Calculus Practice: EOSC 352

11th September, 2009

1. Calculate the gradient of the following scalar fields:

   (a)
   $$\phi(x,y,z) = \sin(x) + \cos(y), \qquad \nabla\phi = \cos(x)\mathbf{i} - \sin(x)\mathbf{j}$$

   (b)
   $$\Psi(x,y,z) = \exp(x^2+y^2+z^2), \qquad \nabla\Psi = 2(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\exp(x^2+y^2+z^2)$$

   (c)
   $$\Phi(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}}, \qquad \nabla\Phi = -\frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}}$$

   (d)
   $$T(x,y,z) = T_0 - (x^2+y^2)/k, \qquad \nabla T = -2(x\mathbf{i}+y\mathbf{j})/k$$

   where T_0 and k are constant parameters.

2. Compute the divergence of the following vector fields:

   (a)
   $$\mathbf{q}(x,y,z) = x^2y\mathbf{i} - y^2x\mathbf{j} + z\mathbf{k}, \qquad \nabla\cdot\mathbf{q} = 1$$

   (b)
   $$\mathbf{u}(x,y,z) = y\cos(x)\mathbf{i} - x\sin(y)\mathbf{j}, \qquad \nabla\cdot\mathbf{u} = -y\sin(x) - x\cos(y)$$

   (c)
   $$\mathbf{q}(x,y,z) = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{4\pi(x^2+y^2+z^2)^{3/2}}, \qquad \nabla\cdot\mathbf{q} = 0 \text{ except at } (x,y,z) = (0,0,0).$$

3. Let
   $$\Phi(x,y,z) = (x^2-y^2)\exp(x)\cos(y) - 2xy\exp(x)\sin(y).$$

   Then $\nabla^2\Phi = 0$.

4. Let $\rho = 1 + x^2$. Compute $\int_V \rho \, \mathrm{d}V$, if V is the triangular prism given by $0 < x < 1 - y$, $0 < y < 1$, $0 < z < 1$. The answer is

$$\begin{aligned}\int_V \rho \, \mathrm{d}V &= \int_0^1 \int_0^1 \int_0^{1-y} 1 + x^2 \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \\ &= \int_0^1 (1-y) + (1-y)^3/3 \, \mathrm{d}y \\ &= \left[-(1-y)^2/2 - (1-y)^4/12\right]_0^1 \\ &= \frac{7}{12}\end{aligned}$$

5. Compute the integral $\int_S \mathbf{q} \cdot \hat{\mathbf{n}} \, \mathrm{d}S$, where S is the part of the surface $z = x^2 + y^2$ that lies above the triangle $0 < x < 1 - y$, $0 < y < 1$, $\hat{\mathbf{n}}$ is the upward-pointing unit normal to the surface, and

$$\mathbf{q}(x, y, z) = z\mathbf{i} + y\mathbf{j} - x\mathbf{k}.$$

The surface has normal

$$\hat{\mathbf{n}} = \frac{\mathbf{k} - 2x\mathbf{i} - 2y\mathbf{j}}{\sqrt{1 + 4x^2 + 4y^2}},$$

and an element of the surface is

$$\mathrm{d}S = \sqrt{1 + 4x^2 + 4y^2} \, \mathrm{d}x \, \mathrm{d}y.$$

On the surface, $z = x^2 + y^2$, so

$$\mathbf{q} = (x^2 + y^2)\mathbf{i} + y\mathbf{j} - x\mathbf{k},$$

and hence

$$\begin{aligned}\int_S \mathbf{q} \cdot \hat{\mathbf{n}} \, \mathrm{d}S &= \int_0^1 \int_0^{1-y} \left[(x^2 + y^2)\mathbf{i} + y\mathbf{j} - x\mathbf{k}\right] \cdot \frac{\mathbf{k} - 2x\mathbf{i} - 2y\mathbf{j}}{\sqrt{1 + 4x^2 + 4y^2}} \sqrt{1 + 4x^2 + 4y^2} \, \mathrm{d}x \, \mathrm{d}y \\ &= \int_0^1 \int_0^{1-y} -2x(x^2 + y^2) - 2y^2 - x \, \mathrm{d}x \, \mathrm{d}y \\ &= \int_0^1 -(1-y)^4/2 - y^2(1-y)^2 - 2y^2(1-y) - (1-y)^2/2 \, \mathrm{d}y \\ &= \left[(1-y)^5/10 - y^3/3 + y^4/2 - y^5/5 - 2y^3/3 + y^4/2 + (1-y)^3/3\right]_0^1 \\ &= \frac{17}{15} \qquad (1)\end{aligned}$$

6. Let

$$\mathbf{q} = x^2 y\mathbf{i} - y^2 x\mathbf{j} + z\mathbf{k},$$

and let S be the surface of the unit sphere, with $\hat{\mathbf{n}}$ its outward-pointing unit normal. Calculate

$$\int_S \mathbf{q} \cdot \hat{\mathbf{n}} \, \mathrm{d}S.$$

Use the divergence theorem. We have

$$\nabla \cdot \mathbf{q} = \frac{\partial x^2 y}{\partial x} + \frac{\partial(-y^2 x)}{\partial y} + \frac{\partial z}{\partial z} = 1,$$

and hence

$$\int_S \mathbf{q} \cdot \hat{\mathbf{n}} \, \mathrm{d}S = \int_v \nabla \cdot \mathbf{q} \, \mathrm{d}V = \int 1 \, \mathrm{d}V = \frac{4}{3}\pi.$$

(The volume of a sphere is $\int V 1 \, \mathrm{d}V = 4\pi/3$ times the radius cubed.)