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VECTOR AND TENSOR CALCULUS

I. ONLY 6 THINGS TO REMEMBER.

4 VECTOR OPERATORS
- GRADIENT $\vec{\nabla}$
- DIVERGENCE $\vec{\nabla}\cdot$
- CURL $\vec{\nabla}\times$
- LAPLACIAN

2 INTEGRAL THEOREMS
- GAUSS DIV. THEOREM
- REYNOLDS TRANSP. THM.

II. DEFN' FIELD: A SCALAR, VECTOR OR TENSOR VALUED FUNCTION WHICH DEPENDS ON SPATIAL POSITION

e.g. $\vec{V}(\vec{r}, t)$ $\quad \rho(x, y, z, t)$
$\vec{V}(x, y, z, t)$ $\quad \rho(\vec{r})$

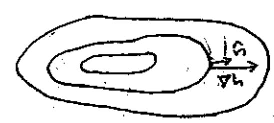
NOTATION
- $\vec{a}$ ≡ ARBITRARY VECTOR VALUED FIELD SUCH AS VELOCITY, VORTICITY, ACCELERATION, FORCE
- ϕ ≡ ARBITRARY SCALAR VALUED FIELD SUCH AS TEMP, PRESSURE, ENTHALPY, VELOCITY POTENTIAL
- $\vec{V}$ ≡ VELOCITY FIELD → A VECTOR FIELD
- $\vec{r}$ ≡ A POSITION VECTOR. THIS IS NOT A FIELD
- $\underset{\sim}{T}$ ≡ TENSOR FIELD SUCH AS STRESS, STRAIN, VELOCITY GRADIENT

III. THE GRADIENT OPERATOR

A) NOTATION:
$\vec{\nabla}\phi$, $\vec{\nabla}\vec{a}$, $\text{grad}\,\phi$, $\text{grad}\,\vec{a}$

$\dfrac{\partial\phi}{\partial x_j}$, $\dfrac{\partial a_i}{\partial x_j}$

e.g. A HILL ; h = ELEVATION



B) GRADIENT OF A SCALAR FIELD
⇒ 1.) $\nabla\phi$ ⇒ A VECTOR
2) $\phi = \phi(x_1, x_2, x_3)$ TAKE TOTAL DIFFERENTIAL

$$d\phi = \left(\frac{\partial\phi}{\partial x_1}\right)dx_1 + \left(\frac{\partial\phi}{\partial x_2}\right)dx_2 + \left(\frac{\partial\phi}{\partial x_3}\right)dx_3$$

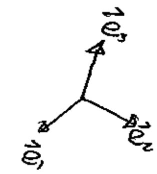
$$d\phi = \vec{\nabla}\phi\cdot d\vec{r} \Rightarrow \underline{\frac{d\phi}{dn} = \vec{\nabla}\phi\cdot\vec{n}}$$; $\vec{r}$ IS POSITION VECTOR

* RATE OF CHANGE OF ϕ IN THE DIRECTION OF $\hat{n}$ IS GIVEN BY $\vec{\nabla}\phi\cdot\hat{n}$; $\hat{n}$ IS THE OUTWARD NORMAL VECTOR

⇒ * $\nabla\phi$ EXTENDS IN THE DIRECTION OF THE GREATEST RATE OF CHANGE OF ϕ AND HAS THAT RATE OF CHANGE FOR ITS LENGTH.

* WE OFTEN CONSIDER THE GRADIEN AS AN OPERATOR:

$$\vec{\nabla} = \text{"del"} = \vec{e}_x \frac{\partial}{\partial x} + \vec{e}_y \frac{\partial}{\partial y} + e_z \frac{\partial}{\partial z} \qquad \text{(CARTESIAN)}$$

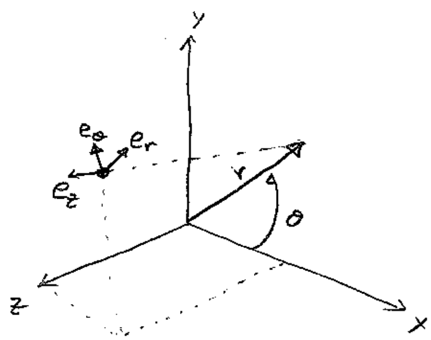


c) $\vec{\nabla}\phi$ IN 3 COORDINATE SYSTEMS.

CARTESIAN (x, y, z)

$$\nabla\phi = \frac{\partial\phi}{\partial x_i} = \vec{\nabla}\phi = \frac{\partial\phi}{\partial x_1}\vec{e}_1 + \frac{\partial\phi}{\partial x_2}\vec{e}_2 + \frac{\partial\phi}{\partial x_3}\vec{e}_3$$

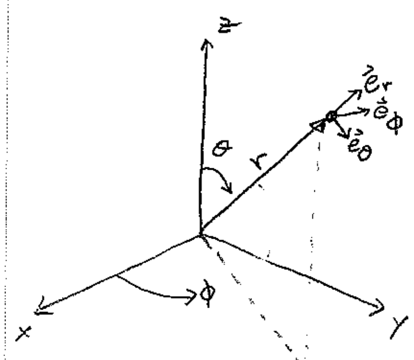
CYLINDRICAL (r, θ, z)



$$\nabla\phi = \frac{\partial\phi}{\partial r}\vec{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\vec{e}_\theta + \frac{\partial\phi}{\partial z}\vec{e}_z$$

SPHERICAL (r, θ, ϕ)

USE P INSTEAD OF ϕ FOR FIELD TO AVOID CONFUSION



$$\vec{\nabla}P = \frac{\partial P}{\partial r}\hat{e}_r + \left(\frac{1}{r}\frac{\partial P}{\partial\theta}\right)e_\theta + \left(\frac{1}{r\sin\theta}\frac{\partial P}{\partial\phi}\right)e_\phi$$

SOME IMPORTANT ISSUES / CAVEATS

- $\frac{\partial\phi}{\partial x_i}$ GIVES CORRECT DERIVATIVE (CORRECT VALUE FOR $\nabla\phi$) ONLY IN CARTESIAN SYSTEM
- HOW DO WE FIND $\vec{\nabla}\phi$ IN NON CARTESIAN COORDINATES
  - LOOK IT UP
  - DERIVE USING THE CHAIN RULE

EXAMPLE: TRANSFORM INTO CYLINDRICAL COORDINATES

$$\vec{\nabla} = \vec{e}_x \frac{\partial}{\partial x} + \vec{e}_y \frac{\partial}{\partial y} + \vec{e}_z \frac{\partial}{\partial z}$$

KEY IDEAS: USE THE CHAIN RULE AND THE FIGURE ON THE PREVIOUS PAGE

① TRANSFORM $\partial/\partial x$

$$\frac{\partial}{\partial x} \Rightarrow \frac{\partial}{\partial x}\, \phi\big(r(x,y,z),\ \theta(x,y,z),\ z(x,y,z)\big)$$

$$\frac{\partial \phi}{\partial x} = \frac{\partial \phi}{\partial r}\left(\frac{\partial r}{\partial x}\right)_{y,z} + \frac{\partial \phi}{\partial \theta}\left(\frac{\partial \theta}{\partial x}\right)_{y,z} + \frac{\partial \phi}{\partial z}\left(\frac{\partial z}{\partial x}\right)_{y,z}$$

$r^2 = x^2 + y^2$ $\qquad$ $x = r\cos\theta$

$\theta = \tan^{-1}(y/x)$ $\qquad$ $y = r\sin\theta$

$z = z$ $\qquad$ $z = z$

$$\left(\frac{\partial r}{\partial x}\right)_{y,z} = \frac{x}{r} = \cos\theta$$

$$\left(\frac{\partial \theta}{\partial x}\right)_{y,z} ; \quad \tan\theta = \frac{y}{x} \xrightarrow[\text{DIFFERENT}]{\text{IMPLICIT}} \frac{1}{\cos^2\theta} = -\frac{y}{x^2}\left(\frac{dx}{d\theta}\right)$$

SEE NEXT PAGE

$$\therefore \left(\frac{\partial \theta}{\partial x}\right)_{y,z} = \frac{-y\cos^2\theta}{x^2} = -\frac{\sin\theta}{r}$$

$$\therefore \frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$

② TRANSFORM $\partial/\partial y$ AND $\partial/\partial z$

$$\frac{\partial}{\partial y} = \left(\frac{\partial r}{\partial y}\right)_{x,z}\frac{\partial}{\partial r} + \left(\frac{\partial \theta}{\partial y}\right)_{x,z}\frac{\partial}{\partial \theta} + \left(\frac{\partial z}{\partial y}\right)\frac{\partial}{\partial z}$$

$$\left(\frac{\partial r}{\partial y}\right)_{x,z} = \frac{y}{r} = \frac{r\sin\theta}{r} = \sin\theta \quad ; \quad \left(\frac{\partial \theta}{\partial y}\right)_{x,z} = \frac{\cos^2\theta}{x} = \frac{\cos\theta}{r}$$

$$\therefore \partial/\partial y = \sin\theta\, \partial/\partial r + \cos\theta/r\ \partial/\partial\theta$$

$\partial/\partial z = \partial/\partial z$ (Z IS SAME IN BOTH COORDINATE SYSTEMS)

SOME REMINDERS

IMPLICIT DIFFERENTIATION

- EXPLICIT AND IMPLICIT FUNCTIONS

$y = f(x)$ ⇒ THE DEPENDENT VARIABLE y IS AN EXPLICIT FUNCTION OF INDEP. VAR x

$s(x,y) = 0$ ⇒ THE DEPENDENT VARIABLE y CAN ONLY BE OBTAINED BY SOLVING THIS EQUATION. THIS IS AN IMPLICIT FUNCTION FOR y IN TERMS OF x.

DIFFERENTIATION OF AN EXPLICIT FUNCTION: APPLY CHAIN RULE TO CALCULATE DERIVATIVES dy/dx W/O MAKING y AN EXPLICIT FUNCTION OF x

$$\frac{d}{dx}(y^2) = \underbrace{\frac{d}{dy}(y^2)}_{\text{DIFF } y^2 \text{ WRT } dy} \cdot \underbrace{\frac{dy}{dx}}_{\text{MULTIPLY BY } \frac{dy}{dx} \text{ TO GET RID OF } dy}$$

e.g. $x + y = 3$

FIND dy/dx: 2 WAYS

1) ALGEBRA → EXPLICIT EXPRESSION

$$y = 3 - x$$

$$\frac{dy}{dx} = -1$$

2) IMPLICIT DIFFERENTIATION

$$\frac{dx}{dx} + \frac{dy}{dx} = \frac{d}{dx}(3)$$

$$1 + \frac{dy}{dx} = 0 \ ; \quad \frac{dy}{dx} = -1$$

e.g. $x^5 + y^3 = 6$

$$\frac{d}{dx}(x^5) + \frac{d}{dy}(y^3) \cdot \frac{dy}{dx} = \frac{d}{dx}(6)$$

$$5x^4 + 3y^2\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-5x^4}{3y^2}$$

e.g.

$$r^2 - x^2 = 0$$

$$\frac{d}{dr}(r^2)\frac{dr}{dx} + \frac{d}{dx}(x^2) = 0$$

$$2r\frac{dr}{dx} + 2x = 0$$

$$\frac{dr}{dx} = \frac{x}{r}$$

③ TRANSFORM $\vec{e}_x$, $\vec{e}_y$ AND $\vec{e}_z$

$$\vec{e}_x = \vec{e}_r \cos\theta - \vec{e}_\theta \sin\theta$$

$$\vec{e}_y = \vec{e}_r \sin\theta - \vec{e}_\theta \cos\theta$$

$$\vec{e}_z = \vec{e}_z$$

④ $$\vec{\nabla} = \vec{e}_x \frac{\partial}{\partial x} + \vec{e}_y \frac{\partial}{\partial y} + \vec{e}_z \frac{\partial}{\partial z}$$

$$= \left(\vec{e}_r \cos\theta - e_\theta \sin\theta\right)\left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right) +$$

$$\left(\vec{e}_r \sin\theta - \vec{e}_\theta \cos\theta\right)\left(\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right) +$$

$$\vec{e}_z \frac{\partial}{\partial z}$$

$\Downarrow$ ALGEBRA

$$\vec{\nabla} = \vec{e}_r \frac{\partial}{\partial r} + \frac{\vec{e}_\theta}{r}\frac{\partial}{\partial\theta} + \vec{e}_z \frac{\partial}{\partial z}$$

C. GRADIENT OF A VECTOR FIELD

$\Rightarrow \vec{\nabla}\vec{a} \Rightarrow$ A TENSOR

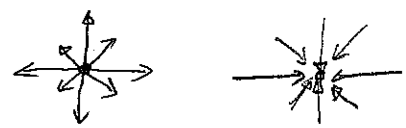
$$\vec{\nabla}\vec{a} = \begin{bmatrix} \frac{\partial a_1}{\partial x_1} & \frac{\partial a_1}{\partial x_2} & \frac{\partial a_1}{\partial x_3} \\ \frac{\partial a_2}{\partial x_1} & \frac{\partial a_2}{\partial x_2} & \frac{\partial a_2}{\partial x_3} \\ \frac{\partial a_3}{\partial x_1} & \frac{\partial a_3}{\partial x_2} & \frac{\partial a_3}{\partial x_3} \end{bmatrix}$$

IV DIVERGENCE OPERATOR

A) NOTATION

$\mathrm{div}\,\vec{a}$, $\mathrm{div}\,\underset{\sim}{T}$, $\vec{\nabla}\cdot\vec{a}$, $\vec{\nabla}\cdot\underset{\sim}{T}$

$\frac{\partial a_i}{\partial x_i}$, $\frac{\partial T_{ij}}{\partial x_j}$



MAGNITUDE OF $\vec{\nabla}\cdot\vec{a}$ GIVES MEASURE OF A SOURCE OR SINK DENSITY

B) PHYSICAL INTERPRETATION: <u>FLUX</u>

$\iiint_V \mathrm{div}\,\vec{a}\, dV \equiv$ NET RATE AT WHICH $\vec{a}$ IS LEAVING THE VOLUME V

e.g.

$\iiint_V \vec{\nabla}\cdot(\rho\vec{V})\, dV \equiv$ NET RATE AT WHICH MASS IS LEAVING A REGION BY FLUID MOTION

C) MATHEMATICAL OPERATION

CARTESIAN $\quad \mathrm{div}\,\vec{a} = \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3}$

CYLIND $\quad \mathrm{div}\,\vec{a} = \frac{1}{r}\frac{\partial}{\partial r}(r a_r) + \frac{1}{r}\frac{\partial}{\partial \theta} a_\theta + \frac{\partial a_z}{\partial z}$

SPHERICAL $\quad \mathrm{div}\,\vec{a} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 a_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(a_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} a_\phi$

D) div (VECTOR) $\longrightarrow$ SCALAR

div (TENSOR) $\longrightarrow$ VECTOR

div (SCALAR) $\longrightarrow$ THERE IS A PROBLEM

IV THE CURL OPERATOR

A) NOTATION

$$\text{curl } \vec{a}, \quad \vec{\nabla} \times \vec{a}, \quad \varepsilon_{ijk} \frac{\partial V_k}{\partial x_j}$$

MAGNITUDE OF $\vec{\nabla} \times \vec{a}$ GIVES MEASURE OF CIRCULATION DENSITY.

B) PHYSICAL INTERPRETATION: ROTATION OF A FLUID ELEMENT

$\vec{\nabla} \times \vec{v}$ = TWICE THE ANGULAR VELOCITY

C) MATHEMATICAL OPERATION

CARTESIAN:

$$\vec{\nabla} \times \vec{v} = \left(\vec{e}_1 \partial/\partial x_1 + \vec{e}_2 \partial/\partial x_2 + \vec{e}_3 \partial/\partial x_3\right) \times \left(v_1 \vec{e}_1 + v_2 \vec{e}_2 + v_3 \vec{e}_3\right)$$

$$= \left(\frac{\partial V_3}{\partial x_2} - \frac{\partial V_2}{\partial x_3}\right)\vec{e}_1 +$$

$$\left(\frac{\partial V_1}{\partial x_3} - \frac{\partial V_3}{\partial x_1}\right)\vec{e}_2 +$$

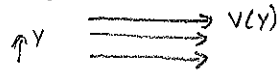
$$\left(\frac{\partial V_2}{\partial x_1} - \frac{\partial V_1}{\partial x_2}\right)\vec{e}_3$$

CYLINDRICAL / SPHERICAL: LOOK THIS UP.

V. TWO SPECIAL CASES OF INTEREST IN FLUIDS

$\vec{\nabla} \times \vec{V} = 0;$ $\equiv$ "IRROTATIONAL VELOCITY FIELD" OR IRROTATIONAL FLOW

e.g.



$\vec{\nabla} \cdot \vec{V} = 0;$ $\equiv$ "SOLENOIDAL VELOCITY FIELD" OR FLOW

FLOW IS DIVERGENCE FREE

e.g.

VI. LAPLACIAN OPERATOR

A) NOTATION

$\nabla^2 \vec{a}, \ \nabla^2 \phi$ $\qquad \dfrac{\partial^2 \phi}{\partial x_i \partial x_i}, \quad \dfrac{\partial^2 a_j}{\partial x_i \partial x_i}$

$\vec{\nabla} \cdot \vec{\nabla} \vec{a}, \ \vec{\nabla} \cdot \vec{\nabla} \phi$

B) PHYSICAL INTERPRETATION: COME BACK TO THIS.

C) MATHEMATICAL OPERATIONS

∇^2 (VECTOR) $\Rightarrow$ VECTOR

∇^2 (SCALAR) $\Rightarrow$ SCALAR.

CARTESIAN:

$$\nabla^2 \vec{a} = \left(\frac{\partial^2 a_1}{\partial x_1^2} + \frac{\partial^2 a_1}{\partial x_2^2} + \frac{\partial^2 a_1}{\partial x_3^2} \right) \vec{e}_1 + \left(\frac{\partial^2 a_2}{\partial x_1^2} + \frac{\partial^2 a_2}{\partial x_2^2} + \frac{\partial^2 a_2}{\partial x_3^2} \right) \vec{e}_2 + \left(\frac{\partial^2 a_3}{\partial x_1^2} + \frac{\partial^2 a_3}{\partial x_2^2} + \frac{\partial^2 a_3}{\partial x_3^2} \right) \vec{e}_3$$

VII. SUMMARY 4 VECTOR OPERATORS

GRAD $\vec{\nabla}\phi \rightarrow$ VECTOR
$\vec{\nabla}\vec{a} \rightarrow$ TENSOR
} RATE OF CHANGE

DIV $\vec{\nabla}\cdot\vec{a} \rightarrow$ SCALAR
$\vec{\nabla}\cdot\underset{\sim}{T} \rightarrow$ TENSOR
} NET OUTFLOW

CURL $\vec{\nabla}\times\vec{a} \rightarrow$ VECTOR } ROTATION OF A FLUID ELEMENT

LAPLACIAN $\nabla^2\vec{a} \rightarrow$ VECTOR
$\nabla^2\phi \rightarrow$ SCALAR.

IN PRACTICE WE LOOK UP EACH OF THESE.

VIII VECTOR IDENTITIES.

$$\vec{\nabla}\cdot(\phi\vec{a}) = \phi\vec{\nabla}\cdot\vec{a} + \vec{a}\cdot\vec{\nabla}\phi$$

$$\vec{\nabla}(\vec{a}\cdot\vec{b}) = (\vec{\nabla}b^T)\vec{a} + (\vec{\nabla}\vec{a}^T)\vec{b}$$

$$\vec{\nabla}\cdot(\underset{\sim}{T}^T\vec{a}) = \underset{\sim}{T} : \vec{\nabla}\vec{a} + \vec{a}\cdot(\vec{\nabla}\cdot\underset{\sim}{T})$$

$$\vec{\nabla}\cdot\phi\underset{\sim}{T} = \phi\vec{\nabla}\cdot\underset{\sim}{T} + \underset{\sim}{T}\vec{\nabla}\phi$$

$$\vec{\nabla}^2\vec{a} = \vec{\nabla}(\vec{\nabla}\cdot\vec{a}) - \vec{\nabla}\times(\vec{\nabla}\times\vec{\nabla})$$

$$\vec{\nabla}\cdot((\vec{\nabla}\vec{a})^T) = \vec{\nabla}(\vec{\nabla}\cdot\vec{a})$$

$$(\vec{\nabla}\vec{u})\vec{u} = \frac{1}{2}\vec{\nabla}(\vec{u}\cdot\vec{u}) - \vec{u}\times(\nabla\times\vec{u})$$

$$\vec{\nabla}\times(\vec{\nabla}\phi) = 0$$

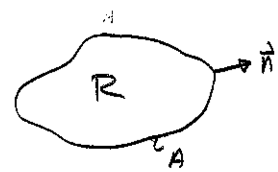
$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{a}) = 0$$

IX TWO INTEGRAL THEOREMS

A. GAUSS DIVERGENCE THM

$$\underset{A}{\iint} \phi \vec{n} \, dA = \underset{\forall}{\iiint} \vec{\nabla} \phi \, d\forall$$

A ↑ AREA OF R; $\forall$ ↑ VOLUME OF R



$$\underset{A}{\iint} (\vec{a} \cdot \vec{n}) \, dA = \underset{\forall}{\iiint} (\vec{\nabla} \cdot \vec{a}) \, d\forall$$

$$\underset{A}{\iint} \underset{\sim}{T} \, \vec{n} \, dA = \underset{\forall}{\iiint} (\vec{\nabla} \cdot \underset{\sim}{T}) \, d\forall$$

$A \equiv$ SURFACE AREA OF R
$\forall \equiv$ VOLUME OF R
$\vec{n} \equiv$ OUTWARD DIRECTED NORMAL VECTOR ON dA

B. REYNOLDS TRANSPORT THM.

LET $\forall(t)$ BE A VOLUME WHICH CHANGES WITH TIME. $A(t)$ IS THE SURFACE AREA OF THE VOLUME. LET Φ BE EITHER A SCALAR OR A VECTOR FIELD $\vec{n}$ IS AN OUTWARD NORMAL ON dA

THEN, FOR ANY TIME t:

$$\frac{d}{dt} \underset{\forall(t)}{\iiint} \Phi \, d\forall = \underset{\forall(t)}{\iiint} \frac{\partial \Phi}{\partial t} \, d\forall + \underset{A(t)}{\iint} \Phi (\vec{v} \cdot \vec{n}) \, dA = \underset{\forall(t)}{\iiint} \left(\frac{D\Phi}{Dt} + \Phi \operatorname{div} \vec{v} \right) d\forall$$