EOS 352 Continuum Dynamics
Fluid flow in porous media

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Overview

These notes cover the following

- porous media and Darcy's law
- rigid porous media and basic poroelasticity
- isotropic and anisotropic permeability

Flow in a tube revisited

As part of the notes on fluid flow, we derived a model for unidirectional fluid flow in a pipe. With a few notational alterations, we can write this model as

$$-\mu \nabla^2 u = f_s - \frac{\mathrm{d}p}{\mathrm{d}s}$$

where ∇ is the Laplacian with respect to the coordinates that describe position in the plane perpendicular to the axis of the pipe (the so-called *antiplane*, and s is distance in the direction of the pipe. f_s is the component of body force along the pipe. u is velocity in the direction of the axis of the pipe, and depends only on the antiplane coordinates but not on s, while the pressure variable p only depends on s, and does so linearly (so $\mathrm{d}p/\mathrm{d}s$ is constant along the pipe). We also assume that $u = 0$ at the walls of the pipe.

For simplicity, suppose the pipe is cylindrical, with radius R. With the implied rotational symmetry, we can then assume that $u = u(r)$, where r is distance from the centreline of the cylinder, and the Laplacian is

$$-\mu\nabla^2 u = -\mu\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}u}{\mathrm{d}r}\right) = f_s - \frac{\mathrm{d}p}{\mathrm{d}s} \tag{1}$$

This can be solved easily by separation of variables

$$\mu r\frac{\mathrm{d}u}{\mathrm{d}r} = \int -\left(f_s - \frac{\mathrm{d}p}{\mathrm{d}s}\right) r\,\mathrm{d}r = -\frac{1}{2}\left(f_s - \frac{\mathrm{d}p}{\mathrm{d}s}\right) r^2 + C$$

$$u = \int -\frac{1}{2\mu}\left(f_s - \frac{\mathrm{d}p}{\mathrm{d}s}\right) r + \frac{C}{\mu r}\,\mathrm{d}r = -\frac{1}{4\mu}\left(f_s - \frac{\mathrm{d}p}{\mathrm{d}s}\right) r^2 + \frac{C}{\mu}\log(r) + D.$$

We have the sole boundary condition $u(R) = 0$, but two constants of integration, C and D. The second condition necessary to find a unique solution is the requirement that the solution not have a singularity at $r = 0$, which is not a boundary as such, but a location where the differential equation (1) is *singular* because the coefficient r inside the differential operator $r^{-1}\,\mathrm{d}/\,\mathrm{d}r(r\,\mathrm{d}/\,\mathrm{d}r)$ vanishes. If we therefore demand that $u(0)$ is finite and $u(R) = 0$, we find

$$u(r) = -\frac{1}{4\mu}\left(f_s - \frac{\mathrm{d}p}{\mathrm{d}s}\right)\left(R^2 - r^2\right).$$

Note 1 *There is in fact an interpretation for what it would mean for C to be non-zero. $\mu\,\mathrm{d}u/\,\mathrm{d}r$ is the shear stress σ_{sr}, if subscript s denotes the component in the direction of the axis of the cylinder, and subscript r denotes the radial component. In other words, for a small element of surface δS perpendicular to the radial direction r, the force parallel to the axis of the cylinder would be $\sigma_{rs}\delta S$. For a cylindrical surface of radius r and length h centered on that axis, the net force parallel to the axis of the cylinder is therefore $\sigma_{rs}2\pi rh$, since $2\pi rh$ is the surface area of the cylinder. If $\sigma_{rs} = \mu\frac{\mathrm{d}u}{\mathrm{d}r} = C/r$, then the force on the cylinder is $2\pi Ch$, and that result holds no matter how small the radius r. In other words, an arbitrarily narrow cylindrical surface centred on the centre axis experiences a force per unit length $2\pi C$, since the length of the cylinder along the centre axis was h, and that force is therefore really a force exerted along the centre line itself (a 'line force').*

The velocity field is therefore parabolic in r, just as the velocity field between two parallel plates computed in the notes on fluid flow was parabolic in distance x from either plate. For a cylindrical pipe, we can also compute the rate at which mass is transported along the pipe. Take a cross-section across the pipe: the rate at which mass passes through that cross-section is

$$Q = \int_S \rho\mathbf{u}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = 2\pi\int_0^R u(r)r\,\mathrm{d}r = \frac{\pi\rho}{8\mu}\left(f_s - \frac{\mathrm{d}p}{\mathrm{d}s}\right)R^4, \tag{2}$$

where ρ is mass density.

There are three things to note here: the mass discharge in the tube is proportional to the sum of pressure gradient and body force component along the pipe, inversely proportional to viscosity μ, and proportional to the fourth power of the tube radius R.

From tubes to porous media: Darcy's law

A *porous medium* is generally a solid that is riddled with microscopic, mutually connected voids that can be filled by a fluid. These voids are called *pore space*, and are distinguished from the solid *matrix*. Think of pore space as being the spaces between individual grains in a pile of sand, or the tubes left by woodworms in a piece of wood, while the sand or wood is the matrix. Often, fluid flow through those void spaces is a topic of interest: for instance, ground water is nothing more than water content in the void spaces of soil and other subsurface material.

When dealing with fluid flow in porous media, we generally do not try to reconstruct all the details of fluid flow at the scale of individual voids, but to take an aggregate view by averaging transport of fluid over many individual voids, but take that average over a scale that is much smaller than the scale of the macroscopic 'system; of interest (for instance, the size of a water-saturated layer of the subsurface or *groundwater aquifer*).

One way to try to get an understanding of this is to imagine that the void space of the porous medium consists of a collection of randomly oriented tubes like the cylindrical pipe considered above. Consider this in one dimension to set the scene. Take a surface are δS and assume that there is a pressure gradient $\mathrm{d}p/\mathrm{d}s$ and component of body force f_s across the surface. Assume also that there a fraction ν of δS is occupied by tubes of radius R, each of them at right angles to δS. That means each tube carries a discharge given by (2). Since the cross-sectional area of a tube is πR^2, then the number n of tubes for a surface area of size δS is

$$n = \frac{\nu \delta S}{\pi R^2}.$$

and the total discharge through the surface is

$$nQ = \frac{\nu \delta S}{\pi R^2} \times \frac{\pi \rho}{8\mu} \left(f_s - \frac{\mathrm{d}p}{\mathrm{d}s} \right) R^4 = \frac{\rho \nu R^2}{8\mu} \left(f_s - \frac{\mathrm{d}p}{\mathrm{d}s} \right) \delta S.$$

In porous media flow, fluid pressure p is generally defined in a similarly smoothed way, where we do not attempt to resolve the scale of individual voids, or tubes in the cartoonish geometry we are considering at the moment. We ignore the fact that p is really only defined in the void space but take it to be a smoothly varying function of position defined everywhere, since any given point should be close to a void space in which the fluid pressure is defined and can be interpolated to the location in pressure.[1] With that in mind, we can then treat the along-tube pressure gradient $\mathrm{d}p/\,\mathrm{d}s$ for tubes perpendicular to δS as being the normal component of ∇p, and similarly the along-tube component of body force f_s is the normal component of a vector-valued body force $\mathbf{f}$, so that

$$nQ = \frac{\rho \nu R^2}{8\mu} (\mathbf{f} - \nabla p) \cdot \hat{\mathbf{n}} \delta S.$$

Recall that a flux is a transport rate per unit surface area; in the present case, we would like to define a mass flux $\mathbf{q}$, averaged over the size of many individual tubes, such that the rate of mass transfer is $\mathbf{q} \cdot \hat{\mathbf{n}} \delta S$. This should equal nQ

$$\mathbf{q} \cdot \hat{\mathbf{n}} \delta S = \frac{\rho \nu R^2}{8\mu} (\mathbf{f} - \nabla p) \cdot \hat{\mathbf{n}} \delta S,$$

which suggests we might put

$$\mathbf{q} = \frac{\rho \nu R^2}{8\mu} (\mathbf{f} - \nabla p) . \tag{3}$$

Since real porous media are not usually collections of cylindrical tubes, the formula (3) is usually generalized as

$$\mathbf{q} = \frac{\rho k}{\mu} (\mathbf{f} - \nabla p) , \tag{4}$$

where the constant k is known as the *permeability* of the porous medium. k is essentially an empirical constant (like thermal conductivity is to heat flux) that represents the small-scale geometry of the pore space: in the case of cylindrical tubes, equation (3) suggests that

$$k = \frac{\nu R^2}{8}; \tag{5}$$

if we change the fluid (and therefore the density ρ and viscosity μ), the permeability remains unchanged.

Equation (4) is usually known as *Darcy's law*, and is widely used to describe fluid flow in porous media. It replaces the much more complicated Navier-Stokes equations as stated in the notes on fluids as a model for average fluid velocity. The driving term $\mathbf{f} - \nabla p$ is usually known as the *hydraulic gradient*, which comes from the following construction: the body force $\mathbf{f}$ is almost always gravity, so $\mathbf{f} = \rho \mathbf{g}$. For constant ρ, we can then generally write $\mathbf{f}$ itself as the negative gradient of a potential. That potential is of the form $\rho g z$ if we treat $\mathbf{g}$ as constant and the z-axis as pointing vertically upwards. In that case, we can write

$$\mathbf{f} - \nabla p = \rho \mathbf{g} - \nabla p = -\nabla(\rho g z + p)$$

[1] There is a mathematical construct called a *multiple scales expansion* that makes this assumption somewhat more systematic, but there is no need to delve into that here.

and defining the *hydraulic potential* Φ as

$$\Phi = \rho g z + p,$$

and the hydraulic gradient as $-\nabla\Phi$, Darcy's law becomes

$$\mathbf{q} = -\frac{\rho k}{\mu}\nabla\Phi :$$

the flow of pore fluid goes down the gradient of hydraulic potential.

Note 2 *Hydrologists (scientists who study groundwater flow) will frequently talk about* hydraulic head*; this is simply the hydraulic potential expressed as a height,*

$$h = \frac{\Phi}{\rho g} = z + \frac{p}{\rho g}.$$

In the derivation above, the fraction ν of the surface δS taken up by tubes can easily be identified with the fraction of volume in the porous medium taken up by voids. That volume fraction is generally called the *porosity*, and more commonly be denoted by ϕ (and we will recycle ν in more sophisticated form in the appendix to these notes).

Note 3 *Permeability is a measure for how much fluid will flow for a given applied pressure gradient. Observe that the tube model* (5) *for porous media predicts that permeability increases with the volume fraction* ϕ *of void space in the medium, which makes sense: more voids should allow more fluid to flow. Crucially, the permeability also depends on the radius of individual tubes as* R^2*. That, too, should correspond to common experience. I can take a pile of grains that has the same fraction of voids (for instance by taking identical spheres and packing them as neatly and closely as is possible) and yet find that water runs through one more easily than through another with the same fraction of voids (for instance, by repeating the experiment with identical spheres, but changing the radius of the spheres): the porous medium consisting larger spheres (which in a sense corresponds to larger pore radii* R*) will allow water to drain more easily. This is why gravel drains more easily than sand, and sand more easily than silt or clay, even though their porosities may not differ by much.*

In case you are not entirely convinced by our construction of Darcy's law, the appendix will flesh out some of the issues may find, in particular with regard to the role of the orientation of individual 'tubes' in the porous medium; the derivation of a version (3) that accounts for tubes with random orientations has to wait until exercise 3 at the end of these notes.

Conservation of mass

We can now set up a mass conservation law for a porous medium. The problem we face in trying to use the formalism for conservation laws we have developed previously is that we now have what is known as a *mixture*: by not resolving the small-scale structure of matrix and pore space, we are effectively saying that, at any location (x, y, z), we simultaneously have an effective solid density and an effective fluid density: recall that ϕ is the fraction of any given volume occupied by pore fluid, so $(1-\phi)$ is occupied by the matrix. A small volume δV around the point (x, y, z) will therefore simultaneously contain a soild mass $\rho_s(1-\phi)\delta V$, where ρ_s is the density of the pure matrix solid, and a fluid mass $\rho\phi\delta V$. Here δV of course must be much smaller than the size of the system we are looking at, but much larger than an individual void or pore.[2]

The problem that arises with the standard Lagrangian volume formulation is how to define that Lagrangian volume: clearly, fluid is moving relative to the matrix, so we can define a volume $V(t)$ that incorporates all the same bits of the solid matrix, but not the same bits of fluid. That is, in fact, the usual construction, since the flux $\mathbf{q}$ defined through Darcy's law above is clearly relative to the matrix (remember that the flow-in-a-pipe solution assumes no slip $u(R) = 0$ at the boundary, which means no slip relative to the matrix, which may itself be moving).

Denoting the matrix velocity by $\mathbf{u}$, conservation of the sediment matrix then becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \rho_s(1-\phi)\,\mathrm{d}V = 0,$$

where aplication of Reynolds' Transport Theorem leads to

$$\int_{V(t)} \frac{\partial[\rho_s(1-\phi)]}{\partial t}\,\mathrm{d}V + \int_{S(t)} \rho_s(1-\phi)\mathbf{u}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = 0$$

and using the divergence theorem gives

$$\int_{V(t)} \frac{\partial[\rho_s(1-\phi)])}{\partial t} + \nabla\cdot[\rho_s(1-\phi)\mathbf{u}]\,\mathrm{d}V = 0.$$

Since $V(t)$ is arbitrary

$$\frac{\partial[\rho_s(1-\phi)]}{\partial t} + \nabla\cdot[\rho_s(1-\phi)\mathbf{u}] = 0$$

[2]This construction is however not as strange as it might seem, but in fact in-built into everything we do in continuum mechanics: we are always assuming that the 'small volumes' δV and surface δS that we are dealing with are small compared with the size over which densities and fluxes vary, but large compared with the size of individual atoms or molecules.

For the fluid, we treat $\mathbf{q}$ as a conductive flux relative to the Lagrangian volume, so we get the global form of conservation of fluid as

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \rho\phi \,\mathrm{d}V + \int_{S(t)} \mathbf{q}\cdot\hat{\mathbf{n}}\,\mathrm{d}S = 0.$$

The same standard procedure for obtaining the corresponding local form as used above gives

$$\frac{\partial(\rho\phi)}{\partial t} + \nabla\cdot(\rho\phi\mathbf{u}) + \nabla\cdot\mathbf{q} = 0.$$

One of the most common cases that we need to consider is that of incompressible solid and fluid phases, so ρ_s and ρ are both constant, while porosity ϕ can change. Conservation of the solid becomes

$$-\frac{\partial\phi}{\partial t} + \nabla\cdot[(1-\phi)\mathbf{u}] = 0 \tag{6}$$

and for the fluid

$$\frac{\partial\phi}{\partial t} + \nabla\cdot(\phi\mathbf{u}) + \nabla\cdot(\rho^{-1}\mathbf{q}) = 0. \tag{7}$$

where

$$\rho^{-1}\mathbf{q} = -\frac{k}{\mu}\nabla\Phi$$

is a *volume flux* (as opposed to mass flux), with units of velocity.[3]

In general, this is as far as we can go without adding another model to determine the velocity field $\mathbf{u}$ of the matrix, and how that model works depends on the material that the matrix is made of. One common situation is that of an *elastic* matrix, where stresses in the matrix depend not on the *strain rate* defined in the notes on fluid dynamics, but on the *strain*, a similarly defined gradient of displacements of material relative to an undeformed reference geometry; this leads to a model called *poroelasticity* or Biot theory, and is beyond the scope of what we are going to do here. A slightly more exotic situation is one in which even the matrix deforms viscously, just with a much larger viscosity than the pore fluid, in which case the model is one of viscous compaction. This is a standard approach to modelling the flow of magma in partially molten regions of the Earth's mantle, where the rock melts in part but not fully, leaving a matrix of solid rock crystals between which the liquid lava flows. In common with other polycrystalline solids in the Earth (such as ice), the solid matrix actually flows viscously over long enough time scales, and it is that viscous deformation which determines $\mathbf{u}$.

Both situations described above are beyond what we are able to treat here. We consider only two simple cases that can be dealt with more straightforwardly, not require the addition of further differential equation models.

[3]Volume flux is, loosely speaking, the volume that passes through unit area in unit time.

Laplace's equation and a diffusion model

The first case we look at in slightly more detail is that of a completely rigid matrix, in which case ϕ is constant and $\mathbf{u}$ vanishes, so conservation of solid is satisfied automatically, while conservation of fluid is

$$\nabla \cdot \mathbf{q} = -\nabla \cdot \left(\frac{\rho k}{\mu} \nabla \Phi \right) = \nabla \cdot \left[\frac{\rho k}{\mu} (\rho \mathbf{g} - \nabla p) \right] = 0$$

If we treat k, μ and ρ as constant (as is reasonable if we are in a uniform material and porosity cannot change as fluid drains), then we obtain Laplace's equation for fluid pressure p, or equivalently, for hydraulic potential Φ:

$$-\nabla^2 \Phi = -\nabla^2 p = 0. \tag{8}$$

The second is a very simple version of poroelasticity, where we assume that the case of porosity varying weakly as a function of changes in fluid pressure: a larger fluid pressure will generally open up void space, for instance by pushing sediment grains apart. The simplest model for this effect is to assume that ϕ is a decreasing function of the so-called *effective pressure* p_e, the difference between a prescribed pressure p_s supported by the solid-fluid mix due to the weight of overlying material, and the pore fluid pressure p:

$$\phi = \phi(p_e) \qquad \text{where} \qquad p_e = p_s - p.$$

The very simplest version simply assumes that there is a compressibility that links ϕ to p_e, as in

$$\phi = \phi_0 - c p_e \tag{9}$$

where ϕ_0 is a constant. If c is very small, density does not vary by much, and we can approximate (see exercise 1)

$$\nabla \cdot [(1 - \phi)\mathbf{u}] \approx (1 - \phi_0) \nabla \cdot \mathbf{u}, \qquad \nabla \cdot (\phi \mathbf{u}) \approx \phi_0 \nabla \cdot \mathbf{u},$$

while, from (9),

$$\frac{\partial \phi}{\partial t} = c \frac{\partial p}{\partial t}$$

since $p_e = p_s - p$ and p_s is fixed. Using the approximations for $\nabla \cdot [(1 - \phi)\mathbf{u}]$ and $\nabla \cdot (\phi \mathbf{u})$ in (6) and (7) and eliminating $\nabla \cdot \mathbf{u}$ between them, we get

$$\frac{c}{1 - \phi_0} \frac{\partial p}{\partial t} + \nabla \cdot (\rho^{-1} \mathbf{q}) = 0,$$

or, substituting for $\mathbf{q}$ using Darcy's law

$$\frac{c}{1 - \phi_0} \frac{\partial p}{\partial t} + \nabla \cdot \left[\frac{k}{\mu} (\rho \mathbf{g} - \nabla p) \right] = 0.$$

Recall that porosity will, in general, be a function of porosity (see for instance equation (3)), $k = k(\phi)$. However, as we have just assumed that ϕ does not vary by much, it is reasonable to approximate k as a constant, $k \approx k(\phi_0)$, in which case

$$\frac{c}{1-\phi_0}\frac{\partial p}{\partial t} - \frac{k}{\mu}\nabla^2 p = 0; \tag{10}$$

in other words, fluid pressure satisfies a diffusion equation.

Exercise 1 *We have made some ad hoc approximations above. These can be justified systematically by non-dimensionalizing the mass conservation equations*

$$\begin{aligned} -\frac{\partial \phi}{\partial t} + \nabla\cdot[(1-\phi)\mathbf{u}] &= 0 \\ \frac{\partial \phi}{\partial t} + \nabla\cdot(\phi\mathbf{u}) + \nabla\cdot(\rho^{-1}\mathbf{q}) &= 0, \end{aligned}$$

combined with the constitutive relations

$$\begin{aligned} \rho^{-1}\mathbf{q} &= \frac{k}{\mu}(\rho\mathbf{g} - \nabla p), \\ \phi &= \phi_0 - c(p_s - p). \end{aligned}$$

Assume permeability is a function of ϕ that we do not specify further at this point, $k = k(\phi)$. Write $x = [x]x^*$, $y = [x]x^*$, $z = [x]z^*$ (thus assuming that the length scales for all three directions are the same), $t = [t]t^*$, $p = [p]p^*$, $\mathbf{u} = [u]\mathbf{u}^*$, $k = [k]\kappa$. Assume that a scale for pressure $[p]$ and a length scale $[x]$ are imposed by boundary conditions and geometry of the system, and are therefore given (rather than being scales that are chosen as part of the process of non-dimensionalizing). Show that the problem can be written in the form

$$\begin{aligned} -\frac{\partial p^*}{\partial t^*} + [1-\phi_0+\gamma(\nu-p^*)]\nabla^*\cdot\mathbf{u}^* - \gamma\mathbf{u}^*\cdot\nabla^* p^* &= 0 \\ \frac{\partial p^*}{\partial t^*} + [\phi_0 - \gamma(\nu-p^*)]\nabla^*\cdot\mathbf{u}^* + \gamma\mathbf{u}^*\cdot\nabla^* p + \nabla^*\cdot\left[\kappa\left(1-\phi_0^{-1}\gamma(\nu-p^*)\right)(\mathbf{g}^*-\nabla^* p^*)\right] &= 0 \end{aligned}$$

where $\nabla^* = \partial/\partial x^*\mathbf{i} + \partial/\partial y^*\mathbf{j} + \partial/\partial z^*\mathbf{k}$. κ can be related to the function k in such a way that $\kappa(1) = 1$. Note that

$$\kappa\left(1-\phi_0^{-1}\gamma(\nu-p^*)\right)$$

stands or 'the function κ evaluated at the quantity in round brackets'.

Derive the corresponding scales $[t]$, $[u]$ and $[k]$ in terms of $[p]$, $[x]$, ϕ_0, c, μ and the function k, and also find expressions for the dimensionless parameters γ, nu and $\mathbf{g}^*$ in terms of $[p]$, $[x]$, ϕ_0, c, μ. Show that the reduced model

$$\begin{aligned} -\frac{\partial p^*}{\partial t^*} + [1-\phi_0]\nabla^*\cdot\mathbf{u}^* &= 0 \\ \frac{\partial p^*}{\partial t^*} + \phi_0\nabla^*\cdot\mathbf{u}^* + \nabla^*\cdot[(\mathbf{g}^*-\nabla^* p^*)] &= 0 \end{aligned}$$

is appropriate provided $c[p] \ll 1$ *and* $p_s/[p]$ *is not large, and that therefore*

$$\frac{\phi_0}{1-\phi_0}\frac{\partial p^*}{\partial t^*} - \nabla^{*2} p^* = 0.$$

Exercise 2 *Based on the expectation pressure satisfies a diffusion equation, hydrologists often use a so-called 'slug test' to estimate the parameters in the diffusion model. A slug test is based on the sudden injection of a finite volume of fluid into a borehole, which will then drain out into the surrounding porous medium. Consider the following set up: you have an aquifer of height* h *in the* z*-direction, which extends to infinity in the* x*- and* y*-directions. That is, the aquifer is all points* (x, y, z) *for which* $0 < z < h$*. There is a borehole along the* z*-axis and a volume* V *of water is injected at* $t = 0$*, when the aquifer has uniform hydraulic potential, with*

$$p = p_0 - \rho g z$$

If the radius of the borehole can be ignored, then the subsequent evolution of water pressure around the borehole can be modelled using polar coordinates (r, θ, z) *(where* $x = r\cos(\theta)$ *and* $y = r\sin(\theta)$*) as*

$$\begin{aligned}
\frac{c}{1-\phi_0}\frac{\partial p}{\partial t} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) - \frac{\partial^2 p}{\partial z^2} &= 0 && \text{for } t > 0,\ r > 0,\ 0 < z < h,\\
p(r, z, 0) &= p_0 - \rho g z && \text{at } t = 0,\\
2\pi r\frac{\partial p}{\partial r} &\to 0 && \text{as } r \to \infty \text{ for } t > 0,\ 0 < z < h,\\
-\frac{\partial(\rho g z + p)}{\partial z} &= 0 && \text{at } z = 0,\ h,\\
2\pi\int_0^h\int_0^\infty c(p - p_0 + \rho g z) r\,\mathrm{d}r\,\mathrm{d}z &= V && \text{for } t > 0,
\end{aligned}$$

where V *is constant. The first equation is the diffusion problem for* p *in the aquifer domain, while the second is the pre-injection pressure distribution as an initial condition. The third and fourth conditions state that no fluid is withdrawn 'at infinity' or at the top or bottom of the aquifer, ensuring that total volume is conserved. The last condition links the pressure change away from its pre-injection distribution to the total injected volume.*

Find a similarity solution of the form

$$p = p_0 - \rho g z + r^{-\alpha}\Pi\left(\frac{r}{t^\beta}\right).$$

If you measured the pressure distribution away from borehole against time, would you be able to uniquely determine k and $c/(1-\phi_0)$ *from those measurements? What if you did not record the volume* V *injected? Would you be able to determine some combination of* k *and* $c/(1-\phi_0)$*, but not both quantities separately?*

Appendix: The effect of tube orientation

One of the problems with the 'derivativion' of Darcy's law that we have presented above using a tube model is that we have assumed that all the tubes are at right angles to the surface element δS. If we were to change the orientation of the surface element δS, that would no longer be the case, even though we end up treating the flux $\mathbf{q}$ as simply proportional to $\rho\mathbf{g} - \nabla p$, without any regard to the alignment of tubes in the porous medium.

We can construct a more sophisticated three-dimensional model that takes account of tube orientation. Suppose first that there are n classes of tubes, all with different orientations, labelled by superscripts i. The ith class of tube as oriented in the direction $\hat{\mathbf{t}}^i$, $\hat{\mathbf{t}}^i$ is a unit vector. *It is important to note that the superscript does not refer to exponentiation here.* We use it to label the tube classes so we can use subscripts to refer to components of vectors and tensors when we revert to the subscript notation we have developed previously. We employ standard vector notation with boldface letters denoting vectors for now to keep things simpler. Note that we could also allow each class to have a different radius R_i (or R^i, but then insisting that the superscript does not refer to exponentiation would become even more fraught, since R is a scalar and can easily be exponentiated, unlike the vector $\hat{\mathbf{t}}^i$). For now we will assume all tubes have the same radius R for simplicity.

Take a volume δV. The number of tubes in that volume belonging to each each class i is proportional to the size of the volume δV, so we can attach a volume density ν_i to each class. In other words, the volume in δV occupied by the ith class of tubes is $\nu_i \delta V$, or equivalently, ν_i is the fraction of total volume taken up by tubes in class I. Obviously the total volume occupied by tubes is $\sum_i \nu_i \delta V$. In other words, the porosity (the fraction of δV occupied by void space, tubes in this case) is

$$\phi = \sum_i \nu_i.$$

We want to know, given the information that we know volume density nu_i, direction $\hat{\mathbf{t}}^i$, and radius R for the tubes, fluid viscosity μ and hydraulic gradient $\rho\mathbf{g} - \nabla p$, can we calculate an effective mass flux $\mathbf{q}$, averaged over surfaces much larger than a single tube, but much smaller than the size of the system.

The discharge in each tube of class i in such a volume can be calculated using the formula (2). If we treat the porous medium as having a continuum fluid pressure p at a scale much larger than that of an individual tube, and that there is also a continuum body force acting on the fluid, $\mathbf{f} = \rho\mathbf{g}$, then the obvious way to formulate the driving term in (2) is

$$f_s - \frac{\mathrm{d}p}{\mathrm{d}s} = (\rho\mathbf{g} - \nabla p) \cdot \hat{\mathbf{t}}^i$$

and the discharge in the tube (in the direction that $\hat{\mathbf{t}}^i$ points in) is

$$Q_i = \frac{\pi}{8\mu} R^4 (\rho_w \mathbf{g} - \nabla p) \cdot \hat{\mathbf{t}}^i$$

Now take a thin slab volume δV with a flat surface δS and thickness δx. Let the normal to δS be $\hat{\mathbf{n}}$. We would like to know how much mass passes through δS per unit time, which is equivalent to knowing how many tubes pass through δS, and what the discharge Q in each is.

Suppose that δx is much thinner than the linear extent of δS, so that we can essentially exclude the possibility of any tubes that pass through δV passing in through the lateral sides; all tubes will pass through δS instad. For each class of tube, we know that the volume occupied by the tubes in δV is $\nu_i \delta V$. For a single tube, the intersection of the tube with the volume has length (along the axis of the tube) δs_i. If we project that length onto the normal to δS, we should get the thickness of the slab, or $\delta x = \delta s_i \hat{\mathbf{t}}^i \cdot \hat{\mathbf{n}}$, so

$$\delta s_i = \frac{\delta x}{\hat{\mathbf{t}}^i \cdot \hat{\mathbf{n}}}.$$

The volume of that tube inside δV is $\pi R^2 \delta s_i$, and the total volume of tubes in this class is

$$n_i \pi R^2 \delta s_i = \frac{n_i \pi R^2 \delta x}{\hat{\mathbf{t}}^i \cdot \hat{\mathbf{n}}}.$$

That volume must equal $\nu_i \delta V$, so

$$n_i = \frac{\hat{\mathbf{t}}^i \cdot \hat{\mathbf{n}}}{\pi R^2 \delta x} \times \nu_i \delta V = \frac{\nu_i \hat{\mathbf{t}}^i \cdot \hat{\mathbf{n}} \delta S}{\pi R^2}$$

since $\delta V = \delta S \delta x$. The total discharge through the ith class of tubes is therefore

$$n_i Q_i = \frac{\nu_i \hat{\mathbf{t}}^i \cdot \hat{\mathbf{n}} \delta S}{\pi R^2} \times \frac{\pi \rho}{8\mu} R^4 \left(\rho \mathbf{g} - \nabla p\right) \cdot \hat{\mathbf{t}}^i = \frac{\rho \nu_i R^2}{8\mu} \left[\left(\rho \mathbf{g} - \nabla p\right) \cdot \hat{\mathbf{t}}^i\right] \hat{\mathbf{t}}^i \cdot \hat{\mathbf{n}} \delta S \qquad (11)$$

The total mass transfer through the surface is obtained by summing over the tube classes,

$$\sum_i n_i Q_i = \sum_i \frac{\rho \nu_i R^2}{8\mu} \left[\left(\rho \mathbf{g} - \nabla p\right) \cdot \hat{\mathbf{t}}^i\right] \hat{\mathbf{t}}^i \cdot \hat{\mathbf{n}} \delta S$$

Recall from the notes on conservation laws that we can think of a flux as being the rate of transfer of a conserved quantity per unit surface area, and that, given a flux $\mathbf{q}$, the rate of transfer through a surface element δS with normal $\hat{\mathbf{n}}$ is $\mathbf{q} \cdot \hat{\mathbf{n}} \delta S$. We can apply this to the transfer of mass if we treat $\mathbf{q}$ as a mass flux. Equating the two formulas for the rate of mass transfer, we get

$$\mathbf{q} \cdot \hat{\mathbf{n}} \delta S = \sum_i \frac{\rho \nu_i R^2}{8\mu} \left[\left(\rho \mathbf{g} - \nabla p\right) \cdot \hat{\mathbf{t}}^i\right] \hat{\mathbf{t}}^i \cdot \hat{\mathbf{n}} \, \mathrm{d}S \qquad (12)$$

and it should be clear that

$$\mathbf{q} = \sum_i \frac{\rho \nu_i R^2}{8\mu} \left[\left(\rho \mathbf{g} - \nabla p\right) \cdot \hat{\mathbf{t}}^i\right] \hat{\mathbf{t}}^i \qquad (13)$$

This should look a little weird. The contribution of the ith tube class gives a contribution to flux in the direction of $\hat{\mathbf{t}}^i$, even if the hydraulic gradient $\rho\mathbf{g} - \nabla p$ does not point in that direction. That should make sense as the flow of fluid is constrained by the direction of the tube, and can differ from the direction of the gradient driving it. What we have potentially is an example of *anisotropy*: where the material properties of a medium, and therefore its response to forcing, depend on the direction in which we are looking in. An obvious example is if we have only a single tube class, in which fluid flow can only be in a single direction $\hat{\mathbf{t}}$ (you can omit the index i and the sum in (13) above.) When you have only a single class, $\mathbf{q}$ is always in the tube direction $\hat{\mathbf{t}}$, but can be driven by hydraulic gradients that are not aligned with the tubes: in fact, as soon as the hydraulic gradient $\rho\mathbf{g} - \nabla p$ has a *component* parallel to the the tube direction, there will be flow.

Where does this leave Darcy's law? If we revert to subscript notation for vectors, (13) can be written as

$$q_i = \frac{\rho}{\mu} K_{ij} \left(\rho g_j - \frac{\partial p}{\partial x_j} \right) \tag{14}$$

with

$$K_{ij} = \sum_l \frac{\nu_l R^2 t_i^l t_j^l}{8} \tag{15}$$

where the superscript l is now used to index tube classes. The spatial indexing in these equations reflects the anisotropy: the index i refers to the direction of flow in q_i, and gets contributions t_i^l from the lth tube direction, while the index j corresponds to the projection of the hydraulic gradient onto the tube direction, which is given by

$$\hat{\mathbf{t}}^l \cdot (\rho\mathbf{g} - \nabla p) = t_j^l \left(\rho g_j - \frac{\partial p}{\partial x_j} \right).$$

We see that permeability is really a tensor; the scalar permeability k advocated earlier is a special case in which

$$K_{ij} = k\delta_{ij}.$$

Then

$$q_i = \frac{\rho k}{\mu} \left(\rho g_i - \frac{\partial p}{\partial x_i} \right),$$

which is the subscript notation version of (4). As exercise 3 shows, this so-called *isotropic* case is indeed what we expect if the tubes are randomly oriented, with no preferred direction.

Even where K_{ij} is not isotropic, it is always symmetric ($K_{ij} = K_{ji}$) and positive semi-definite, meaning that for any vector v_i, $v_i K_{ij} v_j \geq 0$. The former property follows trivially from the definition (15), while the latter is almost as straightforward to demonstrate, since

$$v_i K_{ij} v_j = \sum_l \frac{\nu_l R^2 (t_i^l v_i)(t_j^l v_j)}{8} = \sum_l \frac{\nu_l R^2 (t_i^l v_i)^2}{8}$$

is a sum of squares. $v_i K_{ij} v_j$ can vanish only if there is a direction v_i that is orthogonal to all tube directions.

Appendix: A continuous distribution of tube orientations

The construction of discrete tube classes (with superscript i, later l when reverting to subscript notation) in the previous section is a little bit artificial. In reality, we expect that tubes can point in any direction. Instead of a discrete index i, we can capture this by another density function describing what fraction of tubes point in a given general direction. We will make this more concrete next.

Let us first do this exercise with tubes restricted to be parallel to say the xy-plane. In that case the unit vectors $\hat{\mathbf{t}}$ defining tube orientation all correspond to a point on the unit circle. Here we have the concept of a single angle (say, relative to the x-axis) to to help us define direction, and get some intuition, so we deal with that case first.

Instead of saying that the fraction of volume taken up by tubes of class i with direction $\hat{\mathbf{t}}^i$ is ν_i, we could then say that the fraction of volume taken up by tubes whose direction is at angles between θ and $\theta + \delta\theta$ to the x-axis is $\nu(\theta)\delta\theta$, working on the assumption that the fraction is proportional to the spread of angles $\delta\theta$ we allow for, so long as that spread is small. Each angle θ of course has a unit vector $\hat{\mathbf{n}}(\theta)$ associated with it, and by analogy with the previous section, we would have a total porosity given by adding the volume fractions associated with tubes in ranges θ to $\theta + \delta\theta$, as in

$$\phi = \int_0^{2\pi} \nu(\theta) \, \mathrm{d}\theta.$$

This construction would also suggest that the discharge through the surface δS in note due to tubes with directions between θ and $\theta + \delta\theta$ is, as an equivalent to equation 11,

$$\delta Q = \frac{\rho \nu(\theta) \delta\theta R^2}{8\mu} \left[(\rho \mathbf{g} - \nabla p) \cdot \hat{\mathbf{t}}(\theta) \right] \hat{\mathbf{t}}(\theta) \cdot \hat{\mathbf{n}} \delta S$$

and summing over angles gives the total discharge, so the equivalent of (12) is

$$\mathbf{q} \cdot \hat{\mathbf{n}} \delta S = \int_0^{2\pi} \frac{\rho \nu(\theta) R^2}{8\mu} \left[(\rho \mathbf{g} - \nabla p) \cdot \hat{\mathbf{t}}(\theta) \right] \hat{\mathbf{t}}(\theta) \, \mathrm{d}\theta \cdot \hat{\mathbf{n}} \delta S$$

and the flux is

$$\mathbf{q} = \int_0^{2\pi} \frac{\rho \nu(\theta) R^2}{8\mu} \left[(\rho \mathbf{g} - \nabla p) \cdot \hat{\mathbf{t}}(\theta) \right] \hat{\mathbf{t}}(\theta) \, \mathrm{d}\theta,$$

leading to a version of Darcy's law (14) with

$$K_{ij} = \int_0^{2\pi} \frac{R^2}{8} \nu(\theta) t_i(\theta) t_j(\theta) \, \mathrm{d}\theta.$$

The derivation above was designed for tubes whose orientation remains parallel to the xy-plane, and was only designed to make the step to continuous distributions of tube orientations possible to understand. With $\hat{\mathbf{t}}$ restricted to two dimensions, it was possible to parameterize orientation purely in terms of a single angle. Recall that size of an angle θ is defined as the length of the arc of the unit circle subtended by θ, if you measure angles in radians (which you should!)[4] Specifically, the angle increment $\delta\theta$ that we used to define the volume $\nu(\theta)\delta\theta$ occupied by tubes with orientations between θ and $\theta + \delta\theta$ can be thought of as an arc element around the point on the unit circle that the vector $\hat{\mathbf{n}}((\theta)$ points to.

In three dimensions, the unit vectors $\hat{\mathbf{t}}$ defining correspond to points on the unit sphere. We can define an element of *solid angle* $\delta\Omega$ by taking a patch of the unit sphere around the point that a particular unit vector $\hat{\mathbf{t}}$ points to, and defining the surface area of that patch to be the solid angle increment.[5] To define the orientation of $\hat{\mathbf{t}}$, we of course require two angles, effectively a latitude θ and a longitude φ, which we can view as the standard polar and azimuthal angles in a spherical polar coordinate system. In other words, we have $\hat{\mathbf{t}} = \hat{\mathbf{t}}(\theta, \varphi)$. It is also straightforward to see that, if we define the solid angle $\delta\Omega$ to be the part of the unit sphere that corresponds to polar angles between θ and $\theta + \delta\theta$ and azimuthal angles between φ and $\varphi + \delta\varphi$, then $\delta\Omega = \sin(\theta)\delta\theta\delta\varphi$.

We can then define the fraction of volume occupied by tubes whose associated unit vectors $\hat{\mathbf{t}}$ correspond to points in the patch $\delta\Omega$ in terms of an amended density function ν as $\nu(\theta, \varphi)\delta\Omega$. Following through exactly the same steps as above, making sure that all possible orientations are covered, we get Darcy's law (14) with

$$K_{ij} = \int_0^{2\pi} \int_0^{\pi} \frac{\rho R^2}{8} \nu(\theta, \varphi) t_i(\theta, \varphi) t_j(\theta, \varphi) \sin(\theta)\, \mathrm{d}\theta\, \mathrm{d}\varphi,$$

and the total porosity is now

$$\phi = \int_0^{2\pi} \int_0^{\pi} \nu(\theta, \varphi) \sin(\theta)\, \mathrm{d}\theta\, \mathrm{d}\varphi. \tag{16}$$

Exercise 3 *In an* isotropic *medium, you expect that tubes are equally likely to point in all directions, which means that ν is a constant. In a standard Cartesian coordinate system, the components of $\hat{\mathbf{t}}$ are related to the polar and azimuthal angle as*

$$t_1 = \sin(\theta)\cos(\varphi), \qquad t_2 = \sin(\theta)\sin(\varphi), \qquad t_3 = \cos(\theta).$$

[4] The idea of 360 degrees in a full rotation comes from early estimates of the length of a year, with approximately 360 days to a year so a degree corresponds to the angular progression of the Earth in a single day — not a particularly useful concept for most applications, especially as a year is closer to 365.25 days.

[5] More generally, we can understand a solid angle to be a part of the unit sphere, if a regular angle is a part of the unit circle.

Compute all the components of K_{ij} *assuming that* ν *is constant, and show that* $K_{ij} = k\delta_{ij}$ *for some scalar value* k*. This means you have to work through all combinations of* $i = 1, 2, 3$ *and* $j = 1, 2, 3$*, although as per the comment at the end of the previous section, you should find that* K_{ij} *is automatically symmetric. (*Hint: for off-diagonal matrix components with $i \neq j$, compute the integral with respect to φ first. For diagonal components $i = j$, make sure you know how to integrate $\sin^2(\varphi)$ and $\cos^2(\varphi)$, for which you use the identity

$$\cos(2\varphi) = 1 - 2\sin^2(\varphi) = 2\cos^2(\varphi) - 1$$

To integrate something like $\sin^3(\theta)$*, use* $\sin^3(\theta) = (1 - \cos^2(\theta))\sin(\theta)$*. Express the value of* k *you get in terms of* R *and* ϕ *(which means that you need to solve for* ν *in terms of* ϕ*). How does your answer compare with the simplistic answer* $k = \phi R^2/8$ *we obtained previously?*