Midterm Answers: EOSC 352

30 October, 2009

There are three questions in this exam. Marks available are as indicated, **so choose the level of detail you give and the amount of time you spend on a question accordingly**. Attempt all three questions. You have fifty (50) minutes to complete this exam. Good luck!

1. (4 points) Figure 1 shows eight examples of vector fields. For each of these labelled a-h, state whether the divergence of the vector field at the origin is positive, negative, zero, or of ambiguous sign. *Question for a bonus mark: Two of the vector fields actually have a divergence of $+\infty$ at the origin. Which ones?*
   ANS: a — positive (x-component increases with x, y-component is zero)
   b — zero (x-component is zero, y-component does not depend on y)
   c — positive (x-component increases with x, y-component increases with y)
   d — zero (rotation: along x-axis, x component depends only on y, and along y-axis, y-component depends only on x)
   e — positive (x-component increases with x, y-component increases with y; in fact, there is an abrupt jump in these at the origin, so their derivatives, and hence the divergence, are infinite)
   f — ambiguous (x-component decreases with x, y-component increases with y)
   g — negative (x-component decreases with x along x-axis, y-component decreases with y along y-axis)
   e — positive (x-component increases with x at origin, y-component increases with y at origin; in fact, there is an abrupt jump in these at the origin, so their derivatives, and hence the divergence, are infinite)

2. Let $\rho_X(x, y, z, t)$ be a scalar field representing the density of some conserved scalar quantity X (measured in in units of X per cubic metre — for instance, X could be mass, in which case ρ_X would be the ordinary mass density).
   (a) (1 point) Give a formula for the content of X in a given fixed volume V.
   ANS: $\int_V \rho_X \, \mathrm{d}V$ (which can be derived by splitting V into small volumes ΔV, calculating their X-content as X-density times volume $= \rho_X \Delta V$ and summing.

(b) (3 points) In general, a conserved quantity X can be transported by advection and by conduction. If there is a velocity field $\mathbf{u}(x, y, z, t)$, show from first principles that the rate at which quantity X is transported out of the volume V by advection is given by

$$\int_S \rho_X \mathbf{u} \cdot \hat{\mathbf{n}} \, \mathrm{d}S, \tag{1}$$

where S is the surface of the volume V, and $\hat{\mathbf{n}}$ is the outward-pointing unit normal.

ANS: Split the surface S into small surface elements ΔS. Advection is the transport of X as it moves with bits of matter rather than being exchanged between them. To compute this movement of matter, calculate the volume of material that passes throgh ΔS in a short period of time Δt. Geometrically, this volume is a prism of base area ΔS and side length $u\Delta t$. The sides of the prism make an angle θ with the normal to the base ΔS; this angle is the angle between the velocity vector $\mathbf{u}$ and the unit normal $\hat{\mathbf{n}}$. The height of the prism is then equal to side length times the cosine of the angle θ, or

$$u\Delta t \cos\theta.$$

The volume of the prism is base times height,

$$\Delta S u \Delta t \cos\theta = \Delta S \mathbf{u} \cdot \hat{\mathbf{n}} \Delta t$$

by the definition of a cross product. The content of X in the volume is X-density times volume

$$\rho_X \mathbf{u} \cdot \hat{\mathbf{n}} \Delta S \Delta t$$

Sum over all the ΔS's to get the total amount of X that is carried out of the volume V with bits of matter,

$$\Delta X = \int_V \rho_X \mathbf{u} \cdot \hat{\mathbf{n}} \, \mathrm{d}S . \Delta t$$

The rate at which X leaves V by advection is then

$$\frac{\Delta X}{\Delta t} = \int_V \rho_X \mathbf{u} \cdot \hat{\mathbf{n}} \, \mathrm{d}S.$$

(c) (1 point) If there is also transport by a conductive flux $\mathbf{q}_c$, give a formula for the rate at which X leaves the volume V by conduction.

ANS: For advective transport, the rate of transport is

$$\int_S \mathbf{q}_{\text{advect}} \cdot \ \hat{\mathbf{n}} \, \mathrm{d}S = \int_V \rho_X \mathbf{u} \cdot \hat{\mathbf{n}} \, \mathrm{d}S.$$

as advective flux is $\mathbf{q}_{\text{advect}} = \rho_X \mathbf{u}$. The rate of transport through conductive flux can likewise be calculated as

$$\int_S \mathbf{q}_c \cdot \hat{\mathbf{n}} \, \mathrm{d}S.$$

(d) (1 point) State an equation that relates your answer in part a to the surface integral in (1) and your answer in part c.
ANS: Rate of increase of X-content = - rate at which X flows out through boundary of V. As the rate of outflow is given by the sum of advection and conduction,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V \rho_X \, \mathrm{d}V = -\int_S \rho_X \mathbf{u} \cdot \hat{\mathbf{n}} \, \mathrm{d}S - \int_S \mathbf{q}_c \cdot \hat{\mathbf{n}} \, \mathrm{d}S.$$

(e) (3 points) From this equation, carefully derive a the differential equation that relates $\partial \rho_X / \partial t$ to ρ_X, $\mathbf{u}$ and $\mathbf{q}_c$, stating any assumptions that you make.
ANS: Rearrange and apply the divergence theorem, assuming also that the time derivative can be taken inside the integral:

$$\int_V \frac{\partial \rho_X}{\partial t} + \nabla \cdot (\rho_X \mathbf{u} + \mathbf{q}_c) \, \mathrm{d}V = 0.$$

Now, V is *arbitrary*: this equation must be true for *any* volume V if X is conserved. If the integrand is continuous, we can then make V a very small volume ΔV, over which the integrand is approximately constant. In that case, we have approximately

$$\int_V \frac{\partial \rho_X}{\partial t} + \nabla \cdot (\rho_X \mathbf{u} + \mathbf{q}_c) \, \mathrm{d}V \approx \left[\frac{\partial \rho_X}{\partial t} + \nabla \cdot (\rho_X \mathbf{u} + \mathbf{q}_c) \right] \Delta V = 0$$

But ΔV is non-zero, so we must have

$$\frac{\partial \rho_X}{\partial t} + \nabla \cdot (\rho_X \mathbf{u} + \mathbf{q}_c) = 0.$$

(f) (3 points) If X is mass, we have $\rho_X = \rho$ and the differential equation you derived in part e should take the form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0.$$

Next, take X to be thermal energy. Let $\rho_X = \rho c_p T$, and assume that heat capacity c_p is a constant. Also, let $\mathbf{q}_c = -k \nabla T$. Derive the equation

$$\rho c_p \frac{\partial T}{\partial t} + \rho c_p \mathbf{u} \cdot \nabla T - \nabla \cdot (k \nabla T) = 0.$$

Hint. You may want to use the product rule for divergences, $\nabla \cdot (f\mathbf{g}) = f\nabla \cdot \mathbf{g} + \mathbf{g} \cdot \nabla f$*, where* f *is a scalar field and* $\mathbf{g}$ *is a vector field.*
ANS: We have

$$\frac{\partial \rho_X}{\partial t} + \nabla \cdot (\rho_X \mathbf{u} + \mathbf{q}_c) = 0.$$

Substituting ρ_X and $\mathbf{q}_c$ as indicated, we get

$$\frac{\partial(\rho c_p T)}{\partial t} + \nabla \cdot (\rho c_p T\mathbf{u} - k\nabla T) = 0.$$

Using the product rule, noting that c_p (but *not* ρ) is constant,

$$\rho c_p \frac{\partial T}{\partial t} + c_p T \frac{\partial \rho}{\partial t} + \rho c_p T \mathbf{u} \cdot \nabla T + c_p T \nabla \cdot (\rho \mathbf{u}) - \nabla \cdot (k\nabla T) = 0$$

But

$$c_p T \frac{\partial \rho}{\partial t} + c_p T \nabla \cdot (\rho \mathbf{u}) = c_p T \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u})\right] = 0$$

from the conservation law for mass above. Using this, we get

$$\rho c_p \frac{\partial T}{\partial t} + \rho c_p \mathbf{u} \cdot \nabla T - \nabla \cdot (k\nabla T) = 0$$

as required.

3. (4 points) Suppose you have a temperature field T that satisfies the heat equation,

$$\rho c_p \frac{\partial T}{\partial t} - k\frac{\partial^2 T}{\partial x^2} = 0.$$

Suppose that temperature at $x = 0$ is sinusoidal in time,

$$T(0, t) = T_0 \cos(\omega t),$$

while at a long distane from x,

$$k\frac{\partial T}{\partial x} \to q_{geo}.$$

Try a solution of the form

$$T(x, t) = \mathrm{Re}\left[T_0 \exp(i\omega t + \lambda x)\right] + q_{geo}x/k.$$

Substitute this in the heat equation and from this derive a formula for λ in terms of ω, k, ρ and c_p. Show also that the solution satisfies the two boundary conditions.

ANS: We have[1]

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial t}\left\{\mathrm{Re}\left[T_0\exp(i\omega t+\lambda x)\right]+q_{geo}x/k\right\}$$
$$= \mathrm{Re}\left[\frac{\partial}{\partial t}T_0\exp(i\omega t+\lambda x)\right]+\frac{\partial(q_{geo}x/k)}{\partial t}$$
$$= \mathrm{Re}\left[i\omega T_0\exp(i\omega t+\lambda x)\right] \qquad (2)$$

and

$$\frac{\partial^2 T}{\partial x^2} = \frac{\partial^2}{\partial x^2}\left\{\mathrm{Re}\left[T_0\exp(i\omega t+\lambda x)\right]+q_{geo}x/k\right\}$$
$$= \mathrm{Re}\left[\frac{\partial^2}{\partial x^2}T_0\exp(i\omega t+\lambda x)\right]+\frac{\partial^2(q_{geo}x/k)}{\partial x^2}$$
$$= \mathrm{Re}\left[\lambda^2 T_0\exp(i\omega t+\lambda x)\right] \qquad (3)$$

Substituting these in the heat equation, we get (as in class!)

$$\mathrm{Re}\left[T_0(i\rho c_p\omega - k\lambda^2)\exp(i\omega t+\lambda x)\right]=0$$

which holds if we set

$$i\rho c_p\omega - k\lambda^2 = 0$$

or

$$\lambda = \sqrt{i}\sqrt{\frac{\rho c_p}{k}}.$$

But [2]

$$\sqrt{i} = \pm\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right),$$

[1]Do not be tempted to write things like

$$\frac{\partial T}{\partial t} = i\omega\mathrm{Re}\left[T_0\exp(i\omega t+\lambda x)\right],$$

which would make $\frac{\partial T}{\partial t}$ imaginary. Similarly, it is not true that

$$\frac{\partial^2 T}{\partial x^2} = \lambda^2\mathrm{Re}\left[T_0\exp(i\omega t+\lambda x)\right],$$

as λ turns out to be complex number. In the same vein, you cannot write things like

$$\{\mathrm{Re}\left[T_0\exp(i\omega t+\lambda x)\right] = T_0\exp(\lambda x)Re\exp(i\omega t) = T_0\exp(\lambda x)\cos(\omega t)$$

if λ is complex. In general, if z_1 and z_2 are complex numbers, it is *not* true that $\mathrm{Re}(z_1 z_2) = \mathrm{Re}(z_1)\mathrm{Re}(z_2)$ To see this, try $z_1 = 1+i$, $z_2 = 1+i$. Then $z_1 z_2 = 2i$, $\mathrm{Re}(z_1 z_2) = 0$, but $\mathrm{Re}(z_1)\mathrm{Re}(z_2) = 1$.

[2]Note that it is *not* true to say that

$$\sqrt{i} = \frac{1}{\sqrt{2}} \pm \frac{i}{\sqrt{2}}.$$

so

$$\lambda = \pm\left(\sqrt{\frac{\rho c_p}{2k}} + i\sqrt{\frac{\rho c_p}{2k}}\right)$$

Substituting this into the solution, and taking the real part, we get

$$\begin{aligned} T(x,t) &= \mathrm{Re}\left[T_0\exp(i\omega t + \lambda x)\right] + q_{geo}x/k \\ &= T_0\exp\left(\pm\sqrt{\frac{\rho c_p}{2k}}x\right)\cos\left(\omega t \pm \sqrt{\frac{\rho c_p}{2k}}x\right) \end{aligned}$$

where the same sign out of $\pm$ must be used in both the exponential and cosine functions. Clearly, putting $x = 0$, we get the surface boundary conditon

$$T(0,t) = T_0\cos(\omega t).$$

We can also calculate the heat flux

$$\begin{aligned} -k\frac{\partial T}{\partial x} &= -k\mathrm{Re}\left[\frac{\partial}{\partial x}T_0\exp(i\omega t + \lambda x)\ \right] - k\frac{\partial(q_{geo}x/k)}{\partial x} \\ &= -k\mathrm{Re}\left[\lambda T_0\exp(i\omega t + \lambda x)\ \right] - q_{geo} \\ &= -T_0\sqrt{\frac{\rho c_p k}{2}}\exp\left(\pm\sqrt{\frac{\rho c_p}{2k}}x\right)\left[\cos\left(\omega t \pm \sqrt{\frac{\rho c_p}{2k}}x\right) - \sin\left(\omega t \pm \sqrt{\frac{\rho c_p}{2k}}x\right)\right] - q_{geo} \end{aligned}$$

The first term only dies off as $x \to \infty$ if we choose the $-$ in $\pm$, in which case

$$-k\frac{\partial T}{\partial x} \to -q_{geo},$$

which reproduces the boundary condition at infinity. Choosing this sign, we get

$$\lambda = -\left(\sqrt{\frac{\rho c_p}{2k}} + i\sqrt{\frac{\rho c_p}{2k}}\right)$$

and

$$T(x,t) = T_0\exp\left(-\sqrt{\frac{\rho c_p}{2k}}x\right)\cos\left(\omega t - \sqrt{\frac{\rho c_p}{2k}}x\right).$$

If you choose the minus sign and square, you get

$$\left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)^2 = -i$$

which is obviously not i!

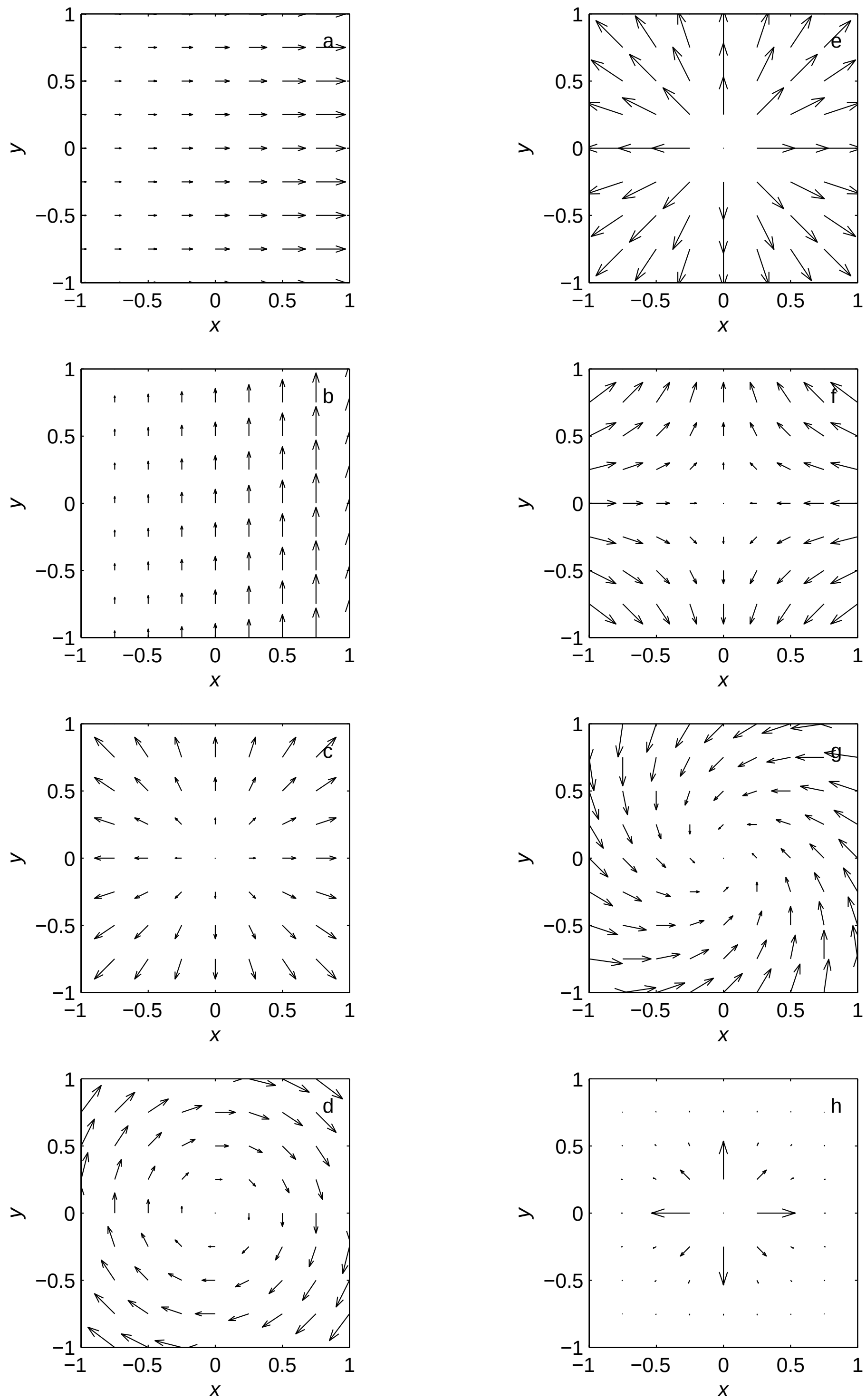

Figure 1: Examples of vector fields. Each panel is labelled by a letter (a-h) in the top right-hand corner, slightly obscured by the arrows showing the vector field.