EOS 352 Continuum Dynamics
Temperature Waves

February 9, 2024

Overview

These notes cover the following

- How to use complex variables to solve the heat equation
- Superposition: exploiting the linearity of the heat equation

Temperature waves

Figure 1 shows two temperature time series measured in Ottawa. The pink line shows air temperature, while the blue line represents temperature 15 cm underground. There are some obvious similarities and differences between these two time series. The most obvious difference is that the daily temperature fluctuations that are clearly visible in the air temperature record are heavily suppressed at 15 cm depth in the ground, whereas the annual temperature signal is much more clearly visible at depth. However, the annual temperature signal is delayed somewhat underground compared with the surface, and negative temperatures (in celsius) are also suppressed.

Here we ask whether these qualitative features of the time series can be explained by theory. In the heat equation, we have a model for temperature evolution based directly on conservation of heat and a couple of empirically grounded constitutive relations for conductive heat flux and heat density. Can this model explain an actual real-world observation? To test this, we have to *solve* the heat equation. In its most general form, this equation takes the form

$$\rho c \frac{\partial T}{\partial t} + \rho c \mathbf{u} \cdot \nabla T - \nabla(\cdot k \nabla T) = a$$

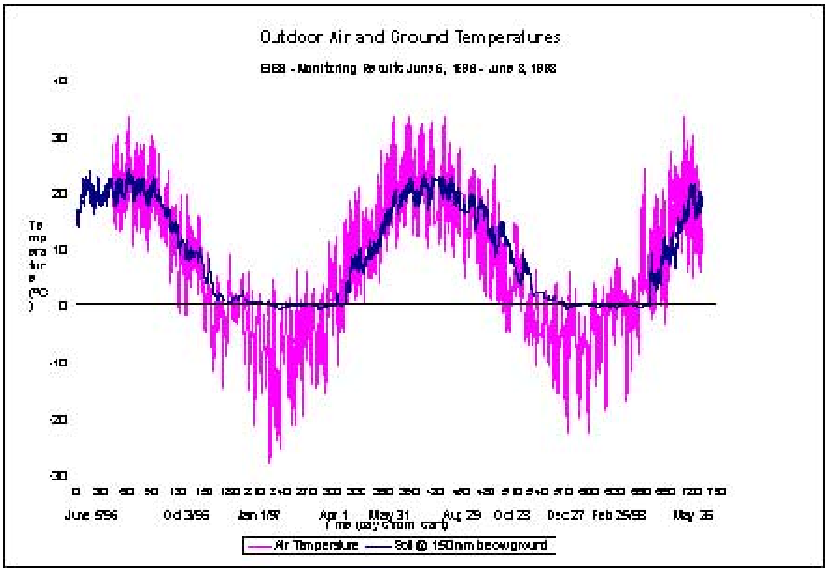

Figure 1: Two time series of temperature taken concurrently in Ottawa, one (pink) showing air temperature, the other (blue) showing soil temperature at 15 cm depth.

It is worth noting that this form of the heat equation with constant c assumes that heat content per unit volume (heat density) is $h = \rho c T$ and is unable to capture latent heat effects associated with phase changes: where there is a phase change, heat density changes discontinuously at constant temperature (e.g. in going from ice to water at O° C), while $\rho c T$ changes continuously (linearly, in fact) with temperature. This is clearly relevant to the time series in figure 1 as soil temperatures appear to become stuck at 0° C in winter, presumably as water in the soil freezes and melts.

We simplify our task somewhat further by making a few additional assumptions. Firstly, we are concerned with heat transport in a solid, where advection is unimportant as velocities are essentially zero (neglecting any small velocities that arise due to thermal expansion etc. as well as heat transport by any water that flows downwards through the soil, which considerably complicates any model). Hence we set $\mathbf{u} = \mathbf{0}$. We also assume that there are negligible heat sources (which could in practice be radioactivity, oxidation reactions or biological activity), and put $a = 0$. Lastly, we assume that temperature depends only on time t and depth in the ground, which we measure by the coordinate x, but not on horizontal displacements (measured by y and z). We also assume that thermal conductivity k is a constant. Then the heat equation becomes

$$\rho c \frac{\partial T}{\partial t} - k \frac{\partial^2 T}{\partial x^2} = 0. \tag{1}$$

Now, stating the heat equation alone isn't enough to specify a solvable problem. Clearly, the temperature variations in the soil shown in figure 1 are driven by temperature variations in the air. The way that this is expressed for the heat equation is through a *boundary condition* at the surface $x = 0$. We are interested in termperature oscillations, and the simplest way to do this is to have a sinusoidal temperature imposed at the surface. We choose to do this in the form

$$T(0,t) = T_0 \cos(\omega t) \tag{2}$$

where T_0 is the amplitude of temperature variations at the surface, and ω is known as the 'angular frequency'. To understand what an angular frequency is, note that the period t_0 of an oscillation is the time period required to go through one cycle. In the cosine above, a single cycle is gone through when the argument of the cosine (i.e., ωt) goes from 0 to 2π. The length of time t_0 that is required to do this is clearly given by $\omega t_0 = 2\pi$. Hence $\omega = 2\pi/t_0 = 2\pi/$ period of oscillation. This can be linked to the ordinary frequency f, which measures the number of cycles per unit time. Ordinary frequency is therefore $1/t_0$, so that $\omega = 2\pi f$.

Exercise 1 *What are the S.I. units of ω? If the cosine wave describes an annual cycle, what is ω? What if the cosine wave describes a daily cycle? A Milankovitch cycle of 40,000 years?*

Now we have to solve the heat equation with this boundary condition. This is where the complex variable material becomes useful. The boundary condition (2) at $x = 0$ can also be expressed in the form

$$T(0,t) = \mathrm{Re}\left[T_0 \exp(i\omega t)\right]. \tag{3}$$

To make use of this, we simply *try* and see if a solution of the form

$$T(x,t) = \mathrm{Re}\left[T_0 \exp(i\omega t + \lambda x)\right]. \tag{4}$$

can be made to satisfy the problem. Clearly, this expression satisfies the boundary condition (3). The remaining question is simply whether it can also be made to satisfy the heat equation (1). To find out, substitute

$$\begin{aligned}\rho c \frac{\partial T}{\partial t} - k\frac{\partial^2 T}{\partial x^2} &= \rho c \frac{\partial}{\partial t}\left\{\mathrm{Re}\left[T_0 \exp(i\omega t + \lambda x)\right]\right\} - k\frac{\partial^2}{\partial x^2}\left\{\mathrm{Re}\left[T_0 \exp(i\omega t + \lambda x)\right]\right\} \\ &= 0.\end{aligned}$$

But from the notes on complex variables, we know that derivatives with respect to a real variable like $\partial/\partial t$ and taking the real part commute, so

$$\begin{aligned}\frac{\partial}{\partial t}\left\{\mathrm{Re}\left[T_0 \exp(i\omega t + \lambda x)\right]\right\} &= \mathrm{Re}\left[T_0 \frac{\partial}{\partial t}\exp(i\omega t + \lambda x)\right] \\ &= \mathrm{Re}\left[i\omega T_0 \exp(i\omega t + \lambda x)\right]\end{aligned}$$

Recall that there is no simple expression for the real part of the product of two complex numbers, $\mathrm{Re}(z_1 z_2)$, so there is no point in trying to simplify further just yet. We similarly have

$$\begin{aligned}\frac{\partial^2}{\partial x^2}\left\{\mathrm{Re}\left[T_0 \exp(i\omega t + \lambda x)\right]\right\} &= \mathrm{Re}\left[T_0 \frac{\partial^2}{\partial x^2}\exp(i\omega t + \lambda x)\right] \\ &= \mathrm{Re}\left[\lambda^2 T_0 \exp(i\omega t + \lambda x)\right]\end{aligned}$$

Putting this back in the heat equation gives

$$\rho c \mathrm{Re}\left[i\omega T_0 \exp(i\omega t + \lambda x)\right] - k\mathrm{Re}\left[\lambda^2 T_0 \exp(i\omega t + \lambda x)\right] = 0$$

But, if a and b are real and z_1 and z_2 are complex numbers, then the properties of Re described in the notes on complex variables ensure that $a\mathrm{Re}(z_1) + b\mathrm{Re}(z_2) = \mathrm{Re}(az_1 + bz_2)$. Applying this here allows us to write

$$\mathrm{Re}\left[(i\rho c\omega - k\lambda^2)T_0 \exp(i\omega t + \lambda x)\right] = 0$$

Now one way in which to ensure that the expression on the left equals zero is to demand that

$$i\rho c\omega - k\lambda^2 = 0. \tag{5}$$

Hence the form of $T(x,t)$ in (4) can be made to satisfy the heat equation by choosing a particular λ, namely that which satisfies (5).

Note 1 *So far we have shown that putting $i\rho c\omega - k\lambda^2 = 0$ ensures that the heat equation is satisfied, but you may be left wondering whether this equality is in fact necessary or not. Note that prior to putting $i\rho c\omega - k\lambda^2 = 0$, we only knew that*

$$\mathrm{Re}\left[(i\rho c\omega - k\lambda^2)T_0 \exp(i\omega t + \lambda x)\right] = 0$$

But x and t are arbitrary (the only constraints being that x must be positive. So we can let $x \to 0$ to find

$$\mathrm{Re}\left[(i\rho c_p\omega - k\lambda^2)T_0 \exp(i\omega t)\right] = 0$$

If t is arbitrary, then we can put $t = 0$ so

$$\mathrm{Re}\left[(i\rho c\omega - k\lambda^2)T_0\right] = 0$$

We can also put $t = \pi/(2\omega)$ so that $\exp(i\omega t) = i$, and

$$\mathrm{Re}\left[i(i\rho c\omega - k\lambda^2)T_0\right] = 0$$

But now consider a complex number $z = a + ib$ about which we know that $\mathrm{Re}(z) = 0$ and $\mathrm{Re}(iz) = 0$. Writing this out, we have $\mathrm{Re}(a + ib) = a = 0$ and $\mathrm{Re}[i(a + ib)] = -b = 0$. Hence $z = a + ib = 0$. In our case, $z = (i\rho c\omega - k\lambda^2)T_0 = 0$. We know that $T_0 \neq 0$, so we must have

$$i\rho c\omega - k\lambda^2 = 0.$$

Formally, we get

$$\lambda = \pm\sqrt{\frac{i\rho c\omega}{k}} = \pm\sqrt{i}\sqrt{\frac{\rho c\omega}{k}}. \tag{6}$$

The snag is that we apparently have to find the square root of i. Is this itself a complex number, or something more complicated? There are two ways of showing that $\sqrt{i}$ is indeed a complex number, and to compute its real and imaginary parts. Version one is simply to assume that we can write

$$\sqrt{i} = a + ib$$

with a and b real. Squaring both sides,

$$i = a^2 - b^2 + 2iab,$$

from which it follows that

$$a^2 - b^2 = 0 \qquad 2ab = 1.$$

The first inequality implies $b = \pm a$. Substituting this into the second inequality gives

$$\pm 2a^2 = 1.$$

But a is, by assumption, real, so $a^2 > 0$ and we must choose the plus sign, so that

$$b = a$$

and

$$2a^2 = 1.$$

Hence $a = \pm 1/\sqrt{2}$ and

$$\sqrt{i} = \pm\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right). \tag{7}$$

Putting this back into (6), we find

$$\lambda = \pm\sqrt{\frac{\rho c\omega}{2k}}\,(1 + i)$$

Note 2 *Another way to find the square root of* i *is to assume it can be written in the polar form* $\sqrt{i} = r\exp(i\theta)$. *Squaring both sides,*

$$i = r^2\exp(i2\theta) = r^2\cos(2\theta) + ir^2\sin(2\theta)$$

so that

$$2^2\cos(2\theta) = 0, \qquad r^2\sin(2\theta) = 1.$$

From the first equality we conclude that $\cos(2\theta) = 0$, *so* $2\theta = \pi/2$ *or* $3\pi/2$. *With* $2\theta = \pi/2$, *the second equality gives* $r^2 = 1$ *or* $r = \pm 1$, *while with* $2\theta = 3\pi/2$, *we get* $r^2 = -1$, *which has no real solutions. Hence* $\theta = \pi/2$, $r = \pm 1$ *and*

$$\sqrt{i} = \pm\exp(i\pi/4) = \pm(\cos(\pi/4) + i\sin(\pi/4))$$

But $\cos(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$, *and we recover (7).*

Now that we have found λ, we can substitute back into (4):

$$T(x,t) = \mathrm{Re}\left\{T_0 \exp\left[i\omega t \pm \sqrt{\frac{\rho c\omega}{2k}}\,(1+i)\right]\right\}. \tag{8}$$

To make sense of this, we need to take the real part. We can re-write

$$\begin{aligned}
T(x,t) &= \mathrm{Re}\left\{T_0 \exp\left[i\omega t \pm \sqrt{\frac{\rho c\omega}{2k}}\,(1+i)\,x\right]\right\}.\\
&= \mathrm{Re}\left\{T_0 \exp\left(\pm\sqrt{\frac{\rho c\omega}{2k}}x\right)\exp\left[i\left(\omega t \pm \sqrt{\frac{\rho c\omega}{2k}}x\right)\right]\right\}\\
&= T_0 \exp\left(\pm\sqrt{\frac{\rho c\omega}{2k}}x\right)\cos\left(\omega t \pm \sqrt{\frac{\rho c\omega}{2k}}x\right). \qquad (9)
\end{aligned}$$

It is important to note here that the same sign out of $\pm$ must be chosen in *both*, the exponential and the cosine: there are only two possibilities for λ, either with a minus in both instances or with a plus.

Note 3 *The above procedure shows the trick to extracting the real part: write the solution in the form*

$$\begin{aligned}
T(x,t) &= \mathrm{Re}\left\{T_0 \exp\left[i\omega t + i\mathrm{Im}(\lambda)x + \mathrm{Re}(\lambda)x\right]\right\}\\
&= \mathrm{Re}\left\{T_0 \exp\left[i(\omega t + \mathrm{Im}(\lambda)x)\right]\exp\left[\mathrm{Re}(\lambda)x\right]\right\}\\
&= T_0 \exp\left[\mathrm{Re}(\lambda)x\right]\cos\left[\omega t + \mathrm{Im}(\lambda)x\right]
\end{aligned}$$

We still need to decide which sign to pick. If we were to pick '+', the solution would be

$$T(x,t) = T_0 \exp\left(\sqrt{\frac{\rho c\omega}{2k}}x\right)\cos\left(\omega t + \sqrt{\frac{\rho c\omega}{2k}}x\right). \tag{10}$$

The important term to look at is the exponential. Because of the positive coefficient, the size of the exponential increases with depth in the ground without bound. The deeper we go underground, the more violent the temperature osciallations expressed by the cosine function become. This is clearly unphysical. But how can this be prevented in the original set-up of the problem? (10) clearly satisfies the heat equation (1) and the boundary conditions (2). The answer is that the heat equation actually requires boundary conditions at another boundary. In our case, there is no actual boundary to sensibly impose, unless we want to model the entire interior of the earth. Instead, the most sensible thing to do is to impose boundary conditions 'at infinity', i.e., a long way from the top of the soil. Physically, a long way down from the top of the soil, we expect a steady geothermal heat flux q_{geo} to be flowing towards the surface. This suggests that we put

$$-k\frac{\partial T}{\partial x} \to -q_{geo}. \tag{11}$$

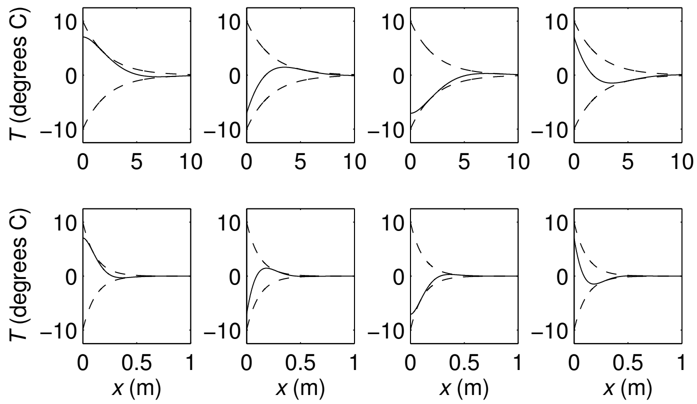

Figure 2: Top row: an annual temperature wave ($\omega = 2\pi/(365 \times 24 \times 3600)$ s^{-1}) with amplitude $T_0 = 10$ C, shown at (left to right) $t = 1.5$ months, 4.5 months, 7.5 months, 10.5 months. Parameter choices are intended to reflect soil, with $k = 0.4$ W m^{-1} K^{-1}, $\rho = 1000$ kg m^{-3}, $c = 800$ W kg^{-1} K^{-1}. The dashed line represent the temperature envelope (minimum and maximum temperatures reached at each x during the cycle). Bottom row: same parameter values, but for a diurnal temperature cycle, $\omega = 2\pi/(24 \times 3600)$ s^{-1}, shown at times $t = 3$ hours, 9 hours, 15 hours, 21 hours.

Note that there is a negative sign on the right because q_{geo} flows *towards* the surface, i.e., in the negative x-direcetion.

Clearly, (11) excludes a solution like (10). If we take the minus sign in (9), we have

$$T(x,t) = T_0 \exp\left(-\sqrt{\frac{\rho c \omega}{2k}} x\right) \cos\left(\omega t - \sqrt{\frac{\rho c \omega}{2k}} x\right). \qquad (12)$$

With this, the heat flux at depth x is

$$-k\frac{\partial T}{\partial x} = \sqrt{\frac{\rho c \omega}{2k}} T_0 \exp\left(-\sqrt{\frac{\rho c \omega}{2k}} x\right) \left[\sin\left(\omega t - \sqrt{\frac{\rho c \omega}{2k}} x\right) - \cos\left(\omega t - \sqrt{\frac{\rho c \omega}{2k}} x\right)\right].$$

Clearly, $-k\partial T/\partial x \to 0$ now as $x \to \infty$, and the solution satisfies (11) when $q_{geo} = 0$. This is a good model when geothermal heat flux is small. The solution is plotted as a function of depth x for various times t in the temperature cycle in figure 2. We will see later how a finite geothermal heat flux can also be accomodated.

Note 4 *This note is an aside that is not necessary to follow the course, but will be of interest to those taking a separate course on partial differential equations (which I recommend!). Many partial differential equations courses teach the method of separation of variables. The temperature wave problem (1) with (2) can also be solved using this method. (1) takes the form*

$$\rho c \frac{\partial T}{\partial t} - k \frac{\partial^2 T}{\partial x^2} = 0$$

Separation of variables for this equation means looking for a solution

$$T(x,t) = \Theta(x)\tau(t),$$

in which the function T that depends on both x and t can be expressed as the product of two functions, each of which depends only on x or t but not on both.[1] *Substituting into the heat equation gives*

$$\rho c \frac{\mathrm{d}\tau}{\mathrm{d}t}\Theta - k\tau\frac{\mathrm{d}^2\Theta}{\mathrm{d}x^2} = 0.$$

We can rearrange this into

$$\frac{\rho c}{k}\frac{1}{\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} = \frac{1}{\Theta}\frac{\mathrm{d}^2\Theta}{\mathrm{d}x^2}.$$

The standard argument now goes that the left-hand side of this equality only depends on t, while the right-hand side depends only on x. As x and t are independent variables, this is only possible if both sides are actually constant (otherwise I could change the value of the left-hand side by changing t while keeping the value of the right-hand side constant by keeping x constant, thereby negating the equality). If both sides are constant, then we can assign this constant a symbol, m:

$$\frac{\rho c}{k}\frac{1}{\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} = m = \frac{1}{\Theta}\frac{\mathrm{d}^2\Theta}{\mathrm{d}x^2}.$$

This gives us two ordinary differential equations, one for τ and one for Θ.

$$\frac{\rho c}{k}\frac{1}{\tau}\frac{\mathrm{d}\tau}{\mathrm{d}t} = m \tag{13a}$$

$$\frac{1}{\Theta}\frac{\mathrm{d}^2\Theta}{\mathrm{d}x^2} = m. \tag{13b}$$

The first is simple to solve by direct integration: we get

$$\frac{\rho c}{k}\log(\tau) = mt + C$$

or

$$\tau(t) = C' \exp\left(\frac{mk}{\rho c}t\right).$$

where $C' = \exp(C)$ is a constant. But we can now restrict the choice of m, as T has to satisfy our boundary conditions. We would like

$$T(0,t) = \Theta(0)\tau(t) = C'\Theta(0)\exp\left(\frac{m}{\rho c}t\right) = \mathrm{Re}\left[T_0 \exp(i\omega t)\right].$$

[1] Note that most functions of two variables cannot be expressed in this way. Try $T(x,t) = x + t$.

There are two exponentials here, but unfortunately also a 'Re', so this doesn't quite work. We dispense with taking the real part for now (and show below why this works), and instead impose the boundary condition

$$T(0,t) = \Theta(0)\tau(t) = C'\Theta(0)\exp\left(\frac{mk}{\rho c}t\right) = T_0\exp(i\omega t). \tag{14}$$

So long as we choose $C'\Theta(0) = T_0$, this can be satisfied provided we also choose

$$m = i\frac{\rho c\omega}{k}. \tag{15}$$

The second equation in (13) can be rewritten in the form

$$\frac{\mathrm{d}^2\Theta}{\mathrm{d}x^2} - m\Theta = 0.$$

The long way to solve this involves integrating factors, but a quick way to find solutions is to assume they take the form

$$\Theta = \exp(\alpha x)$$

Simply substituting gives

$$(\alpha^2 - m)\exp(\alpha x) = 0,$$

and hence

$$\alpha = \pm\sqrt{m} \tag{16}$$

will work.

Now, recall that

$$m = i\frac{\rho c\omega}{k}$$

so that

$$\sqrt{m} = \sqrt{i}\sqrt{\frac{\rho c\omega}{k}}$$

and we can recognize that $\lambda = \sqrt{m}$ from (6):

$$\sqrt{m} = \sqrt{\frac{\rho c\omega}{2k}}(1+i).$$

where we dispense with the $\pm$ because (16) already incorporates both $+\sqrt{m}$ and $-\sqrt{m}$. The solution $T(x,t)$ can then finally be written as

$$\begin{aligned} T(x,t) &= \Theta(x)\tau(t) \\ &= C'\exp(\pm\sqrt{m}x)'\exp\left(\frac{mk}{\rho c}t\right) \\ &= C'\exp\left(i\omega t \pm \sqrt{\frac{\rho c\omega}{2k}}(1+i)x\right). \end{aligned} \tag{17}$$

In order to satisfy (14), we must also have $C' = T_0$. Basically, we have recovered (8), but without the 'Re'. This is because we dropped the 'Re' in the boundary condition (14).

But if a complex function $T(x,t)$ satisfies the heat equation, then so does its real part on its own. This is because taking the real part and differentiation with respect to a real variable commute. In other words, if T satisfies the heat equation, we have

$$\begin{aligned} 0 &= \mathrm{Re}\left[\rho c \frac{\partial T}{\partial t} - k\frac{\partial^2 T}{\partial x^2}\right] \\ &= \rho c \frac{\partial [\mathrm{Re}(T)]}{\partial t} - k\frac{\partial^2 [\mathrm{Re}(T)]}{\partial x^2}, \end{aligned}$$

so that $\mathrm{Re}(T)$ satisfies the heat equation. This (which we knew already by a different route) means that taking the real part of the solution (17) gives us a function that also solves the heat equation. Moreover, taking real parts on both sides of (14) (which T satisfies), we get

$$\mathrm{Re}(T(0,t)) = \mathrm{Re}(T_0 \exp(i\omega t)) = T_0 \cos(\omega t),$$

i.e., we find that (2) is satisfied by $\mathrm{Re}(T)$ rather than T. In fact, the real part of T is

$$\mathrm{Re}(T(x,t)) = T_0 \exp\left(\pm\sqrt{\frac{\rho c \omega}{2k}}x\right)\cos\left(\omega t \pm \sqrt{\frac{\rho c \omega}{2k}}x\right)$$

which is precisely the solution (8) of (1) with (2). Of course, if we also assume that $-k\partial T/\partial x \to 0$ as $x \to 0$ (i.e., (11) with $q_{geo} = 0$) then we must also choose the $-$ sign and we recover (12).

A few properties of the solution

In basic physical terms, we see that the temperature oscillation shown in figure 2 propagates downwards with decreasing amplitude and an increasing time lag. What happens is that heat flowing downwards from the surface when the surface is hot at the maximum in the cycle needs to warm up the soil immediately under the surface. Only once this has happened can the near-surface soil transfer heat downwards to warm up layers of soil that lie further down in the ground. This explains the time lag. However, before deeper layers of soil can be warmed up, the surface tempearture starts to drop again. This causes heat loss back to the surface from the near-surface soils. Less heat is therefore available to warm up layers deeper underground, and this is why the amplitude of the temperature variation decreases with depth.

We can put these observations into a more quantitative framework. Consider first the time lag in warming up deeper layers. This is associated with cosine term

$$\cos\left(\omega t - \sqrt{\frac{\rho c \omega}{2k}}x\right),$$

which is a *travelling wave*: we can write it in the form

$$\cos(a(x - vt))$$

if we put $a = -\sqrt{\rho c\omega/2k}$ and

$$v = \omega\sqrt{\frac{2k}{\rho c\omega}} = \sqrt{\frac{2k\omega}{\rho c}}.$$

The point of writing it in this form is that we can see that the cosine wave moves at a speed v as time t progresses. To understand this, note that the maximum of the wave $\cos(a(x-vt))$ is attained where $a(x-vt) = 0$, i.e. where $x = vt$. So the location at which the cosine wave takes its maximum value moves at speed v downwards into the ground.

We can also calculate the wavelength of the cosine wave. This is the distance x_0 by which we have to increase x to go through an entire cycle of the wave. In other words, it is given by

$$\sqrt{\frac{\rho c\omega}{2k}}x_0 = 2\pi$$

or

$$x_0 = 2\pi\sqrt{\frac{2k}{\rho c\omega}}.$$

The actual solution (12) is the sine wave times an exponentially decreasing envelope function,

$$T_0 \exp\left(-\sqrt{\frac{\rho c\omega}{2k}}x\right),$$

so that the solution is a travelling cosine wave whose amplitude decreases with distance into the ground. A frequently used measure of how fast an exponential function decreases in amplitude is the 'e-folding length' x_e of the envelope. Thishe distance over which the amplitude decreases by a factor e^{-1}, and is therefore given by

$$\sqrt{\frac{\rho c\omega}{2k}}x_e = 1$$

so that

$$x_e = \sqrt{\frac{2k}{\rho c\omega}}.$$

With these observations in mind, we can now understand the time series in figure 1. The e-folding length scale x_e is shorter for larger ω. Recall that ω is angular frequency, equal to $2\pi/$ period of oscillation. For faster temperature oscillations at the surface (e.g. diurnal versus annual), the temperature signal therefore decays faster than for slower oscillations. This is why the annual signal is clearly visible at 15 cm depth (though reduced in amplitude from the surface), whereas the diurnal signal is filtered out quite strongly. We can also understand the time lag between the annual signal at 15 cm depth and at the surface. The cosine wave is a travelling wave propagating into the ground at a finite velocity

$$v = \sqrt{\frac{2k\omega}{\rho c}},$$

and the peak in temperature therefore takes a finite amount of time to reach a point at depth.

Superposition of different frequencies

In our set-up of the problem, we assumed that we had a single angular frequency ω in the problem, whereas the temperature time series in figure 1 clearly contains at least two frequencies. A more realistic boundary condition at the surface than (2) might therefore have been

$$T(0,t) = T_{0,1}\cos(\omega_1 t + \theta_1) + T_{0,2}\cos(\omega_2 t + \theta_2) \tag{18}$$

where we have included a phase shift θ in each cosine because there is no reason why the two oscillations need to reach their maximum at the same time $t = 0$. If we had only one of the two frequencies, it would be a simple matter to adjust the solution derived above in (12) to account for the phase shift. For instance, if we had only the oscillation subscripted with a '1' (so $T_{0,2} = 0$), then a solution of (1) with (18) would be $T(x,t) = T_1(x,t)$ defined by

$$T_1(x,t) = T_{0,1}\exp\left(-\sqrt{\frac{\rho c\omega_1}{2k}}x\right)\cos\left(\omega_1 t - \sqrt{\frac{\rho c\omega_1}{2k}}x + \theta_1\right) \tag{19}$$

and similarly if $T_{0,1} = 0$, a solution to (1) with (18) would be $T(x,t) = T_1(x,t)$, where

$$T_2(x,t) = T_{0,2}\exp\left(-\sqrt{\frac{\rho c\omega_2}{2k}}x\right)\cos\left(\omega_2 t - \sqrt{\frac{\rho c\omega_2}{2k}}x + \theta_2\right). \tag{20}$$

Exercise 2 *Note that the boundary conditions in* (18) *contain* phase angles θ_1 *and* θ_2*, which are assumed to be constant. These were not present in the original version of the problem, in which the boundary condition was* (**??**), $T(0,t) = \cos(\omega t)$. *Consider therefore the slightly generalized version consisting of the heat equation* (1), (11) *with zero geothermal heat flux, and* (2) *with a phase angle,*

$$\rho c\frac{\partial T}{\partial t} - k\frac{\partial^2 T}{\partial x^2} = 0, \tag{21a}$$

$$T(0,t) = T_0\cos(\omega t + \theta), \tag{21b}$$

$$-k\frac{\partial T}{\partial x} \to 0. \tag{21c}$$

Look for a solution

$$T(x,t) = \mathrm{Re}[A \exp(i\omega t + \lambda x)]$$

where A is complex. *Express A in polar form*

$$A = |A| \exp(i\phi)$$

Substitute into (23a) *and show that you still satisfy the heat equation provided*

$$i\rho c\omega - \lambda^2 k = 0.$$

Find the unique value of λ needed to satisfy (23c) *Substitute into* (23b) *and find $|A|$ as well as ϕ. Show that*

$$T(x,t) = T_0 \exp\left(-\sqrt{\frac{\rho c\omega}{2k}}x\right) \cos\left(\omega t - .\sqrt{\frac{\rho c\omega}{2k}}x + \theta\right). \tag{22}$$

Next, consider instead defining a new time variable $t' = t - \theta/\omega$. Let $T'(x,t') = T(x,t)$. By using the chain rule and direct substitution, show that T' satisfies the original problem consisting of (1), (2) *and* (11) *with $q_{geo} = 0$, but with t' repalcing t,*

$$\rho c \frac{\partial T'}{\partial t'} - k \frac{\partial^2 T'}{\partial x^2} = 0, \tag{23a}$$

$$T'(0,t') = T_0 \cos(\omega t'), \tag{23b}$$

$$-k \frac{\partial T'}{\partial x} \to 0. \tag{23c}$$

We already have a solution for this problem

$$T'(x,t') = T_0 \exp\left(-\sqrt{\frac{\rho c\omega}{2k}}x\right) \cos\left(\omega t' - .\sqrt{\frac{\rho c\omega}{2k}}x\right).$$

Show that this is the same as (22).

So how can we cope with both, $T_{0,1}$ and $T_{0,2}$ being non-zero? The important observation is that the heat equation (1) is *linear*, meaning that if $T_1(x,t)$ and $T_2(x,t)$ satisfy the heat equation, then so does their sum $T(x,t) = T_1(x,t) + T_2(x,t)$:

$$\begin{aligned}
\rho c \frac{\partial T}{\partial t} - k \frac{\partial^2 T}{\partial x^2} &= \rho c \frac{\partial (T_1 + T_2)}{\partial t} - k \frac{\partial^2 (T_1 + T_2)}{\partial x^2} \\
&= \rho c \left(\frac{\partial T_1}{\partial t} + \frac{\partial T_2}{\partial t}\right) - k \left(\frac{\partial^2 T_1}{\partial x^2} + \frac{\partial^2 T_2}{\partial x^2}\right) \\
&= \left(\rho c \frac{\partial T_1}{\partial t} - k \frac{\partial^2 T_1}{\partial x^2}\right) + \left(\rho c \frac{\partial T_2}{\partial t} - k \frac{\partial^2 T_2}{\partial x^2}\right) \\
&= 0
\end{aligned}$$

where the last equality follows becayse T_1 and T_2 separately satisfy the heat equation, so the two terms in brackets on the penultimate line both equal zero.

Exercise 3 *The above could have been stated more generally. Show that, if a number of functions $T_i(x,t)$ all satisfy the heat equation, then so does $T = \sum_i c_i T_i$, where the c_i are arbitrary constants.*

Exercise 4 *The concept of linearity in mathematics goes much further than this particular example. Consider an operation $\mathcal{L}$ acting on some object for which multiplication by either real or complex numbers (generically called* scalars *in this context) is defined along with the operation of addition. An example of such an 'object' could be a vector. For instance, for two vectors $\mathbf{a}$ and $\mathbf{b}$, we can define addition $\mathbf{a}+\mathbf{b}$ as well as multiplication by a real (or more exotically, complex) number λ, $\lambda\mathbf{a}$. The object could however be a function, too. For instance, we can define the sum of two functions $f(x)+g(x)$ as well as multiplication of a function by a number λ to form $\lambda f(x)$. The 'operation' can also take many different forms, so long as it produces another object for which addition and multiplication by scalars is defined. For instance, for a vector, the operation could be forming the scalar product with a fixed vector such as $\mathbf{i}$, or for a function, the operation could be taking a derivative, or forming an integral. For instance, we could define $\mathcal{L}(\mathbf{a}) = \mathbf{i}\cdot\mathbf{a}$, or we could have $\mathcal{L}(f) = \int_0^1 f(x)\,\mathrm{d}x$, or $\mathcal{L}(f) = \mathrm{d}f/\mathrm{d}x$. In these examples, the first is an operator acting on a vector $\mathbf{a}$, while the second and third are operators on a function f.*

Whatever the type of object ϕ that the operator acts on, whatever type of scalar c (real or complex) or the operation $\mathcal{L}$, the operation $\mathcal{L}$ is called linear *if*

$$\mathcal{L}(c_1\phi_1 + c_2\phi_2) = c_1\mathcal{L}(\phi_1) + c_2\mathcal{L}(\phi_2)$$

always holds for all ϕ_1, ϕ_2, c_1 and c_2. The advantage of linearity in a set of equations is that, if ϕ_1 and ϕ_2 separately satisfy $\mathcal{L}(\phi_1) = 0$ and $\mathcal{L}(\phi_2) = 0$, then we immediately know that $\mathcal{L}(c_1\phi_1 + c_2\phi_2) = 0$. More generally if a set of ϕ_i, $i = 1,\ldots,n$ satisfy $\mathcal{L}(\phi_i) = 0$ for all i, then $\phi = \sum_i c_i\phi_i$ also satisfies

$$\mathcal{L}(\phi) = \mathcal{L}\left(\sum_i c_i\phi_i\right) = \sum_i c_i\mathcal{L}\left(\phi_i\right) = 0.$$

This is know as the principle of superposition*: a 'weighted' sum (the scalars c_i being the 'weights') of solutions of $\mathcal{L}(\phi) = 0$ is also a solution of the same equation.*

For the following examples, determine whether $\mathcal{L}$ is linear or not. Make sure you are clear what $\mathcal{L}$ is acting on: if $f(x)$ is a function and $\mathcal{L}(f)$ acts on the function f, then linearity is in *f, not in the argument x of the function. In other words, linearity of $\mathcal{L}$ implies that $\mathcal{L}(c_1f_1(x) + c_2f_2(x)) = c_1\mathcal{L}(f_1(x)) + c_2\mathcal{L}(f_2(x))$, and* not *that $\mathcal{L}(f(c_1x_1 + c_2x_2)) = c_1\mathcal{L}(f(x_1)) + c_2\mathcal{L}(f(x_2)$.[2] Because of this, we usually omit the argument of the function f when an operator $\mathcal{L}$ acts on a function f.*

1. *$\mathcal{L}(f) = \mathrm{d}f/\mathrm{d}x$, where f is a function of x.*

[2] This latter result holds if $\mathcal{L}(f)$ is linear in f *and* $f(x)$ is linear in x.

2. $\mathcal{L}(f) = \mathrm{d}^n f/\,\mathrm{d}x^n$, *where n is a fixed integer, and f is a function of x.*

3. $\mathcal{L}(f) = h(x)\frac{\mathrm{d}f}{\mathrm{d}x}$, *where h is a* fixed *function of x.*

4. $\mathcal{L}(f) = \int_a^b f(x)\,\mathrm{d}x$, *where f is a function of x, while a and b are fixed.*

5. $\mathcal{L}(f) = \left[\int_a^b |f(x)|^2\,\mathrm{d}x\right]^{1/2}$, *where f is a function of x, a and b are fixed.*

6. $\mathcal{L}(f) = \nabla f$, *where f is a function of (x, y, z, t).*

7. $\mathcal{L}(\mathbf{q}) = \nabla \cdot \mathbf{q}$, *where $\mathbf{q}$ is a vector field.*

8. $\mathcal{L}(z) = |z|$, *where z is a complex number.*

9. $\mathcal{L}(\mathbf{a}) = \mathbf{a} \cdot \mathbf{i}$, *where $\mathbf{a}$ is a vector.*

10. $\mathcal{L}(f) = \mathrm{Re}(f(x))$, *where f is a complex function of the real variable x, and the scalars c are required to be real.*

11. $\mathcal{L}(f) = \mathrm{Re}(f(x))$, *where f is a complex function of the real variable x, and the scalars c can be complex.*

Now it will come as no suprise if we ask whether the two functions T_1 and T_2 defined in (19) and (20) can be added to solve (1) with (18) as a boundary condition for the general case $T_{0,1} \neq 0$, $T_{0,2} \neq 0$. Clearly, the answer is yes. Each satisfies the heat equation separately, and therefore, by the above, so does

$$T(x,t) = T_1(x,t) + T_2(x,t)$$

Moreover, this sum also satisfies the boundary condition (18), as can be seen by substituting $x = 0$. Hence

$$\begin{aligned} T(x,t) \;\; = \;\; & T_{0,1} \exp\left(-\sqrt{\frac{\rho c \omega_1}{2k}}x\right) \cos\left(\omega_1 t - \sqrt{\frac{\rho c \omega_1}{2k}}x + \theta_1\right) \\ & + T_{0,2} \exp\left(-\sqrt{\frac{\rho c \omega_2}{2k}}x\right) \cos\left(\omega_2 t - \sqrt{\frac{\rho c \omega_2}{2k}}x + \theta_2\right) \end{aligned}$$

Non-zero geothermal heat flux

We have seen how to superpose solutions for different driving frequencies ω, but all of these solutions still have vanishing heat flux as $x \to \infty$. How can we incorporate a finite heat flux q_{geo} in the boundary condition (11) into the solution? In other words, how can we solve (1) with (2) and (11) when $q_{geo} \neq 0$?

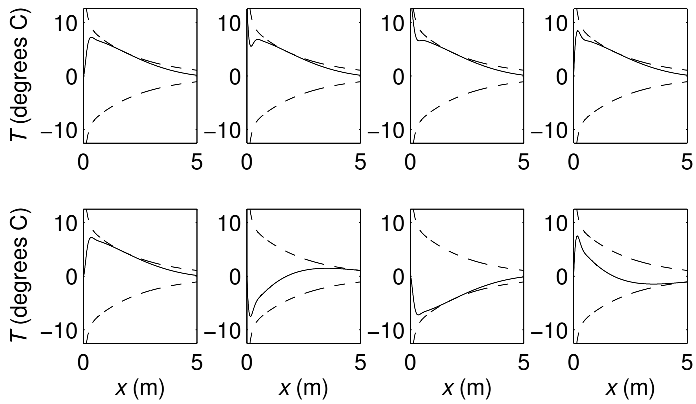

Figure 3: Superposition of annual and diurnal frequencies, both with amplitude 10C. Parameter values are the same as in figure 2. Top row: Snapshots of the temperature field during day 45 of the cycle. Bottom row: snapshots of the temperature field spaced 3 months apart.

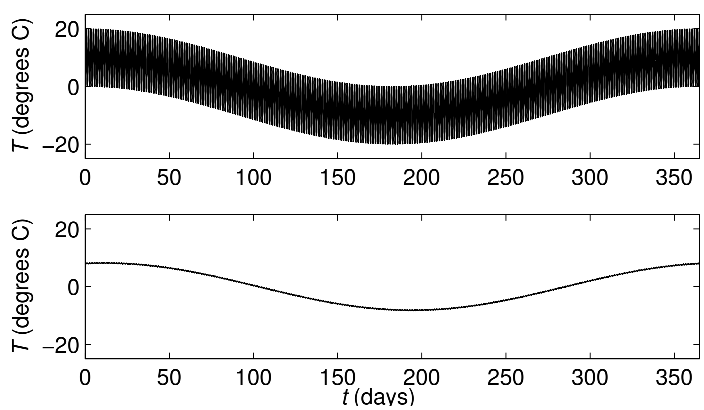

Figure 4: The temperature solution shown in figure 3 plotted against time at depths $x = 0$ (top panel) and $x = 0.45$ m (bottom panel). The attenuation of the high-frequency (diurnal) signal with depth is clearly visible; the phase lag and attenuation of the slow-frequency (annual) signal with depth are also evident, but require a closer look.

The answer is again to use the linearity of the heat equation. Assume that T can be expressed as the sum of two functions T_1 and T_2. From the heat equation, we have

$$\rho c \frac{\partial T_1}{\partial t} - k \frac{\partial^2 T_1}{\partial x^2} + \rho c \frac{\partial T_2}{\partial t} - k \frac{\partial^2 T_2}{\partial x^2} = 0. \tag{24}$$

An obvious function to try for T_1 is the solution (12), so that T_2 then represents a correction that is added on to account for a finite geothermal heat flux:

$$T_1 = T_0 \exp\left(-\sqrt{\frac{\rho c \omega}{2k}} x\right) \cos\left(\omega t - \sqrt{\frac{\rho c \omega}{2k}} x\right).$$

Finding T then amounts to finding T_2. As T_1 itself satisfies the heat, (24) becomes

$$\rho c \frac{\partial T_2}{\partial t} - k \frac{\partial^2 T_2}{\partial x^2} = 0. \tag{25}$$

We still have to deal with the boundary conditions. At the surface, (2) holds:

$$T_1(0, t) + T_2(0, t) = T_0 \cos(\omega t).$$

But, using the form of T_1 above, this simply becomes

$$T_2(0, t) = 0. \tag{26}$$

At infinity, we have (11):

$$-k \frac{\partial T_1}{\partial x} - k \frac{\partial T_2}{\partial x} \to -q_{geo}.$$

But $\partial T_1 / \partial x \to 0$ as $x \to \infty$, so

$$-k \frac{\partial T_2}{\partial x} \to -q_{geo}. \tag{27}$$

Now we have to solve (25) with boundary conditions (26) and (27). Neither boundary condition includes time t explicitly, so we can try to find a steady state solution, $T_2 = T_2(x)$. This turns (25) into

$$-k \frac{\mathrm{d}^2 T_2}{\mathrm{d}x^2} = 0$$

So that, by integrating twice,

$$T = ax + b$$

with a and b constants. But to satisfy the boundary conditions (26) and (27), we must have $a = q_{geo}/k$, $b = 0$:

$$T_2 = \frac{q_{geo} x}{k}.$$

A solution to the full problem is then $T = T_1 + T_2$:

$$T(x, t) = T_0 \exp\left(-\sqrt{\frac{\rho c \omega}{2k}} x\right) \cos\left(\omega t - \sqrt{\frac{\rho c \omega}{2k}} x\right) + \frac{q_{geo} x}{k}.$$

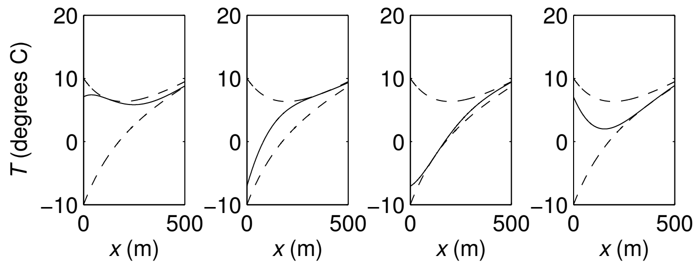

Figure 5: Temperature wave in ice with a period of 2000 years and amplitude 10C in the presence of a geothermal heat flux. Parameter values are $k = 2.1$ W m^{-1} k^{-1}, $\rho = 900$ kg m^{-3}, $c = 2.2 \times 10^3$ W kg^{-1} K^{-1}. Geothermal heat flux is 40 mW m^{-2}. Snapshots of the temperature field are shown 500 years apart. Note that the figures suggest temperatures in excess of 0 °C, which must obviously be nonsense: the temperature here must be measured relative to some mean $\bar{T}$ that is below zero. See exercise 5.

Exercise 5 *We can actually cast what we have done in the last two sections a bit more systematically in terms of the linearity not only of the heat equation, but also of the boundary conditions we apply. Suppose we have a more general problem*

$$\rho c \frac{\partial T}{\partial t} - k \frac{\partial^2 T}{\partial x^2} = a(x) \qquad \textit{for } x > 0 \tag{28a}$$

$$T(0,t) = \bar{T} + \sum_j T_{0,j} \cos(\omega_j t) \textit{at } x = 0, \tag{28b}$$

$$k \frac{\partial T}{\partial x} \to q_{geo} \qquad \textit{as } x \to \infty. \tag{28c}$$

We can actually write the left-hand sides of not only the heat equation, but also the boundary conditions, as linear operators, if we put

$$\mathcal{L}_1(T) = \rho c \frac{\partial T}{\partial t} - k \frac{\partial^2 T}{\partial x^2}, \qquad \mathcal{L}_2(T) = T(0,t), \qquad \mathcal{L}_3(T) = \lim_{x \to \infty} k \frac{\partial T}{\partial x}.$$

1. *Show that $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ are linear operators, in other words, that $\mathcal{L}(\sum_i c_i T_i) = \sum_i c_i \mathcal{L}(T_i)$ for constants c_i and arbitrary functions $T_i(x,t)$*

2. *In the problem above, we do not have equations of the form $\mathcal{L}(T) = 0$. Instead, the heat equation and the boundary conditions take the form $\mathcal{L}_i(T) = f_i$, where f_i is given. The f_i are known as 'inhomogeneous' terms. For instance, for the heat equation, we can put $f_1 = a(x)$, or for the far-field boundary condition, we would put $f_3 = q_{geo}$. The trick is now to find functions T_1, T_2 and T_3 such that each takes care of one of the inhomogeneous terms. Specifically, we would require that T_1 satisfies*

$$\mathcal{L}_1(T_1) = f_1, \qquad \mathcal{L}_2(T_1) = 0, \qquad \mathcal{L}_3(T_1) = 0,$$

with analogous definitions for T_2 *and* T_3 *(so that* $\mathcal{L}_i(T_j) = 0$ *if* $i \neq j$*, and* $\mathcal{L}_i(T_j) = f_i$ *if* $i = j$*). Show that if such functions can be found, then* $T = T_1 + T_2 + T_3$ *satisfies* (28)*.*

3. *Find a solution* $T_1(x)$*, a solution* $T_2(x,t)$ *and a solution* $T_3(x)$ *that satisfy these requirements. You have to assume that* $\int_0^\infty a(x)\,\mathrm{d}x$ *is finite to make this work, and your solution for* T_1 *will probably an integral that looks like* $\int_0^x a(x')\,\mathrm{d}x'$*. The solution for* T_2 *will itself be a sum of terms, which you can find by splitting* $f_2 = f_{2,1} + f_{2,2} + \ldots$ *and putting* $T_2 = T_{2,1} + T_{2,2} + \ldots$ *so that* $\mathcal{L}_1(T_{2,i}) = 0$*,* $\mathcal{L}_2(T_{2,i}) = f_{2,i}$ *and* $\mathcal{L}_3(T_{2,i}) = 0$*.*

4. *Write down a solution to* (28) *as an explicit formula for* $T(x,t)$*.*

5. *So far, we have not mentioned* initial conditions*. In order to specify a unique solution to the heat equation, we need not only boundary conditions, but also initial conditions — meaning, for instance, an initial temperature* $T(x,0)$ *needs to be specified at every point* x *in the region where the heat equation is solved for.*[3] *Suppose we add this to* (28)*,*

$$T(x,0) = T_0(x) \tag{29}$$

where T_0 *is a known function. Show that this initial condition also takes the form* $\mathcal{L}_4(T) = f_4$*. Let* T_1*,* $_2$ *and* T_3 *be the same function defined above, and assume that* $T = T_1 + T_2 + T_3 + T_4$ *solves* (28) *combined with* (29)*. As* $T_1 + T_2 + T_3$ *already solves* (28)*, the role of* T_4 *must be to satisfy the initial conditions. Using the operators* $\mathcal{L}_1$*,* $\mathcal{L}_2$*,* $\mathcal{L}_3$ *and* $\mathcal{L}_4$*, what equations does* T_4 *need to satisfy in order to make* T *a solution?*

[3]This region is also known as the 'domain' on which the heat equation is solved