Midterm: EOSC 352

15th November 2010

There are two questions in this exam. Marks available are as indicated, **so choose the level of detail you give and the amount of time you spend on a question accordingly**. Attempt all questions. You have fifty (50) minutes to complete this exam. Good luck!

1. In class, we considered the diffusion of heat away from a plane into which it was injected at $t = 0$. A similar analysis can be done for injection of heat into a line in a rock at an initially constant background temperature (which we set to zero below). The appropriate dimensionless (scaled) model for temperature $T(r, t)$ as a function of distance r from the line and time t after injection is then

$$\frac{\partial T}{\partial t} - \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) = 0 \qquad \text{for } r \geq 0,\ t > 0. \tag{1a}$$

$$\lim_{t\to 0} T(r,t) = 0 \qquad \text{for any fixed } r > 0 \tag{1b}$$

$$\lim_{r\to\pm\infty} T(r,t) = 0 \qquad \text{for any fixed } t > 0 \tag{1c}$$

$$2\pi\int_0^\infty T(r,t)r\,\mathrm{d}r = 1 \qquad \text{for all times } t > 0 \tag{1d}$$

where the last equation represents the energy injected per unit length of the line. *No need to derive these equations; instead you will be guided through their solution below.*

Look for a similarity solution

$$T(r,t) = t^{-\alpha}\theta\left(\frac{r}{t^\beta}\right) \tag{2}$$

by following these steps (if you get stuck, try skipping to the next step):

(a) (3 points) Let $\xi = r/t^\beta$, and substitute the form of T from (2) into (1a). Transform all derivatives of T into derivatives of θ with respect to ξ.

Using the product and chain rules,

$$\begin{aligned}\frac{\partial T}{\partial t} &= -\alpha t^{-\alpha-1}\theta(\xi) + t^{-\alpha}\theta'(\xi)\frac{\partial \xi}{\partial t}\\ &== \alpha t^{-\alpha-1}\theta(\xi) - \beta t^{-\alpha}\theta'(\xi)\frac{r}{t^{\beta}+1}\\ &= -\alpha t^{-\alpha-1}\theta(\xi) - \beta t^{-\alpha-1}\theta'(\xi)\xi\end{aligned}$$

as $r/t^{\beta+1} = \xi/t$. Similarly,

$$\begin{aligned}\frac{\partial T}{\partial r} &= t^{-\alpha}\theta'(\xi)\frac{\mathrm{d}\xi}{\mathrm{d}r}\\ &= t^{-\alpha-\beta}\theta'(\xi)\\ \frac{\partial^2 T}{\partial r^2} &= \frac{\partial}{\partial r}\frac{\partial T}{\partial r}\\ &= t^{-\alpha-\beta}\theta''(\xi)\frac{\mathrm{d}\xi}{\mathrm{d}r}\\ &= t^{-\alpha-2\beta}\theta''(\xi)\end{aligned}$$

Substituting in the heat equation,

$$-\rho c_p t^{-\alpha-1}\left[\alpha\theta(\xi) + \beta\xi\theta'(\xi)\right] - t^{-\alpha-2\beta}\theta''(\xi) = 0.$$

(b) (3 points) Collect terms so that you have only ξ and t appearing in your equation (but not r), and t appears only as a coefficient in one term in the resulting equation. What does the numerical value of β have to be if a solution of the form (2) can work?
See above. We can rearrange this as

$$-\rho c_p\left[\alpha\theta(\xi) + \beta\xi\theta'(\xi)\right] - t^{1-2\beta}\theta''(\xi) = 0.$$

Now, if θ depends only on ξ but not explicitly on t, then the differential equation for θ had better not contain t explicitly. This is the case if and only if $1-2\beta = 0$ so that $t^{1-2\beta} = 1$. But then

$$\beta = \frac{1}{2}.$$

(c) (2 points) Using the definition of ξ, express t as a function of ξ and r. What does the initial condition (1b) become in terms of r and ξ?
$\xi = r/t^{\beta}$ so $t = (r/\xi)^{1/\beta}$ and $T(r,t) = t^{-\alpha}\theta(\xi) = \xi^{\alpha/\beta}r^{-\alpha/beta}\theta(\xi)$. If $t\to\infty$ at fixed r, then $\xi = r/t^{\beta}\to\infty$ ($t^{-\beta} = t^{-12}\to\infty$ as $t\to 0$. So the initial condition becomes

$$\lim_{\xi\to\infty}\xi^{\alpha/\beta}r^{-\alpha/beta}\theta(\xi) = 0$$

with $r>0$ fixed. But if r is fixed, we can pull $r^{-\alpha/beta}$ out to find

$$\lim_{\xi\to\infty}\xi^{\alpha/\beta}\theta(\xi) = 0.$$

(d) (1 point) What does the boundary condition (1c) become in terms of t and ξ?
Similar to the above: if t is fixed, then $\xi = r/t^\beta \to \infty$ as $r \to \infty$, so

$$\lim_{\xi\to\infty} t^{-\alpha}\theta(\xi) = 0$$

at fixed t. But if t is fixed, then we pull it out of the limit to find

$$\lim_{\xi\to\infty} \theta(\xi) = 0.$$

(e) (3 points) Change variables from r to ξ in the energy conservation constraint (1d). Collect all terms involving t in the resulting equation. Explain why $\alpha = 1$ if a solution of the form (2) is to work.
Change variables (at fixed t) to $\xi = r/t^\beta$ so that $r = t^\beta\xi$, $\mathrm{d}r = t^\beta\,\mathrm{d}\xi$. Then the limits of integration are still 0 to infinity. Also substitute $T = t^\alpha\theta(\xi)$

$$2\pi\int_0^\infty t^{-\alpha}\theta(\xi)t^\beta\xi t^\beta\,\mathrm{d}\xi = 2\pi t^{\alpha-2\beta}\int_0^\infty \theta(\xi)\xi\,\mathrm{d}\xi = 1.$$

But if θ depends only on ξ, so integral does not depend on t. Hence this can only work at all times t if the left-hand side does not contain t explicitly, so that $\alpha - 2\beta = 0$ and $t^{\alpha-2\beta} = 1$. Hence

$$\alpha = 2\beta = 1.$$

(f) (4 points) At this point, you should have derived the ordinary differential equation

$$-\frac{1}{2}\left(2\xi\theta + \xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) - \frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = 0$$

with the additional conditions

$$\xi^2\theta(\xi) \to 0 \qquad \text{as } \xi \to \pm\infty, \tag{3a}$$

$$2\pi\int_0^\infty \theta(\xi)\xi\,\mathrm{d}\xi = 1. \tag{3b}$$

From this, *derive* the equation

$$-\frac{1}{2}\xi^2\theta = \xi\frac{\mathrm{d}\theta}{\mathrm{d}\xi} + C. \tag{4}$$

where C is a constant of integration
The trick is to recognize that we can use the chain rule in reverse:

$$2\xi\theta + \xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi} = \frac{\mathrm{d}(\xi^2\theta(\xi))}{\mathrm{d}\xi}.$$

Then

$$-\frac{1}{2}\frac{\mathrm{d}(\xi^2\theta(\xi))}{\mathrm{d}\xi} - \frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right) = 0$$

Integrating once gives

$$-\frac{1}{2}\xi^2\theta = \xi\frac{\mathrm{d}\theta}{\mathrm{d}\xi} + C.$$

(g) (1 point) Using (3), show that $C = 0$.
The first equality in (3) gives $\xi^2\theta \to 0$ as $\xi \to \infty$. Hence

$$\xi\frac{\mathrm{d}\theta}{\mathrm{d}\xi} \to -C$$

or equally, $\mathrm{d}\xi/\,\mathrm{d}\theta$ behaves as $-C/\xi$ for large ξ. If $C \neq 0$, this however that θ behaves as $-C\log(\xi)$ for large ξ, in which case θ does not tend to zero.

(h) (3 points) Separate variables to find a solution for $\theta(\xi)$ involving one constant of integration.
With $C =)$. we have

$$-\frac{1}{2}\xi^2\theta = \xi\frac{\mathrm{d}\theta}{\mathrm{d}\xi}$$

Separating variables,

$$\frac{1}{\theta}\frac{\mathrm{d}\theta}{\mathrm{d}\xi} = -\frac{\xi}{2}.$$

Integrating once,

$$\log(\theta) = -\frac{\xi^2}{4} + C'$$

where C' is a constant of integration. Equally

$$\theta(\xi) = K\exp\left(-\frac{\xi^2}{4}\right).$$

(i) (2 points) Use one of the conditions in (3) to find that constant of integration.
We have not used the integral constraint yet,

$$2\pi\int_0^\infty \xi\theta(\xi)\,\mathrm{d}\xi = 1.$$

But we can now do the integral on the left. By a change of variable $\xi^2/4 = u$, $\mathrm{d}u = \xi\,\mathrm{d}\xi/2$,

$$2\pi\int_0^\infty K\xi\exp\left(-\frac{\xi^2}{4}\right)\,\mathrm{d}\xi = 4\pi\int_0^\infty K\exp(-u)\,\mathrm{d}u = 4\pi K = 1,$$

so that

$$K = \frac{1}{4\pi}.$$

2. Let r be distance from the origin and $\hat{\mathbf{r}}$ be the unit vector in the radial direction away from the origin. Let ρ, c_p and k be constants, denoting the usual physical quantities. At $t = 0$, let

$$\mathbf{u} = \frac{u_0 r}{r_0}\hat{\mathbf{r}}, \qquad T = \frac{T_0 r_0}{r},$$

where u_0, r_0 and T_0 are constants. Let V be the spherical volume given by $r < r_0$, with S its surface. Note that V is a fixed volume, and does not evolve in time.

(a) (2 points) Compute the rate at which advection carries heat out of the volume V at $t = 0$.
This is $\int_S \rho c_p T \mathbf{u} \cdot \hat{\mathbf{n}}\, \mathrm{d}S$. But, on S, $\mathbf{u} = \mathbf{u}_0 \hat{\mathbf{r}}$, $T = T_0$ and $\hat{\mathbf{n}} = \hat{\mathbf{r}}$, so

$$\rho c_p T \mathbf{u} \cdot \hat{\mathbf{n}} = \rho c_p T_0 u_0 \hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = \rho c_p T_0 u_0$$

because $\hat{\mathbf{r}}$ is a unit vector. Hence the integrand is constant, and $\int_S \rho c_p T \mathbf{u} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = \rho c_p T_0 u_0 \int_S \mathrm{d}S = \rho c_p T_0 u_0 \times 4\pi r_0^2$ (as the surface area of the sphere is $\int_S \mathrm{d}S = 4\pi r_0^2$.

(b) (3 points) Compute the rate at which conduction carries heat out of the volume V at $t = 0$.
This is $\int_S \mathbf{q} \cdot \hat{\mathbf{n}}\, \mathrm{d}S$, where $\mathbf{q} = -k\nabla T$. But T depends only on r, so the gradient is perpendicular to surfaces of constant r (i.e., to spheres centered on the origin, and therefore has direction $\hat{\mathbf{r}}$), and has magnitude $\mathrm{d}T/rdr = -Tr_0/r^2$. On S, $r = r_0$, and so $-k\nabla T$ has magnitude kT_0/r_0, while $\hat{\mathbf{n}} = \hat{\mathbf{r}}$. Hence $\mathbf{q} \cdot \hat{\mathbf{n}} = (kT_0/r)\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = kT_0/r$. Again, the integrand is constant, so $\int_S \mathbf{q} \cdot \hat{\mathbf{n}}\, \mathrm{d}S = kT_0/r_0 \int_S r dS = kT_0/r_0 \times 4\pi r_0^2 = 4\pi k T_0 r_0$.

(c) (1 point) If temperature is in steady state, what is the total rate of heat production in the volume (units of watts, not watts per cubic metre?).
In steady state, total rate of heat production = rate at which heat flows out = rate of flow out through advection + conduction = $4\pi(\rho c_p T_0 u_0 r_0^2 + kT_0 r_0)$.