DEFINITION

MATHEMATICS USED TO DESCRIBE PHYSICAL PHENOMENA IN A CONTINUOUS MEDIA.

ASSUMPTION: CONTINUUM HYPOTHESIS APPLIES!

NOTATION

PHYSICAL ENTITY	EXAMPLE	GIBBS NOTATION	CARTESIAN COORD.	INDICIAL NOTATION
SCALAR	PRESSURE	P	P	P
VECTOR	VELOCITY	$\vec{V}$	$V_1\hat{i}_1 + V_2\hat{i}_2 + V_3\hat{i}_3$	u_i
TENSOR	STRESS	T (INDEPENDENT OF COORD. SYS.)	$\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$	T_{ij}

REFERENCE FRAME: COORDINATE SYSTEMS

LINEAR INDEPENDENCE

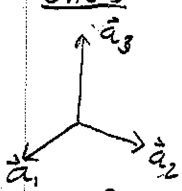
$V_2 = aV_1$ → LINEARLY DEPENDENT

$V_2 \neq aV_1$ → LINEARLY INDEPENDENT

FORMALLY: A SET OF N VECTORS IS LINEARLY INDEPENDENT IFF

$$\sum_{i=1}^{N} \lambda_i \vec{a}_i = 0 \,,$$ WHERE λ_i ARE SCALARS THAT ARE NOT ALL 0.

BASIS



e.g. $\mathbb{R}^3$

CONSIDER A SET OF N INDEPENDENT VECTORS (eg $a_1, a_2, \dots, a_N$). THEN ANY VECTOR IN $\mathbb{R}^N$ (N-DIMENSIONAL SPACE) CAN BE REPRESENTED AS A UNIQUE LINEAR COMBINATION OF THE $\vec{a}$ VECTORS.

WE SAY THAT $(\vec{a}_1, \vec{a}_2, \dots, \vec{a}_N)$ FORMS A BASIS FOR $\mathbb{R}^N$ AND CALL THIS SET OF VECTORS THE BASIS VECTORS.

CONSIDER THE ABOVE EXAMPLE IN $\mathbb{R}^3$ [3D SPACE]:

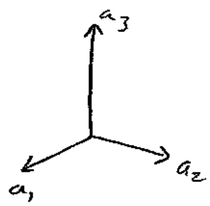
AN VECTOR $\vec{r}$ CAN BE WRITTEN AS $\vec{r}_1 a_1 + \vec{r}_2 a_2 + \vec{r}_3 a_3 = \vec{r}$

COORDINATE SYSTEMS: A SET OF BASIS VECTORS. ABOVE — $(\vec{r}_1, \vec{r}_2, \vec{r}_3)$

COORDINATES: SET OF SCALAR MULTIPLIERS USED TO EXPRESS A VECTOR (a_1, a_2, a_3)

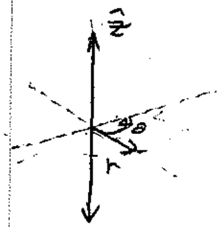
COORDINATE SYSTEMS

RECTANGULAR CARTESIAN



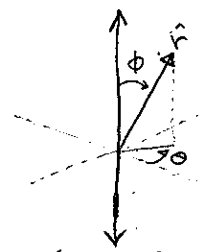
- BASIS VECTORS ARE ORTHOGONAL
- IF WE HOLD 2 COORDINATES CONSTANT AND VARY THE THIRD COORDINATE THE RESULTING LINE (A "COORDINATE CURVE") IS STRAIGHT.

ORTHOGONAL CURVILINEAR COORD. SYSTEMS



(r, θ, z)
CYLINDRICAL

- COORDINATE CURVES IN THESE SYSTEMS WILL BE CURVED.



(r, θ, ϕ)
SPHERICAL

FACTS ABOUT BASIS

- A BASIS FOR $\mathbb{R}^N$ MUST HAVE EXACTLY N VECTORS
- $\mathbb{R}^N$ HAS AN ∞ NUMBER OF POSSIBLE SETS OF BASIS VECTORS FOR $N \geq 2$
- A REPRESENTATION OF A VECTOR UNDER A GIVEN BASIS IS UNIQUE.

SCALAR (ZEROTH-ORDER TENSOR)

A PHYSICAL QUANTITY REQUIRING SPECIFICATION OF A SINGLE NUMBER

e.g. M (MASS), P (PRESSURE), T (TEMPERATURE), t (TIME)

- SCALARS DO NOT DEPEND ON COORDINATE SYSTEMS

- SCALARS CAN BE FUNCTIONS OF A SCALAR VARIABLE

$$P(t)$$

- SCALARS CAN BE FUNCTIONS OF A VECTOR VARIABLE

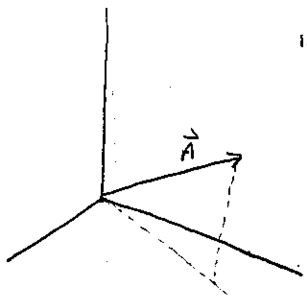
$$P(\vec{r})$$

VECTOR (1ST ORDER TENSOR)

A PHYSICAL QUANTITY REQUIRING SPECIFICATION OF A DIRECTION AND A MAGNITUDE

e.g. $\vec{F}$ (FORCE); $\vec{V}$ (VELOCITY); $\vec{r}$ (POSITION)

- VECTORS DO NOT DEPEND ON COORDINATE SYSTEM



i.e. THIS VECTOR $\vec{A}$ IS THE SAME VECTOR, REGARDLESS OF WHETHER IT IS WRITTEN IN CARTESIAN, CYLINDRICAL, OR ANY OTHER COORDINATE SYSTEM

FORMALLY: VECTORS ARE INVARIANT UNDER A CHANGE IN COORDINATE SYSTEM

COMMENT: COORD SYSTEMS ARE SELF-SERVING FRAMES OF REFERENCE USED TO DESCRIBE VECTORS IN THE SIMPLEST POSSIBLE WAY.

VECTOR FUNCTIONS:

$\vec{V}(t)$: VECTOR VALUED FUNCTION OF A SCALAR VARIABLE
$\vec{V}(\vec{r})$: VECTOR VALUED FUNCTION OF A VECTOR VARIABLE
$\vec{V}(\vec{r}, t)$: etc.

TENSOR (2ND ORDER TENSOR)

A PHYSICAL ENTITY REQUIRING SPECIFICATION OF A MAGNITUDE AND TWO DIRECTIONS.

FORMAL MATH DEFN': A TENSOR IS A TYPE OF LINEAR TRANSFORM FROM $\mathbb{R}^3$ TO $\mathbb{R}^3$. THUS, A TENSOR IS A LINEAR MAP THAT ASSIGNS TO EACH VECTOR $\vec{U}$, A VECTOR $\vec{V}$

$$\underset{\sim}{S}\vec{U} = \vec{V}$$

TENSOR

e.g. STRESS, THERMAL CONDUCTIVITY, STRAIN, MOMENT OF INERTIA, $\vec{\nabla}\vec{V}$ - GRADIENT OF A FLUID VELOCITY FIELD, RATE OF STRAIN

- A TENSOR IS INVARIANT UNDER A COORDINATE TRANSFORMATION
- TENSOR FUNCTIONS

  $\underset{\sim}{T}(t)$, $\underset{\sim}{T}(\vec{r})$, $\underset{\sim}{T}(\vec{r}, t)$

- IT REQUIRES 9 COORDINATES TO SPECIFY A TENSOR (COMPARED TO 3 FOR A VECTOR!)

TENSORS (CONT'D)

THE SIMPLEST WAY TO VIEW A TENSOR IS AS A MATRIX

$\underset{\sim}{S}\vec{U} = \vec{V}$ IN COMPONENT FORM IS EQUIVELANT TO THE MATRIX OPERATION

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

NOTE: DIMENSIONS $\quad 3\times3 \quad\quad 3\times1 \quad\quad 3\times1$

$$S_{11}U_1 + S_{12}U_2 + S_{13}U_3 = V_1 \qquad (1)$$

$$S_{21}U_1 + S_{22}U_2 + S_{23}V_3 = V_2 \qquad (2)$$

$$S_{31}V_1 + S_{32}V_2 + S_{33}U_3 = V_3 \qquad (3)$$

THE ABOVE 3 EQNS FOR V_1, V_2, V_3 ARE WRITTEN IN <u>INDICIAL</u> <u>NOTATION</u> AS

$$S_{ij}U_j = V_i$$

⇒ INDICIAL NOTATION IS, THUS, A SHORTHAND WAY TO WRITE EQUATIONS INVOLVING VECTORS AND TENSORS.

INDICIAL NOTATION (EINSTEIN NOTATION)

EXAMPLE 1. CONSIDER THE DOT PRODUCT OF TWO VECTORS $\vec{a}$ AND $\vec{b}$:

CARTESIAN:

$$\vec{a} = a_1\vec{e}_1 + a_2\vec{e}_2 + a_3\vec{e}_3$$
$$\vec{b} = b_1\vec{e}_1 + b_2\vec{e}_2 + b_3\vec{e}_3$$

$$\vec{a}\cdot\vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

$$= \sum_{i=1}^{3} a_i b_i \quad \text{USING A SUMMATION SIGN}$$

OR

$$= a_i b_i \; ; \quad i \equiv \text{"DUMMY INDEX"}$$

RULE: ANYTIME WE SEE A REPEATED INDEX IN AN INDIVIDUAL TERM WE SUM FROM $1 \to 3$.

e.g. $C_{ij}a_j \Rightarrow C_{i1}a_1 + C_{i2}a_2 + C_{i3}a_3$

EXAMPLE 2

$\vec{F} = m\vec{a}$: A VECTOR EQUATION IMPLYING 3 SCALAR EQUATIONS

$$\underbrace{F_1 = ma_1 \; ; \quad F_2 = ma_2 \; ; \quad F_3 = ma_3}_{\text{INDICIAL NOTATION}}$$

$$F_\ell = m\,a_\ell$$

"FREE INDEX"

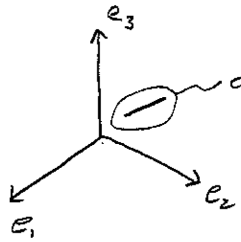
RULE: ANYTIME WE SEE A FREE INDEX CAN REPLACE IT W/ 1, 2, 3
* THE FREE INDEX IMPLIES 3 EQUATIONS.

e.g. $S_{ij}U_j = V_i$

$$S_{i1}U_1 + S_{i2}U_2 + S_{i3}U_3 = V_i$$

$$\left.\begin{aligned} S_{11}U_1 + S_{12}U_2 + S_{13}U_3 &= V_1 \\ S_{21}U_1 + S_{22}U_2 + S_{23}U_3 &= V_2 \\ S_{31}U_1 + S_{32}U_2 + S_{33}U_3 &= V_3 \end{aligned}\right\} \text{3 EQNS}$$

EXAMPLE 3



CONSIDER A LINE SEGMENT OF LENGTH ds

$$ds^2 = dx_1^2 + dx_2^2 + dx_3^2$$

$$ds^2 = dx_m dx_m$$

$\Downarrow$ NOTE

$$ds^2 \neq dx_m^2 \quad \text{NO ~~REPEATED~~ INDEX}$$

$$ds^2 = dx_m dx_m = dx_m dx_n \delta_{mn}$$

WHERE δ_{mn} = KRONECKER DELTA

$$= \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases}$$

EXPANDING:

$$dx_m dx_n \delta_{mn} = \sum_{n=1}^{3} \sum_{m=1}^{3} dx_m dx_n \delta_{mn}$$

$$= \sum_{n=1}^{3} \left[dx_1 dx_n \delta_{1n} + dx_2 dx_n \delta_{2n} + dx_3 dx_n \delta_{3n} \right]$$

$$= dx_1 dx_1 \delta_{11} + \cancelto{0}{dx_2 dx_2 \delta_{12}} + \cancelto{0}{dx_1 dx_3 \delta_{13}}$$
$$+ \cancelto{0}{dx_2 dx_1 \delta_{21}} + dx_2 dx_2 \delta_{22} + \cancelto{0}{dx_2 dx_3 \delta_{23}}$$
$$+ \cancelto{0}{dx_3 dx_1 \delta_{31}} + \cancelto{0}{dx_3 dx_2 \delta_{32}} + dx_3 dx_3 \delta_{33}$$

$$= dx_1 dx_1 + dx_2 dx_2 + dx_3 dx_3$$

EXAMPLE 4: FIND DETERMINANT OF 3×3 MATRIX

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{matrix} a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} \\ - a_{11}a_{23}a_{32} - a_{21}a_{12}a_{33} - a_{31}a_{22}a_{13} \end{matrix}$$

IN INDEX NOTATION

$$|a_{ij}| = \varepsilon_{rst}\, a_{r1}\, a_{s2}\, a_{t3}$$

ε_{rst} = "PERMUTATION SYMBOL" OR "ALTERNATING UNIT TENSOR" OR "SELECTOR TENSOR"

$$= \begin{cases} 0 & \text{IF ANY 2 INDICES ARE} = \\ 1 & \text{IF INDICES PERMUTE LIKE } 1,2,3 \\ -1 & \text{" " " " } 3,2,1 \end{cases}$$

I. E.

$$= \begin{cases} 1 & \text{IF } rst = 1,2,3;\ 2,3,1;\ 3,1,2 \\ -1 & \text{IF } rst = 3,2,1;\ 1,3,2;\ 2,1,3 \\ 0 & \text{OTHERWISE} \end{cases}$$

NOTE: $\varepsilon_{ijk}\,\varepsilon_{ilm} = \delta_{jl}\,\delta_{km} - \delta_{jm}\,\delta_{kl}$

<u>RULES OF INDICIAL NOTATION</u>

a) 3 REPEATED INDICES NOT ALLOWED

e.g. $a_{ij}\, c_i\, b_i \Rightarrow 3 \cdot i$'s $\Rightarrow$ 1 TOO MANY i's (LIKE EYES)

b) # FREE INDICES IN EACH TERM ON EACH SIDE OF AN EQN MUST AGREE IN SYMBOL AND IN NUMBER

$a_{ij} = b_j$ NO. $\qquad a_i b_i + c_j = d_j$ NO.

$a_i b_i = c_j$ NO $\qquad a_j b_i = c_{ij}$ <u>OK</u>

c) # FREE INDICES AND RANK

0 → SCALAR
1 → VECTOR
2 → TENSOR

THE VECTOR CROSS PRODUCT

$\vec{W} = \vec{U} \times \vec{V}$ GIBBS NOTATION

USUAL WAY:

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ U_1 & U_2 & U_3 \\ V_1 & V_2 & V_3 \end{vmatrix} = (U_2 V_3 - V_2 U_3)\hat{i} + (U_3 V_1 - U_1 V_3)\hat{j} + (U_1 V_1 - V_1 U_2)\hat{k}$$

INDEX NOTATION

$$W_i = \varepsilon_{ijk} U_j V_k$$

PROOF:

$$\varepsilon_{ijk} U_j V_k = \varepsilon_{i1k} U_1 V_k + \varepsilon_{i2k} U_2 V_k + \varepsilon_{i3k} U_3 V_k$$

$$= \varepsilon_{i11} U_1 V_1 + \varepsilon_{i12} U_1 V_2 + \varepsilon_{i13} U_1 V_3 + \varepsilon_{i21} U_2 V_1 + \varepsilon_{i22} U_2 V_2 + \varepsilon_{i23} U_2 V_3$$

$$+ \varepsilon_{i31} U_3 V_1 + \varepsilon_{i32} U_3 V_2 + \varepsilon_{i33} U_3 V_3$$

$i=1$

$$\varepsilon_{112} U_1 V_2 + \varepsilon_{113} U_1 V_3 + \varepsilon_{121} U_2 V_1 + \varepsilon_{123} U_2 V_3 + \varepsilon_{131} U_3 V_1 + \varepsilon_{132} U_3 V_2$$

$$= (U_2 V_3 - U_3 V_2)$$

$i=2$

$$= (U_3 V_1 - U_1 V_3)$$

$i=3$

$$= (U_1 V_2 - U_2 V_2)$$

SAME

EXAMPLE 5

PROVE: $(\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c})$

$$\vec{a} \times \vec{b} = \varepsilon_{ijk} a_j b_k$$

$$(\vec{a} \times \vec{b}) \cdot c = \varepsilon_{ijk} a_j b_k c_i$$

PERMUTE:

$$= \varepsilon_{jki} a_j b_k c_i = \vec{a} \cdot (\vec{b} \times \vec{c})$$

EXAMPLE 6

PROVE: $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})$

$$\vec{a}\times\vec{b} = \varepsilon_{ijk} a_j b_k$$

$$\vec{c}\times\vec{d} = \varepsilon_{lmn} c_m d_n$$

$$(\ )\cdot(\ ) = \varepsilon_{ijk} a_j b_k \varepsilon_{lmn} c_m d_n$$

$$= \underbrace{\varepsilon_{ijk}\varepsilon_{lmn}}_{} (a_j b_k c_m d_n)$$

$$(\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km})\, a_j b_k c_m d_n$$

SO

$$\delta_{jm}\delta_{kn}(a_j b_k c_m d_n) - \delta_{jn}\delta_{km}(a_j b_k c_m d_n)$$

PROPERTIES OF δ:

IF $j=m$ OR $k=n$ THEN $\delta_{jm}\delta_{kn} = 1$; $\emptyset$ OTHERWISE

IF $j=n$ OR $k=m$ THEN $\delta_{jn}\delta_{km} = 1$; $\emptyset$ OTHERWISE

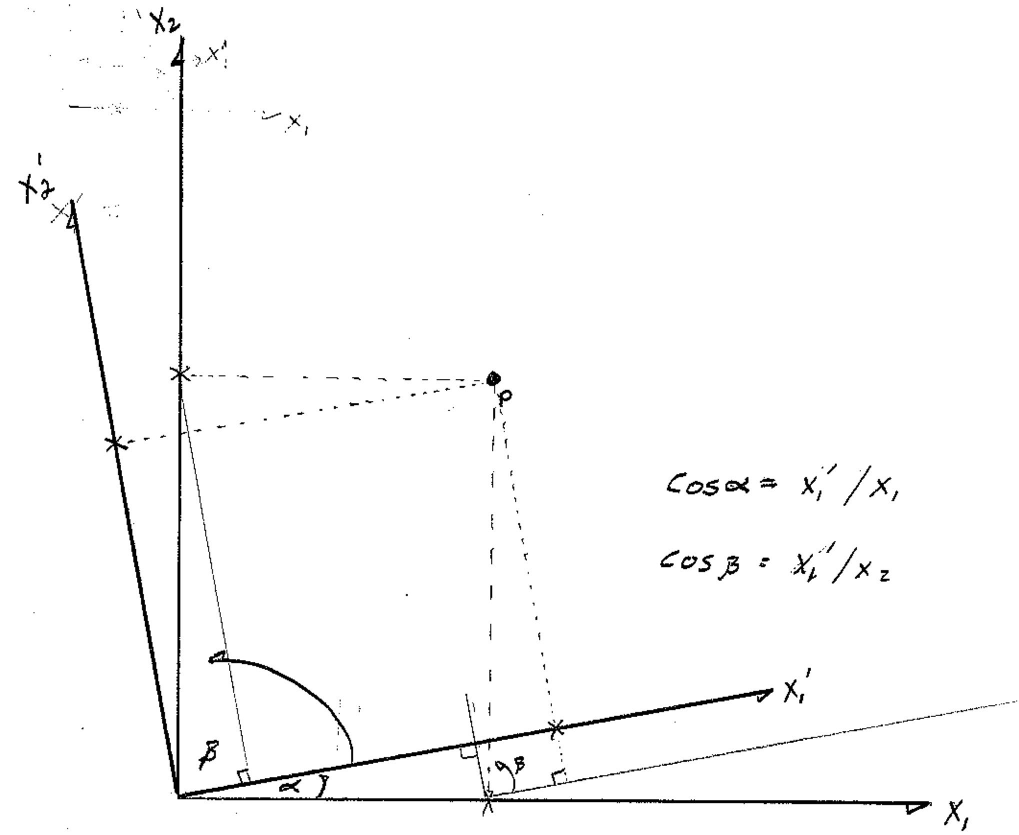
$\therefore$

$$= a_m c_m b_n d_n - a_n d_n b_m c_m$$

$$= (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c}) \quad \checkmark$$

CHANGES IN COORDINATE SYSTEM

GIVEN: A LOCATION P IN (x, y)
FIND: THE LOCATION P IN (x', y')



$\cos\alpha = X_1'/X_1$

$\cos\beta = X_1'/X_2$

$$X_1' = X_1 \cos\alpha + X_2 \cos\beta$$

$\cos\alpha = \cos(X_1, X_1') \equiv$ COS OF ∢ BETWEEN (OLD X-AXIS, NEW X-AXIS)

$\cos\beta = \cos(X_2, X_1') \equiv$ COS OF ∢ BETWEEN (OLD X-AXIS, NEW Y-AXIS)

LET $\cos(X_1, X_1') = C_{11'}$; $\cos(X_2, X_1') = C_{12'}$

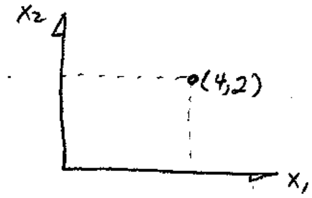
$$X_1' = X_1 C_{11'} + X_2 C_{12'}$$

GENERALIZE THE TRANSFORMATION:

$$X_i' = C_{ji'} X_j$$

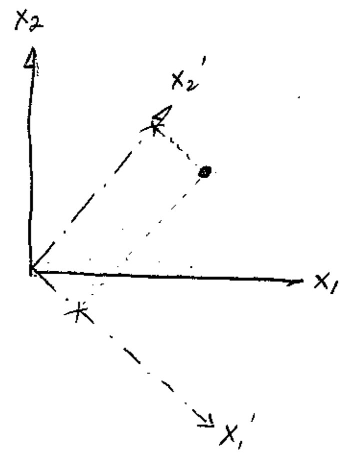
$C_{ji'} \equiv$ MATRIX OF "DIRECTION COSINES" OF ∢ BETWEEN "OLD" AXIS j AND "NEW" AXIS i.

EXAMPLE: COORDINATE TRANSFORMATION



GIVEN:
(4,2) ON (X_1, X_2)

FIND:
X_1', X_2' AFTER A 45 DEGREE ROTATION



$$X_1' = C_{j1'} X_j$$

$$= C_{11'} X_1 + C_{21'} X_2 + C_{31'} X_3$$

$$C_{11'} = \cos(45) = 0.707$$

$$C_{21'} = \cos(135) = -0.707$$

$$C_{31'} = \cos(90) = 0$$

$$X_1' = (4)(0.707) + (2)(-0.707) = 1.414$$

$$X_2' = C_{12'} X_1 + C_{22'} X_2 + C_{32'} X_3$$

$$C_{12'} = \cos(45) = 0.707$$

$$C_{22'} = \cos(45) = 0.707$$

$$C_{32'} = \cos(90) = 0$$

$$X_2' = (0.707)(4) + (0.707)(2) = 4.242$$

$$\boxed{\vec{X}' = \begin{bmatrix} 1.414 \\ 4.242 \\ 0 \end{bmatrix}}$$

COORDINATE TRANSFORMATIONS AS GENERAL WAYS TO <u>DEFINE</u> VECTORS AND TENSORS:

<u>VECTOR</u>: 3 SCALAR QUANTITIES V_i $(i=1,2,3)$; ARE THE SCALAR COMPONENTS OF A VECTOR QUANTITY IF THEY TRANSFORM ACCORDING TO

$$V_j' = C_{ij'} V_i$$

<u>TENSOR (2nd ORDER)</u> 9 SCALAR COMPONENTS, S_{ij} ARE THE SCALAR COMPONENTS OF A 2nd ORDER TENSOR IF THEY TRANSFORM ACCORDING TO:

$$T_{i'j'} = C_{k'i} C_{\ell j'} T_{k\ell}$$

<u>FACTS ABOUT TENSORS</u>

1) A <u>TENSOR TRANSFORMS</u> A VECTOR: $\underset{\sim}{T}\vec{V} = \vec{U}$; $T_{ij}V_j = U_i$

2) <u>LINEAR TRANSFORMATION</u>: LET $\underset{\sim}{S}$ AND $\underset{\sim}{T}$ BE 2nd ORDER TENSORS

$$(\underset{\sim}{S} + \underset{\sim}{T})\vec{u} = \underset{\sim}{S}\vec{u} + \underset{\sim}{T}\vec{u} \quad \text{OR} \quad (S_{ij} + T_{ij})u_j = S_{ij}u_j + T_{ij}u_j$$

$$\underset{\substack{\uparrow \\ \text{SCALAR}}}{d}(\underset{\sim}{S}\vec{u}) = \underset{\sim}{S}(d\vec{u}) \qquad d\, S_{ij}u_j = S_{ij}\, du_j$$

3) <u>TRANSPOSE OF A TENSOR</u>

GIBBS NOTATION: $\underset{\sim}{S}^T$ IS TRANSPOSE OF $\underset{\sim}{S}$.

LET $\underset{\sim}{S}^T$ BE UNIQUELY DEFINED SUCH THAT

$$\underset{\sim}{S}\vec{u} \cdot \vec{V} = \vec{u} \cdot \underset{\sim}{S}^T \vec{V}$$

∴ FOR ALL VECTORS $\vec{u}$ AND $\vec{V}$:

$$(\underset{\sim}{S} + \underset{\sim}{T})^T = \underset{\sim}{S}^T + \underset{\sim}{T}^T$$

$$(\underset{\sim}{S}\underset{\sim}{T})^T = \underset{\sim}{T}^T\underset{\sim}{S}^T$$

$$(\underset{\sim}{S}^T)^T = \underset{\sim}{S}$$

INDICIAL NOTATION:

$$S_{ij}^T = S_{ji}$$

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}^T = \begin{bmatrix} S_{11} & S_{21} & S_{31} \\ S_{12} & S_{22} & S_{32} \\ S_{13} & S_{23} & S_{33} \end{bmatrix}$$

4) PRODUCT OF 2 TENSORS

$$\underset{\sim}{S}\underset{\sim}{T} = \underset{\sim}{S}\cdot\underset{\sim}{T} \Rightarrow (\underset{\sim}{S}\underset{\sim}{T})\vec{u} = \underset{\sim}{S}(\underset{\sim}{T}\vec{u}) \quad \text{FOR ANY VECTOR } \vec{u}$$

$$\underset{\sim}{S}\underset{\sim}{T} \neq \underset{\sim}{T}\underset{\sim}{S} \qquad \text{(GENERALLY)}$$

$$\underset{\sim}{S}\underset{\sim}{T}\vec{u} \Rightarrow S_{ki}T_{ij}u_j = T_{ij}S_{ki}u_j \neq T_{kj}S_{ji}u_i = \underset{\sim}{T}(\underset{\sim}{S}\vec{u})$$

5) IDENTITY TENSOR $\underset{\sim}{I}$

$$\underset{\sim}{I}\vec{v} = \vec{v} \quad \text{FOR ALL } \vec{v}$$

$$\underset{\sim}{I}\underset{\sim}{S}\vec{u} = \underset{\sim}{S}\vec{u}$$

6) INNER PRODUCT OF 2 TENSORS → SCALAR.

$$\underset{\sim}{T} : \underset{\sim}{S} = \text{SCALAR} \qquad T_{ij}S_{ji} = \text{SCALAR}$$

7) SYMMETRIC AND SKEW TENSORS

SYMMETRIC: $S = S^T$ $\qquad S_{ij} = S_{ji}$

SKEW. $S = -S^T$

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & -1 \\ 3 & -1 & 7 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & -2 \\ -1 & 0 & -3 \\ 2 & 3 & 0 \end{bmatrix}$$

SYMMETRIC (ABOUT DIAGONAL) — SKEW (ZEROES ON DIAGONAL)

USEFUL: GIVEN $\underset{\sim}{S}$ IS SYMMETRIC; $\underset{\sim}{W}$ IS SKEW; $\underset{\sim}{T}$ ARBITRARY

a) $\underset{\sim}{S} : \underset{\sim}{T} = \underset{\sim}{S} : \underset{\sim}{T}^T = \underset{\sim}{S} : \{\frac{1}{2}[\underset{\sim}{T} + \underset{\sim}{T}^T]\}$

b) $\underset{\sim}{W} : \underset{\sim}{T} = -\underset{\sim}{W} : T^T = \underset{\sim}{W} : \{\frac{1}{2}[T - T^T]\}$

c) $\underset{\sim}{S} : \underset{\sim}{W} = 0$ (THEY ARE ORTHOGONAL)

8) DECOMPOSITION OF A TENSOR INTO A SYMMETRIC AND A SKEW TENSOR

LET $\underset{\sim}{T}$ BE ANY ARBITRARY TENSOR

$$\underset{\sim}{T} = \underbrace{\tfrac{1}{2}\left(\underset{\sim}{T} + T^T\right)}_{\text{SYMMETRIC}} + \underbrace{\tfrac{1}{2}\left(\underset{\sim}{T} - T^T\right)}_{\text{SKEW}}$$

9) THE DYADIC PRODUCT $\vec{a} \otimes \vec{b}$ OF TWO VECTORS $\vec{a}$ AND $\vec{b}$ IS THE TENSOR THAT ASSIGNS TO EACH VECTOR $\vec{v}$ THE VECTOR $(\vec{b} \cdot \vec{v})\,\vec{a}$:

$$(\vec{a} \otimes \vec{b})\,\vec{v} = (\vec{b} \cdot \vec{v})\,\vec{a}$$

THE DYADIC PRODUCT IS USED TO DEFINE COORDINATES IN A SPECIFIC COORDINATE SYSTEM, IT IS A LINK BETWEEN GIBBS + INDICIAL NOTATION

$$(\vec{a} \otimes \vec{b})_{ij} \Rightarrow a_i b_j$$

10) TRACE, $tr(\cdot)$

THE TRACE IS A LINEAR OPERATION THAT ASSIGNS TO EACH TENSOR $\underset{\sim}{S}$ A SCALAR $tr(\underset{\sim}{S})$ SUCH THAT:

$$tr(\underset{\sim}{S}) = S_{ii}$$

$$\begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix}$$ ADD UP THE DIAGONAL TERMS.