P$$

If the masses of the particles are constant, then

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} &= \sum_P m_P \frac{\mathrm{d}\mathbf{u}_P}{\mathrm{d}t} \\
&= \sum_P \sum_{P'} \mathbf{F}_{PP'} \\
&= [\mathbf{F}_{AB} + \mathbf{F}_{AC} + \ldots] + [\mathbf{F}_{BA} + \mathbf{F}_{BC} + \ldots] + [\mathbf{F}_{CA} + \mathbf{F}_{CB} + \ldots] + \ldots
\end{aligned}$$

It is easy to see then that each force can be paired up with its equal and opposite version, for instance $\mathbf{F}_{AB}$ *with* $\mathbf{F}_{BA}$, $\mathbf{F}_{AC}$ *with* $\mathbf{F}_{CA}$, *and the sum equates to zero. Therefore*

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \mathbf{0}$$

and $\mathbf{p}$ *remains constant.*

When we have body forces, it is difficult to establish that Newton's third law holds; some object at a distance is generating a force on the material in volume $V(t)$ that we are looking at (for instance through a gravitational field), and that material in $V(t)$ will also need to generate an equal and opposite force on the object at a distance. This is tied up with how the body force f_i is generated, and (20) does not automatically ensure that Newton's third law holds.

However, in the absence of body forces, we have only surface forces like $\sigma_{ij}n_j$, and for these we can immediately ensure that Newton's third law holds. Consider for simplicity two Lagrangian volumes $V_A(t)$ and $V_B(t)$ that meet along a common boundary $S_{int}(t)$. The force exerted by volume B on volume A through their common boundary is

$$\int_{S_{int}(t)} \sigma_{ij} n_j^A \,\mathrm{d}S$$

where $\mathbf{n}^A$ is the normal to S_{int} that points out of volume A. Meanwhile, the force exerted by volume A on volume B is

$$\int_{S_{int}(t)} \sigma_{ij} n_j^B \,\mathrm{d}S$$

where $\mathbf{n}^B$ is the normal to S_{int} that points out of volume B. It is easy to see that the normal that points out of B points into A, and therefore that $\mathbf{n}^B = -\mathbf{n}^A$, so that

$$\int_{S_{int}(t)} \sigma_{ij} n_j^B \,\mathrm{d}S = -\int_{S_{int}(t)} \sigma_{ij} n_j^A \,\mathrm{d}S,$$

and the forces are equal and opposite.

Note 6 *The discussion above is analogous to the demonstration at the end of the notes on conservation laws, where we show that the transport of some conserved quantity* Φ *described by the surface integral of a flux* $\int_{S(t)} \mathbf{q}\cdot\hat{\mathbf{n}}\,\mathrm{d}S$ *ensures conservation, because that transport simply takes* Φ *out of one volume and puts it into another at the same rate. Conservation of momentum generalizes that demonstration to the case where* Φ *is a vector.*