EOS 352 Continuum Dynamics
Scaling

April 30, 2014

Overview

These notes cover the following

- Identifying scales from a solution
- Non-dimensional variables
- Scaling without solving a set of equations
- Dimensionless parameters

Temperature waves revisited

Figure 1, shows solutions to the temperature wave problem

$$\rho c \frac{\partial T}{\partial t} - k \frac{\partial^2 T}{\partial x^2} = 0 \qquad \text{on } x > 0 \tag{1a}$$

$$T(0,t) = T_0 \cos(\omega t) \qquad \text{at } x = 0 \tag{1b}$$

$$-k \frac{\partial T}{\partial x} \to 0 \qquad \text{as } x \to \infty \tag{1c}$$

for different physical settings corresponding to different values of ρ, c, k, T_0 and ω. Clearly, the solutions *are* different (look at the axis labels), but they *look* the same when viewed at different scales. In fact, if you were to superimpose the plotted curves on each other, say, by cutting out the top row of figures and placing it over the bottom row, you would find the two curves are *exactly* the same. What lies behind this?

By zooming in and out of the solution, we can obtain the same curves for different parameter values. The important question becomes: what sets the necessary 'zoom

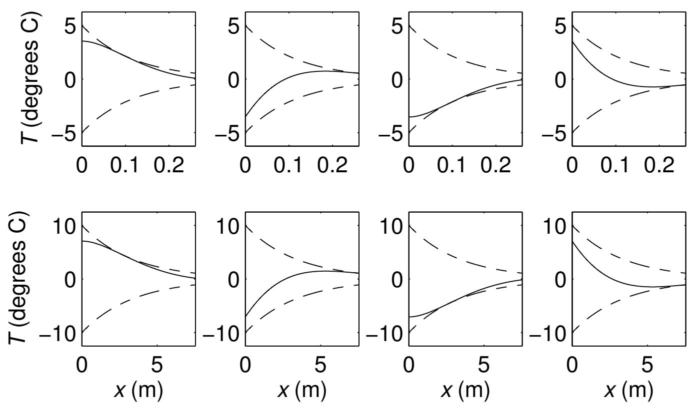

Figure 1: Top row: an diurnal temperature wave ($\omega = 2\pi/(24 \times 3600)$ s^{-1}) in soil ($k = 0.4$ W m^{-1} K^{-1}, $\rho = 1000$ kg m^{-3}, $c = 800$ W kg^{-1} K^{-1}.) with amplitude $T_0 = 5$ C, shown at (left to right) $t = 3, 9, 15, 21$ hours. Bottom row: an annual temperature wave ($\omega = 2\pi/(365 \times 24 \times 3600)$ s^{-1}) in ice ($k = 0.4$ W m^{-1} K^{-1}, $\rho = 1000$ kg m^{-3}, $c = 800$ W kg^{-1} K^{-1}) with amplitude $T_0 = 10$ C, shown at (left to right) $t = 1.5, 4.5, 7.5, 10.5$ months.

level'? Between which limits should we plot along the x- and T-axes? In other words, what are relevant distances into the ground, and relevant temperature variations? Our discussion of the solutions in the notes on temperature waves already give us some clues. The solution takes the form

$$T(x,t) = T_0 \exp\left(-\sqrt{\frac{\rho c\omega}{2k}}x\right)\cos\left(\omega t - \sqrt{\frac{\rho c\omega}{2k}}x\right). \tag{2}$$

Trivially, a natural scale for temperature variations is the amplitude of the temperature oscillation at the surface, T_0. To identify a length scale, we have different choices. Recall that the solution in (2) is the product of an exponential envelope and a travelling sinusoidal wave. As discussed, a natural length scale associated with an exponential is the so-called 'e-folding length scale' In our solution (2), this was given by

$$x_e = \sqrt{\frac{2k}{\rho c\omega}}. \tag{3}$$

Equally, we could let our choice be guided by the wavelength of the sinusoid; this was

$$x_0 = 2\pi\sqrt{\frac{2k}{\rho c\omega}}. \tag{4}$$

The two length scales differ by a factor of $2\pi \approx 6.28\ldots$. So which is it to be? The point here is not the exact numerical value: when we compared the solutions in figure 1, the difference between the different solutions came out of the parameter choices for

ρ, c, k, T_0 and ω. The important point here is that the dependence of *both*, x_e and x_0 on these parameters takes the same form.

In fact, even the factor of 2 that appears inside the square root is beside the point: what we really care about is how the solution changes as we change the parameters ρ, c, k, T_0 and ω, not whether e-folding time or wavelength make a better scale. We can define a length scale that encapsulates the dependence of both, x_e and x_0 on the various different parameters in its simplest form as

$$[x] = \sqrt{\frac{k}{\rho c \omega}}. \tag{5}$$

The notation here, putting the name of the variable x into square brackets $[\cdot]$ is intended to indicate that $[x]$ is a so-called *scale* for the variable x. For given ρ, c, k and ω, $[x]$ is by definition a fixed quantity (a constant) with dimensions of length, and should never be confused with the variable x itself. The notation is purely intended to visually associate the scale $[x]$ with the variable x. In the same notation, we would define a scale for T as

$$[T] = T_0.$$

For completeness, we can also consider what a useful scale for time would be, even though time does not feature in the plots in figure 1. We have previously identified the period of oscillation as $2\pi/\omega$, and so one might be tempted to define a time scale as $[t] = 2\pi/\omega$. However, once more it is only the dependence on the parameter ω that we care about, not the unchanged factor 2π. In this vein, we define

$$[t] = \frac{1}{\omega}.$$

So what is the point in these scales? There are many situations in nature where metres, seconds and kelvins are not ‘natural’ units to use. For instance, if you deal with problems in earth history, seconds are usually an awkward unit of time to use. SI units are simply historically chosen units that have no intrinsic relevance to much of nature (for instance, 1 metre was intended to be 1/10000 of a quarter of the circumference of the earth — which it is near enough — but there is no good reason why this should be a useful unit, other than that it is for instance a useful unit for the height of a human being).

The same issue of natural units also applies to our heat wave problem. Look again at figure 1, and ask what a natural unit to express length and temperatures is. You’ll probably tempted to say ‘10 cm’ for a length scale in the top row, and ‘10 m’ for a length scale in the bottom row. These are however simply the nearest SI-type units you can guess at by looking at the solution graphically. Better perhaps would be to consider the scales $[x]$, $[t]$ and $[T]$ as natural units for distance, time and temperature. To express position, time and temperature in these units, we can define

$$x^* = \frac{x}{[x]}, \qquad t^* = \frac{t}{[t]}, \qquad T^* = \frac{T}{[T]}. \tag{6}$$

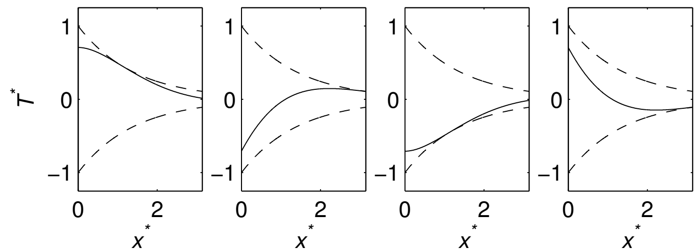

Figure 2: The scaled solution at times $t^* = \pi/4$, $3\pi/4$, $5\pi/4$, $7\pi/4$.

This means for instance that position $x = [x]x^*$ is expressed as a position measurement x^* times the unit $[x]$. This is also what expressing distance in terms of more traditional SI units does: we would ask how many lots of 1 m something is from the origin. Only here we are asking how many lots of $[x]$ something is from the origin.

The quantities x^*, t^* and T^* are usually called *dimensionless variables*, because their dimensions are contained in the units $[x]$, $[t]$ and $[T]$ instead. This to be contrasted with the original, 'dimensional' variables x, t and T. Now we can ask what T^* looks like as a function of x^* and t^*. The answer is

$$T^*(x^*, t^*) = \exp\left(-\frac{x^*}{\sqrt{2}}\right)\cos\left(t^* - \frac{x^*}{\sqrt{2}}\right). \tag{7}$$

Exercise 1 *Show that (7) holds. Do this by substituting $T = [T]T^*$, $x = [x]x^*$, $t = [t]t^*$ into (2), and then substituting the formulae for $[x]$, $[t]$ and $[T]$ above.*

So what do we learn from this? Primarily that, when expressed in terms of the dimensionless variables defined above, the solution (2) *always* takes the same form, regardless of the particular choice of ρ, c, k, ω and T_0. What changes between different parameter values is the size of the units. This is the reason why the plots in figure 1 look the same. When x and T are scaled as above, they *are* the same.

There is another point here, however: consider what happens when x^* takes values 'around one' (say, from 0 to 2 or so; this concept of being 'around one' is often written as $x^* = O(1)$, read: 'x^* is of order one'). Physically, this means that we are looking at dimensional depths below the surface comparable to the length scale $[x]$. What we find here is that the temperature T^* will vary by an amount close to unity over a single cycle. This means that the dimensional temperature T varies by an amount comparable to the temperature scale $[T] = T_0$ during a single cycle (figure 2). Physically, one could say that, at these depths, the temperature field 'sees' the effect of the imposed surface variations. If, by contrast, we make x^* very large compared with one, so that dimensional depths are much bigger than $[x]$, then temperature variations will be much smaller than the temperature scale $[T]$, i,e, T^* will vary by an amount much less than one. The temperature field at these greater depths no longer 'sees' the effect of the surface variations. The point of finding a scale $[x]$ was to identify how the typical depths to which temperature variations can penetrate into the ground depend on the parameters ρ, c, ω and k.

If we look at the definition in (5), we can make some obvious observations: the depth to which temperature variations penetrate increases with conductivity k (which facilitates heat transport) but decreases with ρc (which suppress heat transport to greater depths, because heating up the same depth of soil requires more heat and therefore leaves less heat to be conducted deeper). The penetration depth also decreases with the frequency $\omega/(2\pi)$ of the surface variations: rapid surface temperature variations mean that there is less time over which heat flows into the ground before the direction of flow is reversed, and this does not allow heat to penetrate as deeply. One useful feature of having identified the scales above is that we can put these observations — which one could possibly have guessed at — into quantitative form.

Scales from model equations

In the temperature wave example, finding scales was relatively easy because we had a complete solution available. Suppose however that we didn't. Could we still identify natural length, time and temperature scales, using only the original equations (1)? The way to do this is to simply assume that there is a sensible choice of scales $[x]$, $[t]$ and $[T]$, but that this choice is simply not known yet. Define dimensionless variables once more as in (6), and simply substitute into the original equations (1), treating $[x]$, $[t]$ and $[T]$ as constants. There is a bit of basic calculus to be done: we are basically performing a change of variables, and hence we have to express the derivatives in (1a) in terms of t^* and x^*. But this is straightforward. Take $\partial\cdot/\partial t$: we have, from the chain rule,

$$\frac{\partial\cdot}{\partial t} = \frac{\partial\cdot}{\partial t^*}\frac{\partial t^*}{\partial t}$$

But $t^* = t/[t]$ with $[t]$ a constant, so $\partial t^*/\partial t = 1/[t]$, and

$$\frac{\partial\cdot}{\partial t} = \frac{1}{[t]}\frac{\partial\cdot}{\partial t^*}.$$

As $[T]$ is also a constant, we get

$$\frac{\partial T}{\partial t} = \frac{1}{[t]}\frac{\partial([T]T^*)}{\partial t^*} = \frac{[T]}{[t]}\frac{\partial T^*}{\partial t^*}.$$

Similarly, we can deal with secod derivatives. We have

$$\frac{\partial\cdot}{\partial x} = \frac{1}{[x]}\frac{\partial\cdot}{\partial x^*}.$$

To deal with the second derivatives, we note that

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{1}{[x]}\frac{\partial T}{\partial x^*}\right).$$

As $[x]$ is again simply a constant, we get

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{[x]} \times \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x^*}\right) = \frac{1}{[x]} \times \frac{1}{[x]}\frac{\partial}{\partial x^*}\left(\frac{\partial T}{\partial x^*}\right) = \frac{1}{[x]^2}\frac{\partial^2 T}{\partial x^{*2}}.$$

Lastly, we substitute $T = [T]T^*$, and get

$$\frac{\partial^2 T}{\partial x^2} = \frac{[T]}{[x]^2}\frac{\partial^2 T^*}{\partial x^{*2}}.$$

Exercise 2 *Show that, if $f = [f]f^*$ and $y = [y]y^*$, then*

$$\frac{\mathrm{d}^n f}{\mathrm{d}y^n} = \frac{[f]}{[y]^n}\frac{\mathrm{d}^n f^*}{\mathrm{d}y^{*n}}.$$

With $x = [x]x^*$, $t = [t]t^*$, $T = [T]T^*$, the equations in (1) therefore become

$$\frac{\rho c[T]}{[t]}\frac{\partial T^*}{\partial t^*} - \frac{k[T]}{[x]^2}\frac{\partial^2 T^*}{\partial x^{*2}} = 0 \qquad \text{on } [x]x^* > 0 \tag{8a}$$

$$[T]T^*(0, t^*) = T_0 \cos(\omega[t]t^*) \qquad \text{at } [x]x^* = 0 \tag{8b}$$

$$-\frac{k[T]}{[x]}\frac{\partial T^*}{\partial x^*} \to 0 \qquad \text{as } [x]x^* \to \infty \tag{8c}$$

We would like to pick our scales to simplify these equations as much as possible.

The ideal situation would presumably to have as many of the coefficients in the equations simplify so that they ‘disappear’. It would be tempting therefore to say for instance that

$$\frac{\rho c[T]}{[t]} = 1,$$

as this would set the first coefficient to unity, which would make it ‘disappear’. However, we have to remember that $[x]$ must have units of length, $[t]$ must have units of time, and $[T]$ must have units of temperature. A bit of algebra will then show that $\rho c[T]/[t]$ in fact has units of W m^{-3} and so cannot equal one. You might be tempted to say that it should therefore be set equal to 1 W m^{-3}. This however misses the point: why should this quantity equal unity in SI units, when we are trying to get away from SI units?

The natural procedure is in fact to manipulate the equations above until all they are grouped into terms that are dimensionless (meaning they have no units). Take for instance the heat equation, (8a). The terms $\partial T^*/\partial t^*$ and $\partial^2 T^*/\partial x^{*2}$ are both dimensionless, while $\rho c[T]/[t]$ and hence (to be dimensionally consistent) $k[T]/[x]^2$ have units of W m^{-3}. The obvious procedure is then to divide both sides of the equation by one of these last two terms, say $k[T]/[x]^2$:

$$\frac{\rho c[x]^2}{k[t]}\frac{\partial T^*}{\partial t^*} - \frac{\partial^2 T^*}{\partial x^{*2}} = 0 \qquad \text{on } x^* > 0, \tag{9a}$$

where (with $[x] > 0$) we have also manipulated $[x]x^* > 0$ to read $x^* > 0$. The term

$$\frac{\rho c[x]^2}{k[t]}$$

is then simply a number (with no dimensions) that depends only on the parameters ρ, c and k, and on the choice of scales $[x]$ and $[t]$. In fact, it is a product of powers of these quantities, if we recall that dividing by $k[t]$ is the same as multiplying by $k^{-1}[t]^{-1}$. A number of this kind is usually called a *dimensionless group*. Similarly, the other two equations in (8) can be manipulated into

$$T^*(0,t^*) = \frac{T_0}{[T]}\cos(\omega[t]t^*) \qquad \text{at } x^* = 0, \tag{9b}$$

$$-\frac{\partial T^*}{\partial t^*} \to 0 \qquad \text{as } x^* \to \infty. \tag{9c}$$

Although they may seem less obvious, there are exactly two additional dimensionless groups here, namely $\omega[t]$ and $T_0/[T]$.

Now, we have the ability to choose the three scales $[T]$, $[x]$ and $[t]$ in any way we please (with the only restriction that $[x]$ should not be negative). The obvious procedure is to choose these scales in such a way that the three dimensionless groups above are equal to unity, if this is possible:

$$\frac{\rho c[x]^2}{k[t]} = 1, \qquad \omega[t] = 1, \qquad \frac{T_0}{[T]} = 1. \tag{10}$$

A bit of algebra then shows that

$$[T] = T_0, \qquad [t] = \frac{1}{\omega}, \qquad [x] = \sqrt{\frac{k}{\rho c \omega}}. \tag{11}$$

Reference to the previous section will show that we have recovered exactly the same scales as we did by working off the known solution, using only the original partial differential equation model and some algebra. This is the real power of scaling (or *non-dimensionalisation*, as it is also called): it allows us to learn something about a physical system (such as relevant length scales) without having to solve the problem.[1]

With this choice of scales, (9) finally becomes the much simpler-looking model

$$\frac{\partial T^*}{\partial t^*} - \frac{\partial^2 T^*}{\partial x^{*2}} = 0 \qquad \text{on } x^* > 0, \tag{12a}$$

$$T^*(0,t^*) = \cos(t^*) \qquad \text{on } x^* = 0, \tag{12b}$$

$$-\frac{\partial T^*}{\partial x^*} \to 0 \qquad \text{as } x^* \to \infty. \tag{12c}$$

[1] It is worth noting here that the scaling procedure here does not predict the factors $2\pi/\sqrt{2}$ or $\sqrt{2}$ that would translate the length scale $[x]$ into the wavelength x_0 or e-folding length scale x_e above: this is something that does require a full solution, but it also motivates why we didn't include these factors in our length scale $[x]$ even when we did have a full solution available: we were only interested in the dependence on the parameters ρ, c, k, ω and T_0.

This is straightforward to solve using the methods we have already seen: in fact, it is simpler than before because we don't have to keep track of all the different parameters. In a way, that often time-consuming part of the solution has been taken care of. We can simply try a solution of the form

$$T(x^*, t^*) = \mathrm{Re}\left[\exp(it^* + \lambda x^*)\right],$$

which automatically satisfies the boundary condition (12b) at the surface. To ensure that it also satisfies (12a), we require that

$$\mathrm{Re}\left[(i - \lambda^2)\exp(it^* + \lambda x^*)\right] = 0,$$

or equally

$$\lambda^2 = i.$$

Hence

$$\lambda = \pm\frac{1+i}{\sqrt{2}}.$$

But to satisfy the boundary condition (12c), we must choose the negative sign,

$$\lambda = -\frac{1+i}{\sqrt{2}}.$$

With this, it is easy to show that

$$T^*(x^*, t^*) = \exp\left(-\frac{x^*}{\sqrt{2}}\right)\cos\left(t^* - \frac{x^*}{\sqrt{2}}\right) \tag{13}$$

as before: reassuringly, scaling after finding a solution and finding a solution from the scaled model equations (12) yield the same answer.

Dimensionless parameters and approximations

One of the remarkable observations about the temperature wave problem above is that it always yields the same scaled solution, no matter what the actual parameter values for ρ, c, k, ω and T_0 were. This is not the case for all differential equation models. A simple example arises from the temperature wave problem when we do not ignore the geothermal heat flux in (1c), and instead write

$$-k\frac{\partial T}{\partial x} \to -q_{geo} \qquad \text{as } x \to \infty$$

The equivalent of (8) is then instead

$$\frac{\rho c[T]}{[t]}\frac{\partial T^*}{\partial t^*} - \frac{k[T]}{[x]^2}\frac{\partial^2 T^*}{\partial x^{*2}} = 0 \qquad \text{on } [x]x^* > 0 \tag{14a}$$

$$[T]T^*(0, t^*) = T_0\cos(\omega[t]t^*) \qquad \text{at } [x]x^* = 0 \tag{14b}$$

$$-\frac{k[T]}{[x]}\frac{\partial T^*}{\partial x^*} \to -q_{geo} \qquad \text{as } [x]x^* \to \infty \tag{14c}$$

and the procedure of rearranging the parameters and scales into dimensionless groups yields the following analogue of (9):

$$\frac{\rho c[x]^2}{k[t]}\frac{\partial T^*}{\partial t^*} - \frac{\partial^2 T^*}{\partial x^{*2}} = 0 \qquad \text{on } x^* > 0, \tag{15a}$$

$$T^*(0,t^*) = \frac{T_0}{[T]}\cos(\omega[t]t^*) \qquad \text{at } x^* = 0, \tag{15b}$$

$$\frac{\partial T^*}{\partial t^*} \to \frac{q_{geo}[x]}{k[T]} \qquad \text{as } x^* \to \infty. \tag{15c}$$

Importantly, this produces four rather than three dimensionless groups, namely $\rho c[x]^2/k[t]$, $\omega[t]$, $T_0/[T]$ and $q_{geo}[x]/(k[T])$. But we have only three scales to choose, so we can only ensure that three of these dimensionless groups equal unity. We are at liberty to once more make the choices in (10). This leaves us with one potentially non-unity dimensionless group, which we can now (using (11)) rewrite as

$$\frac{q_{geo}[x]}{k[T]} = \frac{q_{geo}}{T_0\sqrt{\rho c k \omega}}.$$

It is customary to denote a non-unity dimensionless group (usually called a *dimensionless parameter*, just as the original quantities ρ, c, k etc were called parameters) by a single letter (often from the Greek alphabet), for instance

$$\gamma = \frac{q_{geo}}{T_0\sqrt{\rho c k \omega}}. \tag{16}$$

With this, the equivalent of (12) becomes

$$\frac{\partial T^*}{\partial t^*} - \frac{\partial^2 T^*}{\partial x^{*2}} = 0 \qquad \text{on } x^* > 0, \tag{17a}$$

$$T^*(0,t^*) = \cos(t^*) \qquad \text{on } x^* = 0, \tag{17b}$$

$$\frac{\partial T^*}{\partial x^*} \to \gamma \qquad \text{as } x^* \to \infty. \tag{17c}$$

with solution

$$T^*(x^*,t^*) = \exp\left(-\frac{x^*}{\sqrt{2}}\right)\cos\left(t^* - \frac{x^*}{\sqrt{2}}\right) + \gamma x^* \tag{18}$$

We see that γ appears in the same place in (17) where geothermal heat flux apears in the dimensional counterpart of these equations, namely (1a), (1b) and (14a). For this reason,it can naturally be described as a 'dimensionless geothermal heat flux'. But can we expand on what the magnitude of γ means physically? One answer is to look at what happens when γ is small. By small, we mean small compared with unity (this is often written as $\gamma \ll 1$). Then, when x^* is not large (that is, when the exponential term is not small and the temperature field experiences the effect of the

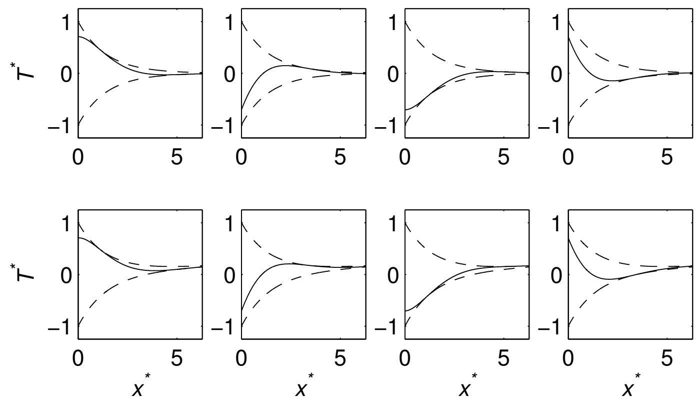

Figure 3: Top row: the solution (18) with $\gamma = 0$ at times $t^* = \pi/4,\ 3\pi/4,\ 5\pi/4,\ 7\pi/4$. Bottom row: the same but with $\gamma = 0.025$. Not that the solutions only differ very slightly, and then only for larger x^*

surface), the effect of geothermal heat flux on the temperature field is negligible, and the solution (18) is well-approximated by (13),

$$T^*(x^*, t^*) \approx \exp\left(-\frac{x^*}{\sqrt{2}}\right)\cos\left(t^* - \frac{x^*}{\sqrt{2}}\right).$$

This is ultimately the basis for neglecting geothermal heat flux altogether in our model, as we did in writing down (1). This is illustrated in figure 3.

We could have guessed at this from the definition of $\gamma = q_{geo}[x]/k[T]$. If q_{geo} is a constant heat flux, then we can write from Fourier's law that $|q_{geo}| = k\Delta T/\Delta x$, where ΔT is the temperature change over a distance Δx. It follows by re-arranging that $q_{geo}[x]/k$ is the corresponding temperature difference over a distance $\Delta x = [x]$. Therefore $\gamma = q_{geo}[x]/(k[T])$ is the ratio of the temperature change due to geothermal heat flux over a distance $[x]$ to the temperature variation $[T]$ imposed by the boundary conditions at the surface. When this ratio is small, the temperature variation at the surface dominates the temperature field. What we see here is that the smallnes of the parameter γ gives us a quantitative basis for making the approximation that geothermal heat flux is small.

In fact, one of the main advantages of identifying the parameter γ in the model equations (17) is that we can identify when geothermal heat flux is unimportant without ever solving the problem. We can simply say that a very small value of γ suggests that a good model is instead to replace γ by zero, that is to replace (17c) by

$$\frac{\partial T^*}{\partial x^*} \to 0 \qquad\qquad \text{as } x^* \to \infty,$$

which immediately leads to (12) and the solution (13). Being able to make approximations on a quantitative basis before attempting a solution is in fact one of the main motivations for trying to scale model equations.

Note 1 *We could also ask what happens if γ is large, that is, when γ is much bigger than one (this is often written as $\gamma \gg 1$). By the same reasoning as above, we now expect the temperature field to be dominated by geothermal heat flux. In fact, (18) is then well-approximated by*

$$T^*(x^*, t^*) \approx \gamma x^*. \tag{19}$$

However when x^* is now around one, T^* is actually not close in value to one, but is instead close in value to γ and is therefore large. This means that the dimensional temperature $T = [T]T^*$ is not in fact close to $[T] = T_0$, but instead is similar in value to $[T]/\gamma$. Using the definition of γ, we see that this is equal to $[T]/\gamma = q_{geo}[x]/k$. In this case, one can argue that a natural scale for temperature is in fact not $[T] = T_0$ but $[T]_{alt} = q_{geo}[x]/k$.

Suppose we were to start over by defining dimensionless variables through $T^{**} = T/[T]_{alt}$, $x^{**} = x/[x]_{alt}$, $t^{**} = t/[t]_{alt}$, where we use the double asterisk and the subscript 'alt' (for 'alternative') to distinguish these scales and dimensionless variables from our previous choice. We would then get (9) with each dimensionless variable carrying two rather than one asterisk, and each scale carrying a subscript 'alt'. In terms of the choice of scales from the dimensionless groups $\rho c[x]^2/k[t]$, $\omega[t]$, $T_0/[T]$ and $q_{geo}[x]/(k[T])$ in (15), recognizing that $q_{geo}[x]/k$ is a better scale for temperature than T_0 would then correspond to setting

$$\frac{\rho c[x]_{alt}^2}{k[t]_{alt}} = 1, \qquad \omega[t]_{alt} = 1, \qquad \frac{q_{geo}[x]}{k[T]_{alt}} = 1.$$

This has the solution

$$[t]_{alt} = \frac{1}{\omega}, \qquad [x]_{alt} = \sqrt{\frac{k}{\rho c \omega}}, \qquad [T]_{alt} = \frac{q_{geo}[x]}{k} \tag{20}$$

In terms of these new dimensionless variables, the equivalent of the equations (17) is

$$\frac{\partial T^{**}}{\partial t^{**}} - \frac{\partial^2 T^{**}}{\partial x^{**2}} = 0 \qquad \text{on } x^{**} > 0, \tag{21a}$$

$$T^{**}(0, t^{**}) = \delta \cos(t^{**}) \qquad \text{on } x^{**} = 0, \tag{21b}$$

$$-\frac{\partial T^{**}}{\partial x^{**}} \to 1 \qquad \text{as } x^{**} \to \infty. \tag{21c}$$

where

$$\delta = \frac{T_0}{[T]} = \frac{T_0\sqrt{\rho c k \omega}}{q_{geo}}. \tag{22}$$

Straightforwardly, (21) can be shown to have solution

$$T^{**}(x^{**}, t^{**}) = \delta \exp\left(-\frac{x^{**}}{\sqrt{2}}\right) \cos\left(t^{**} - \frac{x^{**}}{\sqrt{2}}\right) + x^{**}. \tag{23}$$

Now we know that we made this alternative choice of scales when the original parameter $\gamma = q_{geo}/(T_0\sqrt{\rho c k \omega})$ was large. It is easy to see from (16) and (22) that

$$\delta = \gamma^{-1},$$

so that $\delta \ll 1$ when $\gamma \gg 1$. It follows that the solution (23) is approximately

$$T^{**}(x^{**}, t^{**}) \approx x^{**}, \tag{24}$$

which is essentially the same as (19) except for the double asterisks and the fact that the factor γ is missing. Now, when x^{**} is around one, T^{**} is no longer large, but is itself comparable to one, so the new choice of temperature scale makes more sense.

One could have arrived at this equally by looking at the model (21) rather than its solution, and considering what happens when δ is very small. An obvious thing to do in that case is to replace (21b) by

$$T^{**}(x^{**}, 0) = 0 \qquad \text{at } x^{**} = 0.$$

so that

$$\frac{\partial T^{**}}{\partial t^{**}} - \frac{\partial^2 T^{**}}{\partial x^{**2}} = 0 \qquad \text{on } x^{**} > 0, \tag{25a}$$

$$T^{**}(0, t^{**}) = 0) \qquad \text{on } x^{**} = 0, \tag{25b}$$

$$\frac{\partial T^{**}}{\partial x^{**}} \to 1 \qquad \text{as } x^{**} \to \infty. \tag{25c}$$

This obviously has the solution $T^{**}(x^{**}, t^{**}) = x^{**}$ that we just found in (24). The advantage in using $\delta \ll 1$ before we even start solving anything is that we get a simpler set of equation, and don't have to go through the whole complex variable procedure to solve this.

Exercise 3 *In note 1 immediately above, recall that $\delta = 1/\gamma$. An important feature of this new choice of scales is that the solution in (23) takes the same form as (18), except that all variables have two rather than one asterisk, and that in going from (18) to (23), the right-hand side has been divided by γ. In fact, if we look at (20), we see that the scales for time and distance are unchanged, i.e. $[t]_{alt} = [t]$, $[x]_{alt} = [x]$, but that the scale for temperature is changed as*

$$[T]_{alt} = [T] \times \frac{q_{geo}[x]}{kT_0} = \gamma[T].$$

What this means is that the old and new dimensionless variables can in fact be related through

$$x^{**} = x^*, \qquad t^{**} = t^*, \qquad T^{**} = \gamma T^*. \tag{26}$$

Often, if you already have a scaled model but you find that one of your parameters is large (like γ above) and suggests that the original choice of scales was not really *appropriate, it can be quicker to define new dimensionless variables not in terms of a new set of scales and the original dimensionless variables. Instead, one can define them in terms of the dimensionless variables you already have multiplied by a power of the parameter in question. This is called a* rescaling*.*

Suppose we had not gone through the argument in the note above, but had simply tried a rescaling

$$x^{***} = \gamma^a x^*, \qquad t^{***} = \gamma^b t^*, \qquad T^{***} = \gamma^c T^*.$$

Change variables in (17) to show that

$$\gamma^{b-2c}\frac{\partial T^{***}}{\partial t^{***}} - \frac{\partial^2 T^{***}}{\partial x^{***2}} = 0 \qquad \textit{on } x^{***} > 0, \tag{27a}$$

$$T^{***}(0, t^{***}) = \gamma^c \cos(\gamma^{-b} t^{***}) \qquad \textit{on } x^{***} = 0, \tag{27b}$$

$$\frac{\partial T^{***}}{\partial x^{***}} \to \gamma^{1+c-a} \qquad \textit{as } x^{***} \to \infty, \tag{27c}$$

and hence that (21) follows if we choose $a = b = 0$, $c = -1$.