EOS 352 Continuum Dynamics
Similarity solutions

February 13, 2017

Overview

These notes cover the following

- Form of a typical similarity solution
- Transforming a partial differential equation and initial / boundary conditions to a similarity variable
- Identifying exponents in the definition of the similarity variable
- Solving the resulting ordinary differential equation

A model problem

Consider what happens when hot magma suddenly intrudes into a host (or 'country') rock along a narrow sill or dyke. Initially, the host rock will be cold, and will only be warmed over time by the conduction of heat away from the sill or dyke. This warming can be important if it leads to changes in the structure of the host rock. Some of the relevant questions one can ask are: how hot does the host rock get at a certain distance from the sill or dyke? How far away from the sill or dyke will it heat up enough to get chemically altered? One way to look at this is to construct a model.

We consider heat conduction into the host rock in one dimension. We assume that there is no advection as the rock is stationary. Let the dyke lie in the yz-plane, and the x-axis be perpendicular to from the dyke. We also assume that temperature T depends on x and t only. With density ρ, heat capacity c and thermal conductivity k constant and no heat production in the rock, we have

$$\rho c \frac{\partial T}{\partial t} - k \frac{\partial^2 T}{\partial x^2} = 0. \tag{1}$$

We assume that the injection of magma occurs at time $t = 0$, so (1) holds for $t > 0$. At time $t = 0$, assume that the host rock is at a uniform temperature T_0 everywhere away from the dyke or sill. For simplicity, suppose the dyke or sill is very narrow, so that we can write this initial condition as

$$T(x, 0) = T_0 \qquad \text{for } x \neq 0. \tag{2}$$

We also assume that there is no heat flux in or out at large distances, so

$$\lim_{x \to \pm\infty} \frac{\partial T}{\partial x} \to 0 \qquad \text{for all } t > 0. \tag{3}$$

So far, we have constructed a problem to which $T(x, t) \equiv T_0$ seems to be a valid solution (substitute to verify this). The reason is that we have not provided any information about the energy injected at $t = 0$. By making the dyke or sill infinitely narrow, we have assumed that this heat is initially confined in an infinitely narrow space (i.e., the plane $x = 0$). Presumably, this can't quite be right as a finite amount of heat in an infinitely small volume must mean an infinite temperature. But the assumption provides a useful starting point. The remaining problem is how to specify the amount of heat injected. To understand how to do this, consider the column of material that lies above and below a base area A in the yz-plane (i.e, a column of material perpendicular to the plane of the dyke or sill that extends infinitely to $x \to \pm\infty$. As T is assumed to depend only x, there is no conduction of heat out of the sides of the column, so the total amount of heat contained in the column must remain constant at all times t. We concern ourselves only with the amount caused by a temperature excess beyond the background temperature T_0. This amount of heat is, at any time t, given by

$$H = \int_V \rho c (T - T_0) \, \mathrm{d}V = A \rho c \int_{-\infty}^{\infty} (T(x, t) - T_0) \, \mathrm{d}x \tag{4}$$

Following the argument above, the amount of energy H should remain constant in time. That is, the integral in (5) should not depend on t. However, at time $t = 0$, the temperature T is equal to T_0 everywhere except at $x = 0$, and this allows us to identify H as the amount of energy contained in a part of the dyke or sill that has surface area A.[1] We can then define $E_0 = H/A$ as the amount of energy injected per unit surface area of the dyke. The relevant information that tells us about heat injection in the problem is therefore that

$$\rho c \int_{-\infty}^{\infty} (T(x, t) - T_0) \, \mathrm{d}x = E_0 \qquad \text{for all } t > 0 \tag{5}$$

with E_0 constant.

[1] Here, having squashed the dyke into a plane of zero thickness, we ignore the fact that the dyke or sill actually has two surfaces, i.e. left/right or top/bottom

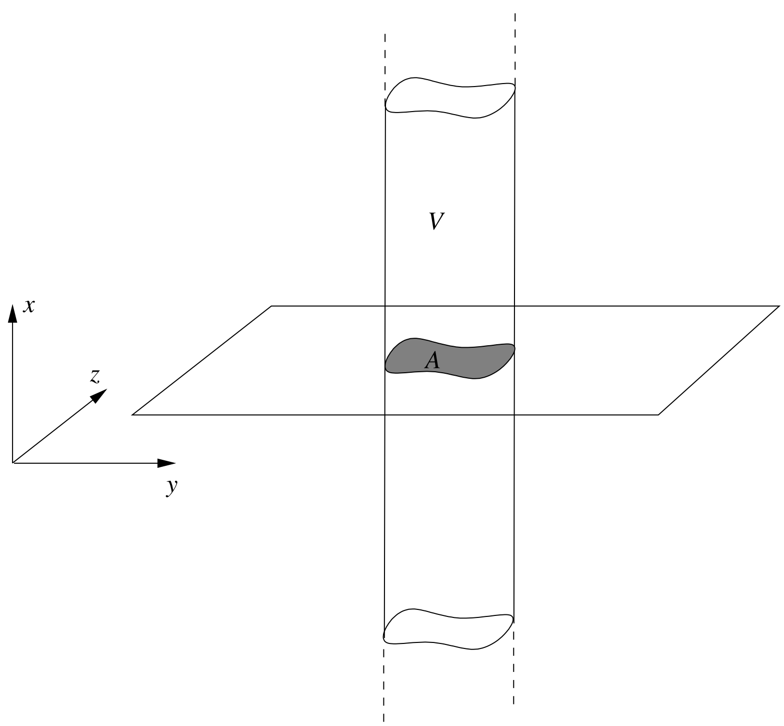

Figure 1: A column of material V lying above and below base area A.

Exercise 1 *Let V be the column of material given by $0 < y < L$, $0 < z < L$, $-R < x < R$, where L and R are fixed (i.e., do not depend on time). Consider conservation of heat in the absence of advection or heat production,*

$$\rho c \frac{\partial T}{\partial t} - \nabla \cdot (k \nabla T) = 0.$$

If $T = T(x,t)$, show that the 'excess' heat content $\int_V \rho c(T - T_0)\, \mathrm{d}V$ of the column changes as

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V \rho c(T - T_0)\, \mathrm{d}V = -kL^2 \left(\left.\frac{\partial T}{\partial x}\right|_{x=R} - \left.\frac{\partial T}{\partial x}\right|_{x=-R} \right).$$

Show (by taking the limit $R \to \infty$) that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^{\infty} \rho c(T(x,t) - T_0) = 0.$$

Note 1 *There is a potential flaw in assuming that the heat equation (1) holds* for all *x when $t > 0$: we are therefore assuming that this equation also holds at $x = 0$, the location of the dyke itself, so that the dyke effectively behaves as part of the host rock. This may be wrong if dyke initially contains liquid magma that releases a significant amount of heat during solidification. If that is the case, then the temperature of the dyke may remain at the melting point until all the latent heat is conducted away, which the present model cannot account for.*

Exercise 2 *Another physical setting that the model can be applied to is sudden heat* production *along a plane. This can occur during an earthquake, where heat is produced on a fault (which can be idealized as a plane) by a sudden slip in the presence of friction. If τ (the Greek letter tau) denotes the amount of friction per unit area of the fault (units of N m^{-2}) and the slip distance is d, what is E_0?*

We can contrast the dyke/sill problem with the temperature wave problem we looked at before. The latter was an example of the heat equation forced by a boundary condition that varies over time. In fact, we considered a particular type of time-varying boundary condition by considering 'sinusoidal' (sine-wave-like) variations by imposing $T(0,t) = T_0 \cos(\omega t)$. The problem we are looking at here is different in that we apply boundary conditions that are fixed, and instead force the problem with an initial condition. That is, the temperature field is not driven by variations at a boundary over time, but by the state in which it started out.

The form of a similarity solution

To be definite, and to make clear some technical points, we re-state the problem here. We have

$$\rho c \frac{\partial T}{\partial t} - k\frac{\partial^2 T}{\partial x^2} = 0 \qquad \text{for } t > 0 \tag{6a}$$

$$\lim_{t\to 0} T(x,t) = T_0 \qquad \text{for any fixed } x \neq 0 \tag{6b}$$

$$\lim_{x\to\pm\infty} \left.\frac{\partial T}{\partial x}\right|_{(x,t)} = 0 \qquad \text{for any fixed } t > 0 \tag{6c}$$

$$\rho c \int_{-\infty}^{\infty} (T(x,t) - T_0)\,\mathrm{d}x = E_0 \qquad \text{for all } t > 0. \tag{6d}$$

The reason why we have re-written the initial condition (2) in the form (6b) is to make it clear what we mean by $T(x,0) = T_0$: We expect some odd behaviour at time $t = 0$, as the temperature at $x = 0$ (the location of the infinitely narrow dyke[2] containing a finite amount of energy) must then be infinite. The easiest way to define an initial temperature field $T(x,0)$ is to take the limit $t \to 0$ in the function $T(x,t)$ while staying at a fixed position x. Similarly, the limit in the boundary condition (3) should be taken at a given point in time. The relevance of these technicalities will become clearer below.

Now, as with the temperature wave solutions, the approach we will pursue here is to *guess* a general form of the solution that we might expect, and figure out what is required to make this guess work: in the case of temperature waves, with $T(x,t) = \mathrm{Re}\,[T_0 \exp(i\omega t + \lambda x)]$, a particular choice of λ was required. Here, we would like to use some intuition for what the solution should do to motivate a guess as to its general form.

At time $t = 0$, all the excess heat (associated with temperatures above T_0) is concentrated in the dyke, so initially there will be an infinite temperature gradient causing conduction out of the dyke. This should cause a region of elevated temperatures to form immediately around the dyke as the heat initially contained in the dyke spreads out. However, we still expect heat to be relatively concentrated, so temperature gradients are still fairly high. This will cause further conduction , spreading heat further away from the dyke. As the total amount of heat in the system stays the same, this however means that temperatures close to the dyke have to fall. So what we expect is a temperature distribution $T(x,t)$ that is initially concentrated near $x = 0$ but then spreads out. The maximum temperature will presumably always be at the dyke location $x = 0$, but this will drop over time.

Now suppose that the temperature distribution retained the same spatial *shape* over time, but that this shape got spread out sideways, and its amplitude got reduced

[2] or sill, but to keep the language simpler, we'll talk about dykes from now on.

at the same time. How can we try to capture this mathematically? One form to try would be to write

$$T(x,t) = T_0 + t^{-\alpha}\theta\left(\frac{x}{t^\beta}\right) \tag{7}$$

where θ is a function of the single variable x/t^β. This is often called the 'similarity variable' and denoted by a single symbol, the Greek letter ξ (read: xi):

$$\xi = \frac{x}{t^\beta}, \tag{8}$$

in which case (7) can be written as

$$T(x,t) = T_0 + t^{-\alpha}\theta(\xi). \tag{9}$$

The term T_0 is simply there because we are interested in temperatures in excess of the background. But why does the second term conform to our intuition developed above?

The function θ describes the general spatial 'shape' of the temperature field, as it encapsulates the dependence of T on position x. What changes over time t is that the shape gets more 'spread out'. To see this, suppose that θ describes a temperature field that drops off with distance from the dyke, so θ decreases with ξ (when $\xi > 0$). Now suppose we look at this at different times t, and ask how far out we have to go before θ drops to a given level. The function $\theta(\xi)$ will always take the same value a fixed value of ξ. Suppose the given level of θ is attained at some fixed value $\xi = \xi_0$. With (8), this means that in terms of physical distance, the given level is attained at $x = t^\beta \xi_0$: assuming $\beta > 0$, the physical distance increases with time t in a way that is proportional to t^β.

Similarly, the factor $t^{-\alpha}$ represents the fact that, as the temperature field spreads out, its amplitude must also drop off. At the dyke location $x = 0$, the temperature field will take the form $T_0 + t^{-\alpha}\theta(0)$, so (with $\alpha > 0$) this represents a dropping-off in temperature over time. These observations are the reason why a solution of the form (7) is called a *similarity solution*: its shape always remains the same (or *self-similar*) but its amplitude and width change.

Perhaps the best way to see how this works is to actually solve the problem, and then see what the solution does. Remember that we are simply going to *try* a solution of the form (7), and that we are still not guaranteed that this will work. In terms of solving the problem (6), trying this form will have the advantage that, if this approach works, we have to find a function θ of a single variable ξ, rather than a function $T(x,t)$ of two variables. Alongside θ, we also have to find suitable exponents α and β (think of this as similar to the way we wrote the solution for the temperature wave in the form $T(x,t) = \mathrm{Re}\,[T_0 \exp(i\omega t + \lambda x)]$, where we had to find the right value of λ).[3]

[3]In both cases, the temperature waves and the present dyke/sill problem, there are more general

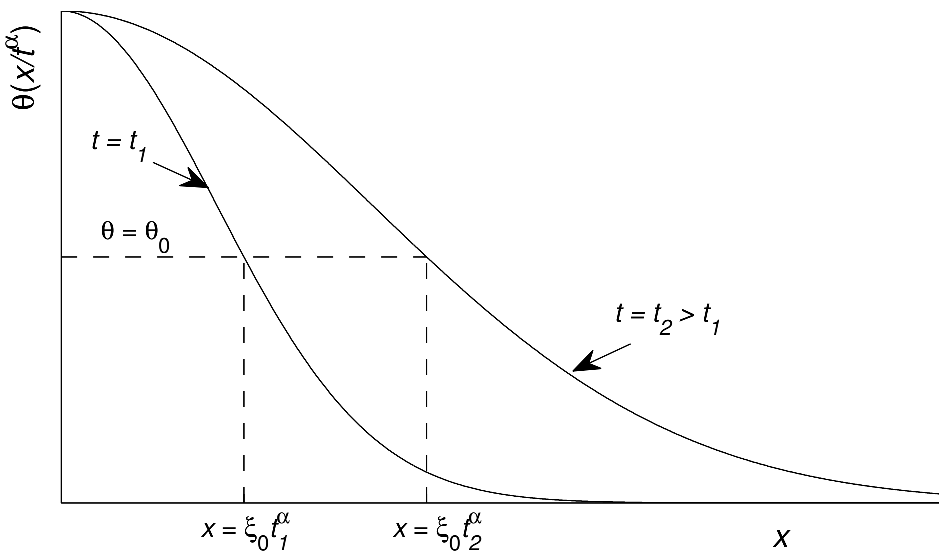

Figure 2: The position x at which the function $\theta(x/t^\alpha)$ attains some given value $\theta = \theta_0$ (depicted by the horizontal dashed line in the plot) depends on the value of t. When t takes a larger value t_2, the corresponding position x is larger than for a smaller time t_1.

Transforming to the similarity variable

To test whether a solution of the form (9) with ξ defined through (8) can be made to work, we have to substitute this into (6). Take the heat equation (6a) first. To substitute for T, we note that with (9) and (8), we have

$$\begin{aligned}\frac{\partial T}{\partial t} &= \frac{\partial(t^{-\alpha}\theta(\xi))}{\partial t} \\ &= -\alpha t^{-\alpha-1}\theta(\xi) + t^{-\alpha}\theta'(\xi)\frac{\partial \xi}{\partial t} \\ &= -\alpha t^{-\alpha-1}\theta(\xi) - \beta x t^{-\alpha-\beta-1}\theta'(\xi),\end{aligned}$$

mathematical methods that can be used to *prove* that a particular form of solution must hold. For the temperature wave problem, the relevant approach is to use so-called *Fourier Transforms*, which is beyond the scope of this course. Many classes on partial differential equations or on applied complex analysis will covers this; likewise, a text on mathematical methods for physicists will provide an introduction to Fourier Transforms. For the present problem, it turns out that one can use scaling methods to show that the solution *must* take the form $T(x,t) = T_0 + t^{-\alpha}\theta\left(\frac{x}{t^\beta}\right)$ advocated in (7).

where the prime denotes differentiation with respect to ξ. We also have

$$\begin{aligned}\frac{\partial T}{\partial x} &= \frac{\partial(t^{-\alpha}\theta(\xi))}{\partial x}\\ &= t^{-\alpha}\theta'(\xi)\frac{\partial \xi}{\partial x}\\ &= t^{-\alpha-\beta}\theta'(\xi),\end{aligned}$$

and so

$$\begin{aligned}\frac{\partial^2 T}{\partial x^2} &= \frac{\partial\big(t^{-\alpha-\beta}\theta'(\xi)\big)}{\partial x}\\ &= t^{-\alpha-\beta}\theta''(\xi)\frac{\partial \xi}{\partial x}\\ &= t^{-\alpha-2\beta}\theta'(\xi).\end{aligned}$$

Substituting into (6a), we get

$$\rho c\left[-t^{-\alpha-1}\theta(\xi) - \beta x t^{-\alpha-\beta-1}\theta'(\xi)\right] - k t^{-\alpha-2\beta}\theta''(\xi) = 0. \tag{10}$$

We see that we get something that looks like an ordinary differential equation for θ in terms of ξ, and this in a way is the primary motivation for this approach: ordinary differential equations are easier to handle than partial differential equations. Now, the whole point here is that, by assumption, θ is only allowed to depend on x and t through the similarity variable $\xi = x/t^{\beta}$, but cannot depend on x and t separately. It follows that we should not have x and t appearing separately in the equation (10) (through coefficients like $t^{-\alpha-1}$, $xt^{-\alpha-\beta-1}$ and $t^{1-2\alpha}$), but only in the form of ξ. To make this happen, we can first write x in terms of ξ and t as $x = \xi t^{\beta}$ and substitute:

$$\begin{aligned}\rho c\left[-t^{-\alpha-1}\theta(\xi) - \beta \xi t^{\beta} t^{-\alpha-\beta-1}\theta'(\xi)\right] - k t^{-\alpha-2\beta}\theta''(\xi) &=\\ \rho c\left[-t^{-\alpha-1}\theta(\xi) - \beta t^{-\alpha-1}\xi\theta'(\xi)\right] - k t^{-\alpha-2\beta}\theta''(\xi) &= 0.\end{aligned}$$

Now we still have a lot of coefficients involving t left, and we can gather them all together by dividing both sides by $t^{-\alpha-1}$:

$$-\rho c\left[\alpha\theta(\xi) + \beta\xi\theta'(\xi)\right] - k t^{1-2\beta}\theta''(\xi) = 0.$$

Now, in order for θ not to depend explicitly to ξ, we have to make the one remaining coefficient involving t disappear as well. The only way to do this for general $t > 0$ is to set

$$1 - 2\beta = 0 \tag{11}$$

so that

$$\beta = \frac{1}{2}, \tag{12}$$

and

$$\rho c\left[\alpha\theta(\xi)+\frac{1}{2}\xi\theta'(\xi)\right]+k\theta''(\xi)=0. \tag{13}$$

We see immediately that β cannot be chosen arbitrarily to make a similarity solution work, but what about α?

We still have to deal with the initial and boundary conditions in (6). Take (6b) next. Substituting, we get

$$\lim_{t\to 0}\left[T_0+t^{-\alpha}\theta\left(\frac{\xi}{t^\beta}\right)\right]=T_0 \qquad \text{for any fixed } x\neq 0.$$

Again, because we want to find θ as a function of the similarity variable $\xi = x/t^\beta$ only, we should transform this into an expression involving ξ. We know that $\beta > 0$, and therefore the limit $t \to 0$ implies $\xi \to \infty$ if $x > 0$, and $\xi \to -\infty$ if $x < 0$. Because the limit is taken with x fixed, we can also write $t = |x|^{1/\beta}|\xi|^{-1/\beta}$ inside the limit, and treat x as a constant. Cancelling the T_0 on both sides,

$$\lim_{\xi\to\pm\infty}|x|^{-\alpha/\beta}|\xi|^{\alpha/\beta}\theta(\xi)=0.$$

But x is fixed as the limit is taken and so can be taken outside the limit, giving

$$\lim_{\xi\to\pm\infty}|\xi|^{\alpha/\beta}\theta(\xi)=0. \tag{14}$$

This effectively gives us a boundary condition on θ, but does not yet help find the exponent α. Still, we have two more equations in (6) to go.

Next, take (6c). This now takes the form

$$\lim_{x\to\pm\infty}t^{-\alpha-\beta}\theta'\left(\frac{x}{t^\beta}\right)=0. \qquad \text{for any fixed } t>0.$$

Again, we want to write this in terms of $\xi = x/t^\beta$. For fixed $t > 0$, the limit $x \to \pm\infty$ clearly corresponds to the limit $\xi \to \pm\infty$. Taking the fixed $t^{-\alpha-\beta}$ outside the limit as well, we have

$$\lim_{\xi\to\pm\infty}\theta'(\xi)=0. \tag{15}$$

We get a second boundary condition on θ, but still no information on α.

This brings us to the last piece of information in (6), namely (6d): Substituting, we get

$$\rho c\int_{-\infty}^{\infty}t^{-\alpha}\theta\left(\frac{x}{t^\beta}\right)\,\mathrm{d}x=E_0 \qquad \text{for all } t>0.$$

Again, transform to $\xi = x/t^\beta$, so $\mathrm{d}x = t^\beta\,\mathrm{d}\xi$. We get

$$\rho c t^{-\alpha+\beta}\int_{-\infty}^{\infty}\theta(\xi)\,\mathrm{d}\xi=E_0 \qquad \text{for all } t>0.$$

Now the right-hand side is independent of t, and so the left-hand side also cannot depend on t. The only way to make this work is to put

$$-\alpha + \beta = 0 \tag{16}$$

and hence

$$\alpha = \beta = \frac{1}{2}, \tag{17}$$

in which case

$$\rho c \int_{-\infty}^{\infty} \theta(\xi)\,\mathrm{d}\xi = E_0 \tag{18}$$

We see that, like β, α also cannot be chosen arbitrarily. However, with the choice $\alpha = \beta = 1/2$, we now have a sensible problem. This consists of (14) with $\alpha = 1/2$,, which we can write as

$$\lim_{\xi \to \pm\infty} \xi\theta(\xi) = 0, \tag{19}$$

as well as (15), (18) and (13) with $\alpha = 1/2$:

$$\frac{\rho c}{2}\left[\theta(\xi) + \xi\theta'(\xi)\right] + k\theta''(\xi) = 0. \tag{20}$$

Solution

All that remains now is to solve for $\theta(\xi)$, and look at the actual solution to understand how this works. We start with (20). This is a second-order differential equation with non-constant coefficients, and even if you have taken a basic differential equations course, you may not immediately be able to see what to do with it. There is a trick here, which often becomes useful in similarity solution problems: we recognize that we can use the product rule in reverse. In particular, we notice that

$$\theta(\xi) + \xi\theta'(\xi) = \frac{\mathrm{d}(\xi\theta)}{\mathrm{d}\xi}.$$

Apply the product rule to the right-hand side to see this.

Exercise 3 *Show that*

$$n\theta(\xi) + \xi\theta'(\xi) = \frac{1}{\xi^{n-1}}\frac{\mathrm{d}(\xi^n\theta)}{\mathrm{d}\xi}.$$

Using this, we can write (20) in the form

$$\frac{\rho c}{2}\frac{\mathrm{d}(\xi\theta)}{\mathrm{d}\xi} + k\frac{\mathrm{d}^2\theta}{\mathrm{d}\xi^2} = 0$$

We can now integrate once with respect to ξ to find

$$\frac{\rho c}{2}\xi\theta(\xi) + k\theta'(\xi) = C$$

The constant of integration can be found by taking the limit of $\xi \to \pm\infty$. We have from (15) that $\theta'(\xi) \to 0$, while $\xi\theta(\xi) \to 0$ from (19). This requires that

$$C = 0$$

and hence

$$\frac{\rho c}{2}\xi\theta(\xi) + k\theta'(\xi) = 0$$

Separating variables, we get

$$\frac{1}{\xi}\frac{\mathrm{d}\theta}{\mathrm{d}\xi} = -\frac{\rho c \xi}{2k}.$$

Integrating both sides with respect to ξ,

$$\log[\theta(\xi)] = -\frac{\rho c \xi^2}{4k} + C.$$

Exponentiating, we get

$$\theta(\xi) = \tilde{C}\exp\left(-\frac{\rho c \xi^2}{4k}\right) \tag{21}$$

where $\tilde{C} = \exp(C)$. We can find the constant $\tilde{C}$ in principle by applying (18):

$$\rho c \int_{-\infty}^{\infty} \tilde{C}\exp\left(-\frac{\rho c \xi^2}{4k}\right) \mathrm{d}\xi = E_0,$$

so that

$$\tilde{C} = \frac{E_0}{\rho c \int_{-\infty}^{\infty} \exp\left(-\frac{\rho c \xi^2}{4k}\right) \mathrm{d}\xi}.$$

The integral on the right can be computed in principle (an analytical form requires a few tricks that I will not go into here), but this is not particularly important to understand the solution. The full solution is in fact $T(x,t) = T_0 + t^{-\alpha}\theta(\xi)$, or, with $\alpha = 1/2$, $\xi = x/t^{1/2}$,

$$T(x,t) = T_0 + \tilde{C}t^{-1/2}\exp\left(-\frac{\rho c x^2}{4kt}\right) \tag{22}$$

We want to see what this looks like. For any fixed time t, the solution is a bell-shaped curve (a so-called 'Gaussian curve') centred on the original dyke location at $x = 0$, with T tending to T_0 at large distances x from the dyke. We can also try to understand how this curve evolves over time. One measure of the 'width' of the bell-shaped curve is the distance x at which the bell-shaped part represented by

$$\tilde{C}t^{-1/2}\exp\left(-\frac{\rho c x^2}{4kt}\right)$$

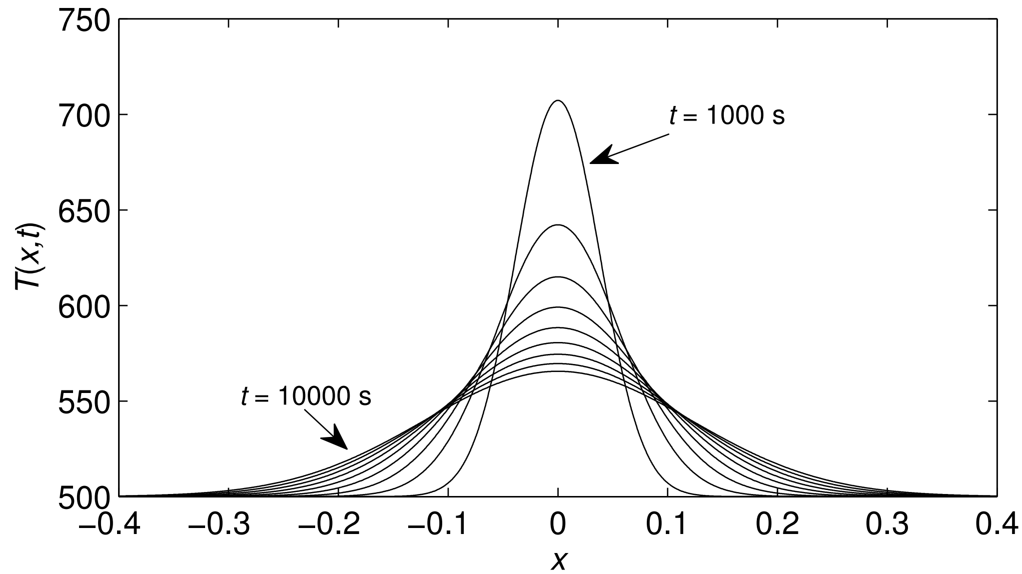

Figure 3: The similarity solution (22) plotted at 1000 s intervals from $t = 1000$ s to $t = 10000$ s. Parameter values are appropriate for rock, $\rho = 2.7 \times 10^3$ kg m^{-3}, $c = 1000$ J kg^{-1} K^{-1}, $k = 2$ W m^{-1} K^{-1}, $T_0 = 500$ K and $E_0 = 5.4 \times 10^7$ J m^{-3} (which would be the amount of heat stored in a 1 cm wide layer of rock with the same values of ρ and c and a temperature exceeding T_0 by 2000 K). This gives $\tilde{C} = 6.6 \times 10^3$ K s$^{1/2}$.

reaches half its maximum height. The maximum height is clearly attained at $x = 0$, and the height itself is $\tilde{C}t^{-1/2}$. So, as expected the height of the curve decreases over time. The width at which half that height is attained corresponds to the values of x at which the exponential equals 1/2, or equally, where the argument of the exponential reaches log(1/2). Let these values be denotes by x_{width}, so that

$$-\frac{\rho c x_{\text{width}}^2}{4kt} = \log(1/2).$$

Solving this gives

$$x_{\text{width}} = \pm\sqrt{\frac{4k\log(2)}{\rho c}}t^{1/2}.$$

The width of the bell-shaped curve therefore increases as $x_{\text{width}} \propto t^{1/2}$, just as its height decreases as $\propto t^{-1/2}$. Together, the decrease in amplitude and increase in width ensure that the area under the curve (which is essentially the total energy of the system per unit surface area of the dyke, divided by ρg) remains constant.

Exercise 4 *We could have formulated the problem differently: instead of insisting*

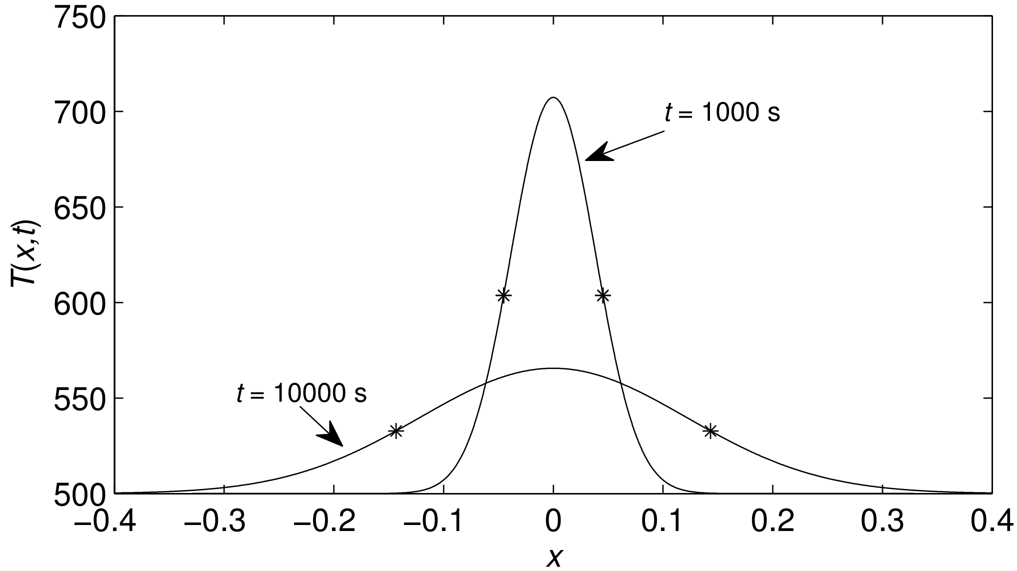

Figure 4: The solutions at $t = 1000$ s and $t = 10000$ s shown in figure 3, with the 'widths' $x_{\rm width}$ indicated by stars.

that flux at infinity is zero, we could have insisted on a fixed temperature T_0 at infinity,

$$\begin{aligned} \rho c \frac{\partial T}{\partial t} - k \frac{\partial^2 T}{\partial x^2} &= 0 && \textit{for } t > 0 \\ \lim_{t \to 0} T(x,t) &= T_0 && \textit{for any fixed } x \neq 0 \\ \lim_{x \to \pm\infty} T(x,t) &= 0 && \textit{for any fixed } t > 0 \\ \rho c \int_{-\infty}^{\infty} (T(x,t) - T_0)\, \mathrm{d}x &= E_0 && \textit{for all } t > 0. \end{aligned}$$

Again assume a solution of the form (7), and transform the problem to the similarity variable ξ in (8). Show that we again get $\alpha = \beta = 1/2$, and that the problem now takes the form

$$\frac{\rho c}{2} \left[\theta(\xi) + \xi \theta'(\xi)\right] + k\theta''(\xi) = 0, \tag{23a}$$

$$\lim_{\xi \to \pm\infty} \xi \theta(\xi) = 0, \tag{23b}$$

$$\lim_{\xi \to \pm\infty} \theta(\xi) = 0, \tag{23c}$$

$$\rho c \int_{-\infty}^{\infty} \theta(\xi)\, \mathrm{d}\xi = E_0. \tag{23d}$$

Note that (23b) in fact implies (23c): if $\xi\theta(\xi)$ tends to zero as ξ tends to $\pm\infty$, then surely so must $\theta(\xi)$. Integrate (23a) again to show that

$$\frac{\rho c}{2}\xi\theta(\xi) + k\theta'(\xi) = C$$

Use (23b) on its own to show that we must have $C = 0$. Hint: if $C \neq 0$, then $\theta'(\xi)$ approaches a finite limit. Why is this not possible?