Midterm Practice: EOSC 352

27 October, 2009

1. (3 pt points) Figure 1 shows the gradient of a scalar field. Sketch the shape of several contours (with approximately fixed contour interval).

2. Let $\rho_X(x, y, z, t)$ be a scalar field representing the density of some conserved scalar quantity X (measured in in units of X per cubic metre — for instance, X could be mass, in which ρ_X would be the ordinary mass density).

   (a) (3 pt points) If there is no conduction and only advection by a velocity field $\mathbf{u}(x, y, z, t)$, show from first principles that the rate at which quantity X is transported out of a volume V is given by

   $$\int_S \rho_X \mathbf{u} \cdot \hat{\mathbf{n}} \, \mathrm{d}S, \tag{1}$$

   where S is the surface of the volume V, and $\hat{bmn}$ is the outward-pointing unit normal.

   (b) (3 pt points) Derive from first principles an expression for the total amount of X in the volume V.

   (c) (2 pt points) State an equation that relates the surface integral in (1) to your answer in part b.

   (d) (3 pt points) From this equation, carefully derive the differential equation

   $$\frac{\partial \rho_X}{\partial t} + \nabla \cdot (\rho_X \mathbf{u}) = 0,$$

   stating any assumptions that you make.

3. (3 pt points) Suppose you have a temperature field T that satisfies the heat equation,

   $$\rho c_p \frac{\partial T}{\partial t} - k \frac{\partial^2 T}{\partial x^2} = 0.$$

   Suppose that the temperature field $T(x, 0)$ at an initial time $t = 0$ has a sinusoidal profile,

   $$T(x, 0) = T_0 \cos(\kappa x),$$

where κ is a real number (inversely proportional to wavelength). Try a solution of the form

$$T(x,t) = \mathrm{Re}[T_0 \exp(i\kappa x + \sigma t)]$$

Substitute this into the heat equation. What does σ have to be if this form of T satisfies the heat equation? Qualitatively (in one sentence), how does the temperature profile evolve over time, and how does the rate at which this happens depend on wavelength $\lambda = 2\pi/\kappa$.

4. (3 pt points) By direct differentiation, verify that

$$T(x,t) = t^{-1/2}\theta_0 \exp\left(-\frac{\rho c_p x^2}{4kt}\right)$$

satisfies the heat equation

$$\rho c_p \frac{\partial T}{\partial t} - k\frac{\partial^2 T}{\partial x^2} = 0.$$

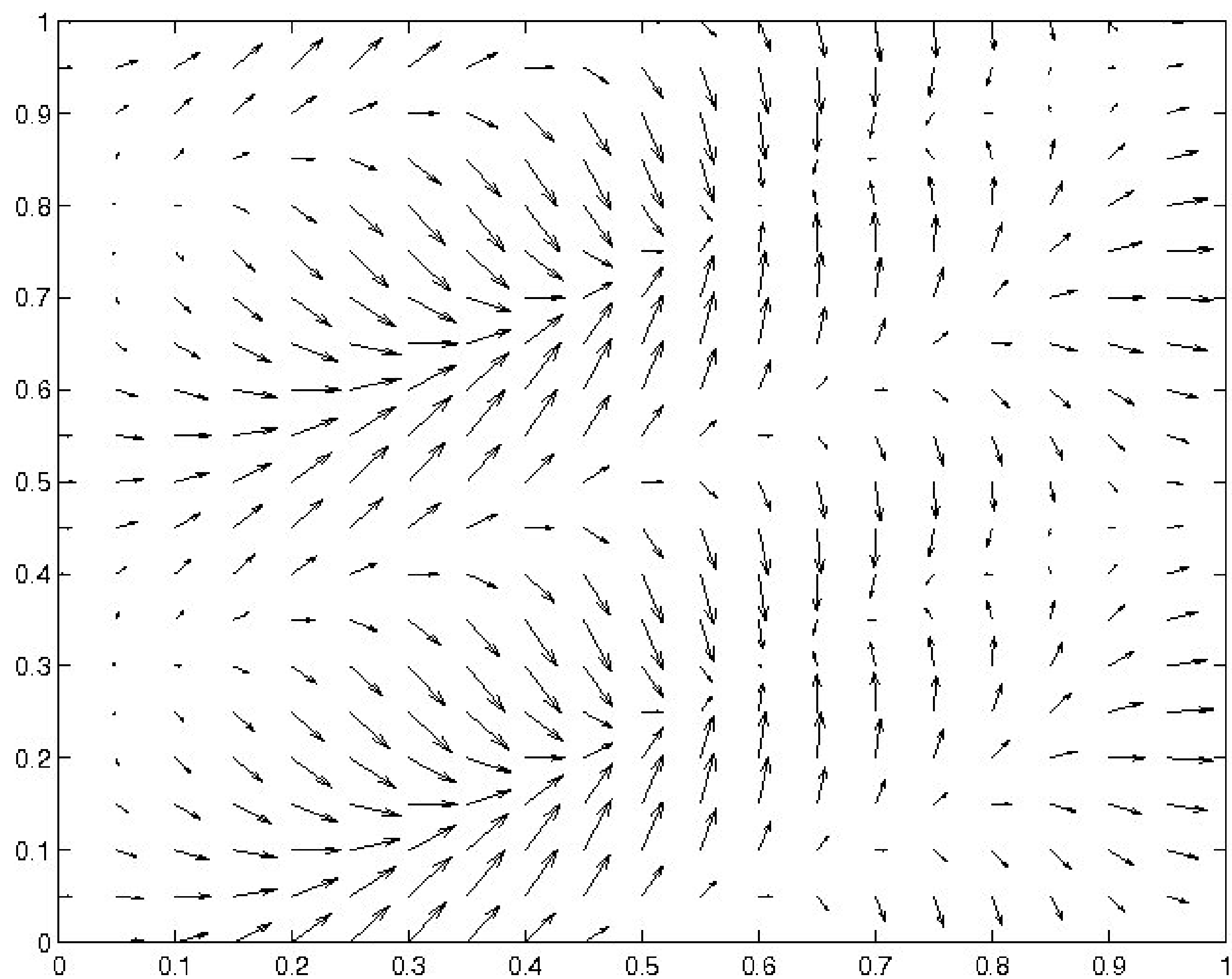

Figure 1: Gradient of a scalar field.