A hydrologically coupled higher-order flow-band model of ice dynamics with a Coulomb friction sliding law

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The influence of hydrologic transience and heterogeneity on basal motion is an often-neglected aspect of numerical ice-flow models. We present a flow-band model of glacier dynamics with a Coulomb friction sliding law that is coupled to a model of the basal drainage system by means of subglacial water pressure. The ice-flow model contains “higher-order” stress gradients from the Stokes flow approximation originally conceived by Blatter (1995). The resulting system of nonlinear equations is solved using a modified Picard iteration that is shown to improve the rate of convergence. A parameterization of lateral shearing is included to account for the effects of three-dimensional geometry. We find that lateral drag has a discernible effect on glacier speed even when glacier width exceeds glacier length. Variations in flow-band width are shown to have a greater influence on flow line speed than either different valley cross-sectional shapes or the presence or absence of glacier sliding along valley walls. Modeled profiles of subglacial water pressure depart significantly from pressures prescribed as a uniform fraction of overburden, thus producing profiles of glacier sliding that are distinctly different from those that would be described by a sliding law controlled by overburden pressures. Simulations of hydraulically driven glacier acceleration highlight the value of including a representation of basal hydrology in models aiming for improved predictive capability of glacier dynamics.


1. Introduction

The projected glacier and ice sheet contribution to eustatic sea level reported in the latest IPCC report [Solomon et al., 2007] contains considerable uncertainty due primarily to dynamical ice-flow processes not currently included in ice sheet and climate models. These processes include horizontal-stress gradients, subglacial hydraulic processes and their coupling to ice dynamics, grounding line migration, and calving front mechanics [Lipscomb et al., 2009]. The imperative to develop more authentic numerical models of the dynamical behavior observed in tidewater outlet glaciers and ice streams is therefore a major priority. Recent strides have been made at the ice sheet scale by Pollard and DeConto [2009] and Bueler and Brown [2009]. Pollard and DeConto [2009] model the West Antarctic ice sheet using the shallow ice approximation (SIA) for grounded ice and an ice-shelf model for floating ice. The effects of grounding-line dynamics and ice-shelf buttressing are incorporated, and basal sliding is permitted when the bed reaches the pressure melting point, yet there is no explicit treatment of basal hydrology. Bueler and Brown [2009] include both vertical shear stresses and membrane stresses in a hybrid approach where the shallow shelf approximation (SSA) is used as a sliding rule within a model that uses the SIA. Modeled basal melt rate drives a diffusion equation used to describe pore water pressure in the basal sliding formulation. The SIA is appropriate for ice flow with little or no slip [Hutter, 1983], and the viscous membrane model of the SSA is applicable for ice flow undergoing rapid slip [MacAyeal, 1989]. Simulations of the Greenland Ice Sheet using this hybrid ice sheet model [Bueler and Brown, 2009] compare favorably to remotely sensed ice-surface velocity measurements (E. Bueter et al., Modeled and observed fast flow in the Greenland ice sheet, submitted to Geophysical Research Letters, 2010).

In this study we focus on valley glaciers, where glacier depths and widths can be of comparable magnitudes, in which case the shallow ice and shallow shelf assumptions do not apply. Glacier and ice sheet models that endeavor to combine aspects of fast and slow sliding within the same set of partial differential equations have become collectively known as “higher-order” models [e.g., Schoof and Hindmarsh, 2010]. These models include mechanical effects neglected by the SIA, principally horizontal stress gradients but up to and including all the terms in the full Stokes equations. The higher-order stress gradients play a major role in the vicinity of bed topographic variations.
and pronounced slopes [e.g., Le Meur et al., 2004], for glacier geometries with high-aspect ratios [e.g., Baral et al., 2001; Pattyn et al., 2008], and for outlet glaciers and ice streams especially when basal sliding is considerable [e.g., Price et al., 2002]. In recent years the use of such models has become prevalent, even prompting a community intercomparison project [Pattyn et al., 2008].

Unlike models that employ the SIA, these so-called higher-order models are able to incorporate and distribute the boundary-layer stress generated by sliding at the base of the ice mass. This can have a leading-order effect on the mechanics of the flow. In spite of this, previous implementations of these models have frequently assumed a no-slip condition [e.g., Hvidberg, 1996; Johnson and Staiger, 2007] or employed a simplified representation of basal sliding [e.g., Price et al., 2008; Pattyn et al., 2006; Delcourt et al., 2008]. Although basal sliding is an essential component of fast ice flow [e.g., Kamb et al., 1985; Alley et al., 1986; Engelhardt and Kamb, 1998], incorporation of basal mechanics into glacier and ice sheet models remains a major challenge. The model described in this paper makes use of some of the latest theoretical work on sliding [Schoof, 2005; Gagliardini et al., 2007; Schoof, 2010].

Sliding velocity and basal drag depend critically on basal water pressure and subglacial water storage. Numerous field studies have shown that daily and seasonal changes in subglacial hydrology play a major role in modulating ice dynamics [e.g., Iken and Bindschadler, 1986; Anderson et al., 2004; Joughin et al., 2008]. This essential feature is almost always neglected or primitively incorporated in current model configurations [cf. Le Brocq et al., 2009]. The hydrological coupling in the model we present allows basal effective pressure to vary in time and space, influencing sliding, and thus permitting stress redistribution to take place over the glacier bed.

Reduced dimensional models have a widespread use within Earth system science, with examples including column models of the atmosphere and ocean mixed layer [e.g., Randall et al., 1996; Niiler and Kraus, 1977]. In glaciology, two-dimensional models of glacier flow (both planform and flow line) continue to be valuable tools for understanding glacier dynamics and guiding the development of more comprehensive models. The reduced size and complexity provide a flexible framework for testing parameterizations and examining the influence of various components within the model.

In this paper we outline and demonstrate the use of a two-dimensional ice dynamics model that parameterizes three-dimensional effects. The model incorporates higher-order stress gradients, and is coupled to a physically motivated model of the subglacial drainage system through the effective pressures influence on a Coulomb friction sliding law. The intended applications of the model include assessing the role subglacial hydrology plays in the dynamics of outlet glaciers.

2. Governing Equations

The system of equations that forms our numerical ice-flow model are based upon a force-balance equation together with a constitutive relationship that relates strain rate to stress. We assume invariance in the across-flow direction to develop a flow line model. In a Cartesian coordinate system \((x, z)\) the \(x\) axis is horizontal and orientated along the flow line and the \(z\) axis is vertical and perpendicular to \(x\). The flow line model is then adapted to a flow-band by specifying a flow-unit half-width \(W = W(x)\), transverse to the flow line. Derivations of the working equations are described below.

2.1. Force Balance

The three-dimensional force-balance equation is stated as

\[
\rho \frac{Du}{Dt} = \nabla \cdot \sigma + \rho g,
\]

where \(u = (u, v, w)\) and \(g = (0, 0, -g)\) are the components of the velocity and gravitational acceleration, respectively, \(\sigma\) denotes the stress tensor, and \(\rho\) represents the density of ice. For Stokes flow, used to describe the creep of ice, accelerations are negligible (see Froude number argument of Greve and Blatter [2009, p. 64]), and the left hand side of equation (1) is zero.

Following the pioneering model of Blatter [1995] we apply an approximation of the Stokes equation that is second-order accurate in the aspect ratio of the ice mass, regardless of the amount of slip at the base of the glacier [Schoof and Hindmarsh, 2010]. This provides a good description of glacier flow regimes that include little or no slip as well as fast sliding [Schoof and Hindmarsh, 2010]. Under this approximation the vertical normal stress is assumed hydrostatic. Writing the stress tensor in terms of its components, \(\sigma_{ij}\), we obtain the following:

\[
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} = 0,
\]

\[
\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} = 0,
\]

\[
\sigma_{zz} = -\rho g (s - z),
\]

where \(b \leq z \leq s\), with \(b\) and \(s\) denoting the bed and surface elevations, respectively. The definition of deviatoric stress,

\[
\sigma_{ij}' = \sigma_{ij} - \frac{1}{3} b_{ij} (\sigma_{xx} + \sigma_{yy} + \sigma_{zz}),
\]

together with equation (4) gives

\[
2\sigma_{xx} = 3\sigma_{xx} + \sigma_{yy} - \rho g (s - z),
\]

\[
2\sigma_{yy} = 3\sigma_{yy} + \sigma_{xx} - \rho g (s - z).
\]

Combining equations (6) and (7) we obtain

\[
\sigma_{xx} = 2\sigma_{xx} + \sigma_{yy} - \rho g (s - z),
\]

\[
\sigma_{yy} = 2\sigma_{yy} + \sigma_{xx} - \rho g (s - z).
\]
Table 1. Constants and Other Model Parameters

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Value</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rho )</td>
<td>ice density</td>
<td>910</td>
<td>kg m(^{-3})</td>
</tr>
<tr>
<td>( g )</td>
<td>gravitational acceleration</td>
<td>9.81</td>
<td>m s(^{-2})</td>
</tr>
<tr>
<td>( n )</td>
<td>Glen’s flow law exponent</td>
<td>3</td>
<td>–</td>
</tr>
<tr>
<td>( e_0 )</td>
<td>viscosity regularization</td>
<td>( 1 \times 10^{-30} )</td>
<td>y r(^{-1})</td>
</tr>
<tr>
<td>( A )</td>
<td>flow law parameter</td>
<td>( 1 \times 10^{-16} )</td>
<td>Pa (^{-r} )</td>
</tr>
<tr>
<td>( \lambda_{\text{max}} )</td>
<td>wavelength of bedrock bumps</td>
<td>2</td>
<td>m</td>
</tr>
<tr>
<td>( \theta_{\text{max}} )</td>
<td>maximum bed obstacle slope</td>
<td>0.5</td>
<td>–</td>
</tr>
<tr>
<td>( C )</td>
<td>Coulomb friction law constant</td>
<td>0.84</td>
<td>–</td>
</tr>
<tr>
<td>( b^* )</td>
<td>critical sheet thickness</td>
<td>0.15</td>
<td>m</td>
</tr>
<tr>
<td>( K_{\text{max}} )</td>
<td>maximum hydraulic conductivity</td>
<td>( 2.5 \times 10^3 )</td>
<td>m s(^{-1})</td>
</tr>
<tr>
<td>( K_{\text{min}} )</td>
<td>minimum hydraulic conductivity</td>
<td>( 2.5 \times 10^3 )</td>
<td>m s(^{-1})</td>
</tr>
<tr>
<td>( k_x )</td>
<td>hydraulic conductivity parameter</td>
<td>20</td>
<td>–</td>
</tr>
<tr>
<td>( k_y )</td>
<td>hydraulic conductivity parameter</td>
<td>0.9</td>
<td>–</td>
</tr>
<tr>
<td>( \rho_w )</td>
<td>water density</td>
<td>1000</td>
<td>kg m(^{-3})</td>
</tr>
<tr>
<td>( Q_0 )</td>
<td>geothermal flux</td>
<td>0.07</td>
<td>W m(^{-2})</td>
</tr>
<tr>
<td>( L )</td>
<td>specific latent heat of fusion</td>
<td>( 3.35 \times 10^5 )</td>
<td>J kg(^{-1})</td>
</tr>
</tbody>
</table>

which, substituted into equations (2) and (3), yields

\[
\frac{\partial}{\partial x} \left( 2\sigma_{xx}' + \sigma_{yy}' \right) + \frac{\partial \sigma_{xy}'}{\partial y} + \frac{\partial \sigma_{xy}'}{\partial z} = \rho g \frac{\partial s}{\partial x},
\]

\[
\frac{\partial \sigma_{xx}'}{\partial x} + \frac{\partial}{\partial y} \left( 2\sigma_{yy}' + \sigma_{xx}' \right) + \frac{\partial \sigma_{yy}'}{\partial z} = \rho g \frac{\partial s}{\partial y},
\]

This is referred to as a higher-order model due to the retention of horizontal stress gradients in the force-balance equations. Under the classification of Hindmarsh [2004] this model corresponds to the class of models that includes a multilayer longitudinal stress scheme labeled LMLa. Note that this is not the full-stress solution, as we have assumed the vertical normal stress to be hydrostatic, but the pressure (the sum of the normal stresses) does depart from the hydrostatic pressure because of the longitudinal deviatoric stress:

\[
P_l = \sigma_{xx}' + \sigma_{yy}' - \rho g (s - z).
\]

In the solution to the full Stokes equations the pressure would include the integrated horizontal gradient in vertical shear stress (see Price et al. [2007] for a full-stress solution). Situations in which the ice pressure is not hydrostatic can occur in the presence of bedrock perturbations and with abrupt changes in basal sliding conditions; in these situations a hydrodynamic component to the pressure becomes important.

2.1.1. Constitutive Relation

[12] Glacier ice deforms as a non-Newtonian fluid. For isotropic ice, this nonlinear behavior is governed by a power law rheology known as the generalized Glen’s flow law [Paterson, 1994]. The dual form of this relation, in terms of viscosity, is as follows:

\[
\sigma_{ij}' = 2\nu \dot{\epsilon}_{ij},
\]

where \( \dot{\epsilon}_{ij} \) is the strain rate tensor defined as

\[
\dot{\epsilon}_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right),
\]

and \( \nu \) denotes the strain rate-dependent effective viscosity given by

\[
\nu = \frac{1}{2} A^{-1/n} (\epsilon + \dot{\epsilon}_0)^{(1-n)/n},
\]

with \( \dot{\epsilon}_0 = \frac{1}{2} \dot{\epsilon}_{ij} \dot{\epsilon}_{ij} \) (the second invariant of the strain rate tensor squared) and \( \dot{\epsilon}_0 \) a small number used to avoid singular behavior in the case of vanishing strain rate [Meier, 1958]. The flow-law parameter \( A \) is temperature-dependent and is often determined based on a modified Arrhenius relation [e.g., Paterson and Budd, 1982]; we use a value representative of ice at \(-2^\circ\)C (see Table 1). The flow-law exponent \( n \) is set equal to 3.

2.1.2. Two-Dimensional Reduction

[13] Reducing to the along-flow and vertical dimension we obtain a flow line geometry where ice is assumed to flow in an infinite plane (as we resolve only one horizontal direction). In reality glaciers are confined to channels and are subjected to lateral shearing and across flow compression or extension. We therefore parameterize the lateral drag \( \sigma_{xy} \) by making the following assumptions: \( \dot{\epsilon}_{xz} = 0 \) and \( \dot{\epsilon}_{yz} = \frac{u_z}{W} \) (i.e., \( u(y) \) varies linearly across the flow line [cf. Dupont, 2004]). If the shape of the glacier valley is known, then \( W = W(x, z) \) can also be used. The lateral drag is then parameterized as

\[
\sigma_{xy} = 2\nu \dot{\epsilon}_{xy} \approx \frac{2\nu u_z}{W} \frac{\partial W}{\partial x},
\]

and we have \( \frac{\partial \sigma_{xy}}{\partial x} \approx -\frac{\nu (u - u_z)}{W} \). To take account of converging or diverging flow the stress term \( \sigma_{yy}' \) is retained by assuming \( \sigma_{yy}' \approx \frac{\nu (u - u_z)}{W} \) [Nye, 1959], where \( u_L = u_L(x, z) \) is a sliding velocity along the valley sidewalls (see section 2.2). The flow-band adaption is parameterized as

\[
\sigma_{yy} = 2\nu \dot{\epsilon}_{xy} \approx \frac{2\nu u_L}{W} \frac{\partial W}{\partial x},
\]

2.1.3. Velocity and Viscosity

[14] We follow Colinge and Rappaz [1999] and Pattyn [2002] in using the constitutive relation to reformulate the momentum equations as a second-order differential equation for velocity. Using equations (14), (15), (16), and (17) the viscosity of ice is computed as

\[
\nu = \frac{1}{2} A^{-1/n} \left[ \left( \frac{\partial u}{\partial x} \right)^2 + \frac{u_z}{W} \frac{\partial W}{\partial x} \right] + \frac{u}{W} \frac{\partial u}{\partial x} \frac{\partial W}{\partial x} + \frac{1}{4} \left( \frac{\partial^2 u}{\partial x^2} + \frac{1}{4} \left( \frac{u - u_z}{W} \right)^2 + \dot{\epsilon}_0 \right)^{(1-n)/n},
\]

and employing equations (10), (13), (14), (16), and (17) the velocity \( u \) is determined by solving

\[
-\frac{u}{W^2} \left( \frac{\nu}{W^2} + \frac{\partial u}{\partial x} \right) + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial x} + 4 \frac{\partial^2 u}{\partial x^2} + \frac{\nu}{W^2} \frac{\partial^2 u}{\partial x^2} \frac{\partial W}{\partial x} = \rho g \frac{\partial s}{\partial x} - \frac{\nu u_L}{W^2} - \frac{2}{\partial W} \left( \frac{\nu u_L}{W} \frac{\partial W}{\partial x} \right),
\]
A stress-free surface holds at the ice-atmosphere interface [Blatter, 1995]:

$$\frac{\partial s}{\partial x} \left( 4 \frac{\partial u}{\partial x} + 2u \frac{\partial W}{\partial x} \right) - \frac{\partial u}{\partial z} = 0 \quad \text{at} \quad z = s. \quad (20)$$

### 2.2. Sliding

The motion of ice at the bed is usually parameterized in a sliding law that relates basal drag, $\tau_b$, to the sliding velocity, $u_b$. The simplest form is a linear relation $\tau_b = \beta \cdot u_b$, where the friction parameter $\beta$ can vary between zero, which implies no traction and a stress-free basal boundary condition, and $\beta^2 \to \infty$, which would result in a no-slip condition and zero basal velocity. This relation is often used in higher-order models [e.g., Pattyn et al., 2006; Pattyn, 2008; Pattyn et al., 2008] as it is easy to implement and capable of producing a large range of values for the sliding velocity.

In order to incorporate the effect of cavitation on sliding, the effective pressure ($N$) at the base of the ice has been widely used in sliding laws, for example,

$$u_b = A_s \tau_b w N^{-q}, \quad (21)$$

where $A_s$, $p$, and $q$ are positive constants that are empirically determined. This power law formulation [Bindschadler, 1983] has also been utilized in higher-order models [e.g., Pattyn, 2002, 2003]. Difficulties with this formulation can arise because basal drag is a function of the basal velocity. Often this issue is bypassed by assuming that the driving stress, $\tau_D$, (which depends only on local geometry) is a good approximation of the basal drag. This is only correct if both lateral and longitudinal stresses are negligible at the glacier base, which is seldom the case when a glacier is sliding.

A fundamental problem with relations of the form (21) is that they allow for a physically unrealistic situation in which arbitrarily large shear stresses can be generated at the bed, regardless of the value of $N$ [Schoof, 2005]. Schoof [2005] thus proposed a phenomenological sliding rule in the form of a Coulomb friction law, which is implemented in our model:

$$\tau_b = C \left( \frac{u_b}{u_b + N w} \right)^{1/n} N, \quad \Lambda = \frac{\lambda_{max} A}{m_{max}}, \quad (22)$$

where $C$ is a constant, $\lambda_{max}$ a dimensional wavelength for the dominant bumps along the bedrock, and $m_{max}$ the maximum slope of those bumps. As above, the boundary condition for basal velocity depends on the basal velocity solution. Equation (22) is treated as a nonlinear Robin-type boundary condition, which because of its nonlinearity, forms part of an iterative solution procedure (see section 2.5). This friction law (equation (22)) is a more physically sound model of glacier sliding, as it predicts bounded basal drag, and accounts for cavitation, which can occur during sliding over both hard and soft beds. Before the onset of cavitation when velocities are low (and effective pressures high), $\tau_b$ is independent of effective pressure ($\tau_b \propto u_b^{1/n}$); as velocities rise (and effective pressures decrease) $\tau_b/N$ increases monotonically with $u_b/N$. $\tau_b/N$ approaches an asymptote when $u_b/N \sim \Lambda$ and there is significant cavitation.

The vertically resolved lateral velocity along the sidewalls, $u_L$, used in the lateral drag parameterization (16) is computed by application of a similar friction law,

$$\sigma_{xy} = C \left( \frac{u_B m_{max}}{u_B m_{max} + N w_{max} A} \right)^{1/n} N_L, \quad (23)$$

where $N_L = N_L(x, z) = P_s - P_g$ is the effective pressure along the sidewalls. Following Schoof [2006] the water pressure, $P_w$, decreases with height, reaching zero at the top of the water table, $z_t$:

$$P_w = \begin{cases} P_w^b - \rho_w g (z - b) & \text{if} \quad z < z_t \\ 0 & \text{if} \quad z \geq z_t \end{cases} \quad (24)$$

Note $N_L(b) = N = \max(N_c, P_i(b) - P_w^b)$ where $N_c \geq 0$ is some critical effective pressure, $P^b_w$ is the water pressure at the bed, and $z_t = b + (\rho_w g - N)/\rho_w$.

Importantly, the subglacial water pressure $P_w^b$ in equations (21)-(23) is usually unknown and therefore prescribed or parameterized [e.g., Pattyn, 1996]. However, the magnitude, spatial extent, and transient nature of basal sliding is strongly dependent on this variable, motivating the coupling of ice dynamics to an evolving subglacial drainage system. Here we link the sliding law to subglacial hydrology through the subglacial water pressure only and other aspects of hydrological coupling, such as the drowning of bedrock protrusions, are not considered [cf. Creyts and Schoof, 2009].

### 2.3. Subglacial Drainage

A one-dimensional version of the subglacial component of the hydrology model of Flowers and Clarke [2002] is utilized here, to model basal water-pressure explicitly. This is a diffusive construct that avoids specification of a particular system morphology. Considering conservation of mass in a subglacial “water sheet” (superscript “s”) with a local subglacial water thickness, $h^s$,

$$\frac{\partial h^s}{\partial t} + \frac{\partial Q^s}{\partial x} = Q_G + u_b \tau_b - b^s, \quad (25)$$

where $Q^s$ denotes the vertically integrated subglacial water flux, $Q_G$ is the geothermal flux, $L$ is the latent heat of fusion of ice, and $b^s$ an additional water source term (e.g., due to surface input). Adopting a nonlinear Darcian framework to describe the flux,

$$Q^s = -\frac{K h^s}{\rho_w g} \frac{\partial h^s}{\partial x}, \quad (26)$$

with fluid potential $\psi^s = P^s + \rho_w g b$, and the subglacial water pressure, $P^s_w$, related to sheet thickness $h^s$ by an empirical expression, $P^s_w = P_i(b) \left( \frac{h^s}{L} \right)^{1/2}$ [Flowers and Clarke, 2002], where $h^c$ is the critical water thickness, such that $P^s_w = P_i(b)$. A physical interpretation of $h^c$ is given in section 3.4.3. The effective hydraulic conduc-
tivity, $K$, can fluctuate in space and time as a function of $h^i = h^i(x, t)$ according to the following parameterization:

$$
\log(K) = \frac{1}{\pi} \left[ \log(K_{\text{max}}) - \log(K_{\text{min}}) \right] \arctan \left[ k_0 \left( \frac{K^i}{K^0} - k_0 \right) \right] + \frac{1}{2} \left[ \log(K_{\text{max}}) + \log(K_{\text{min}}) \right],
$$

(27)

where $k_0$ modulates the abruptness of the transition from $K_{\text{min}}$ to $K_{\text{max}}$ and $k_0$ determines its position. This is a means of capturing the dynamic evolution of drainage system efficiency, but is limited in that it does not truly describe “fast” or conduit-based drainage [e.g., Röthlisberger, 1972; Raymond et al., 1995].

2.4. Evolving Free Surface

[22] Although we will only consider fixed geometry in this paper, we present the model equations for surface evolution for the sake of completeness:

$$
\frac{\partial h}{\partial t} = -\frac{1}{W} \frac{\partial (\bar{n}hW)}{\partial x} + M,
$$

(28)

where $\bar{n} = \frac{1}{z} \int_0^z u \, dz$ is the depth-averaged horizontal velocity and $M$ is the surface mass balance rate. The evolution of the free surface is computed by converting (28) into a diffusion equation,

$$
\frac{\partial h}{\partial t} - \frac{\partial}{\partial x} \left( D \frac{\partial h}{\partial x} \right) = \frac{\partial}{\partial x} \left( \frac{D}{W} \frac{\partial h}{\partial x} \right) + \frac{\partial}{\partial x} \left( \frac{D}{W} \frac{\partial h}{\partial x} \right) + M,
$$

(29)

with diffusivity $D = -\bar{n}h \left( \frac{\partial h}{\partial x} \right)^{-1}$. This commonly adopted framework is advantageous for reasons of numerical stability [Hindmarsh and Payne, 1996], but is not applicable for domains with very flat surfaces (as $\frac{\partial h}{\partial x} \to 0$ then $D \to \infty$).

[23] In the case of a nonrectangular glacier channel, ice discharge given by $Q = \bar{n}hW$ in equation (28) is no longer justified; conservation of mass requires us to calculate the total discharge at some point $x$ along the flow line as follows:

$$
Q = 2 \int_{z=b}^{z=a} \int_{y=0}^{y=W(z)} u(y, z) \, dz \, dy,
$$

(30)

and the change in cross-sectional area $S$ by

$$
\frac{\partial S}{\partial t} = -\frac{\partial Q}{\partial x} + 2MW(z = s),
$$

(31)

where $S = 2 \int_0^s W(z) \, dz$, which depending on the channel shape, can be written as a function of $h$. We interpolate between the modeled centerline velocity $u_l(z)$ and the sidewall velocity $u_l(z)$ computed with the friction law to determine $u_l(y, z)$; for example, for an ice stream or “plug” flow profile:

$$
u(y, z) = \left( \frac{y}{W(z)} \right)^{n+1} u_l(z) + \left( 1 - \left( \frac{y}{W(z)} \right)^{n+1} \right) u(z).
$$

(32)

2.5. Solution Procedure

[24] The system of equations is solved on a fixed grid; in applications where we allow an evolving geometry (not considered in this paper) the grid is regenerated at each time step to coincide with the domain boundaries. The transformed equations under this coordinate framework are presented in Appendix A. The equations are discretized on a staggered grid that is irregular in the vertical and horizontal direction using second-order finite difference stencils. Due to the nonlinear nature of equation (19), a Picard iteration (successive substitution) of $\nu$ and $u$ is used to obtain convergence. A subspace iteration procedure is also employed which significantly reduces computation time (see section 3.2). For a stable numerical implementation of the basal boundary condition (equation (22)) we assume that $\dot{\varepsilon}_b$ is the only strain rate contributing to the basal drag:

$$
\frac{1}{A} \left( \frac{\partial Q}{\partial z} \right)^{k+1} - \frac{C^n N^n_{\text{max}}}{u^b_0 m_{\text{max}} + N^n_{\text{max}} A} u^{k+1}_b = 0,
$$

(33)

where $k$ is the Picard iterate. Equation (19) together with the basal boundary condition (equation (33)) and the surface boundary condition (equation (20)) form the matrix equation

$$
\mathbf{A}(u^*) \mathbf{u}^* = \mathbf{b}(u^*),
$$

(34)

where the solution is $u^*$, and where the matrix $\mathbf{A}$ and the vector $\mathbf{b}$ depend on the velocity from the previous iteration $u^i$. The initial estimation of $u$ at $k = 0$, i.e., $u_0$, is provided by the solution from the previous time step or at $t = 0$ by the SIA:

$$
u(z) = u(b) - 2(\rho g)^{\frac{1}{2}} \frac{\partial u}{\partial x} \int_b^z A(s - z)^n \, dz.
$$

(35)

The next iterate is calculated as

$$
u^{k+1} = \nu^k + \gamma (\nu^* - \nu^k);
$$

(36)

by letting $\gamma = 1$, we have $\nu^{k+1} = \nu^*$ (i.e., a Picard iteration). However, an adaptive underrelaxation scheme attempts to produce a swifter convergence:

$$
\gamma = \begin{cases} 
\gamma_{\text{min}} + (1 - \gamma_{\text{min}}) \exp \left[ -\alpha (\bar{\delta} - \bar{\varepsilon}) \right] & \text{for } \bar{\delta} > \bar{\varepsilon} \\
1 & \text{for } \bar{\delta} < \bar{\varepsilon},
\end{cases}
$$

(37)

where $\bar{\delta} = ||u^* - \nu^k||$, $\bar{\varepsilon}$ is tol, $\gamma_{k+1} = \gamma_k \bar{\varepsilon}$, and

$$
\alpha = \frac{-\ln \left( \frac{\bar{\delta} - \gamma_{\text{min}}}{1 - \gamma_{\text{min}}} \right)}{\bar{\delta} - \bar{\varepsilon}}.
$$

(38)

Such numerical schemes have been used in other disciplines such as in groundwater hydrology [e.g., Durbin and Delfneros, 2007].

[25] At each iterate $k$ the lateral sliding parameterization (equation (23)) is updated with its own iteration loop (successive substitution, $u_{l_{i+1}} = u_{l_i}$):

$$
\frac{1}{2A} \left( \frac{u^i_l - u_{l_{i+1}}^i}{W} \right) - \frac{C^n N^n_{\text{max}}}{u^i_l m_{\text{max}} + N^n_{\text{max}} A} u_{l_{i+1}}^i = 0.
$$

(39)
Our results for exercise E are plotted for the Little Ice Age geometry (Figure 1) under both no-slip and local free-slip boundary conditions. Results of these exercises show our model to fall within the standard deviation generated by other higher-order non-full-Stokes models (Table 2). In many instances the solution from our model remains close to the full Stokes solutions with the departure being greatest for high-aspect ratios [cf. Pattyn et al., 2008]. Our results for exercise E are plotted along with other model results in Figure 2.

### 3. Results and Discussion

[26] We conduct a number of numerical tests to demonstrate solutions and highlight the novel aspects of our model. We begin with intercomparison exercises (section 3.1) and then confirm the advantages of using the subspace solver (section 3.2), examine the lateral shear stress parameterization (section 3.3), and demonstrate the modeled interaction of basal sliding with spatial and temporal changes in water pressure (section 3.4). Reference constants and model parameters used in all simulations, unless otherwise stated, are provided in Table 1. In all simulations presented in this paper we use a fixed geometry and do not consider surface or terminus evolution.

#### 3.1. Model Intercomparison

[27] As a means of comparing our model results against those of similar models we have conducted exercises that form part of the Ice Sheet Model Intercomparison Project—Higher-Order Models (ISMIP-HOM) [Pattyn et al., 2008]. The model intercomparison consisted of five steady state diagnostic experiments (labeled A to E) and one time-dependent prognostic test (labeled F). These experiments examine stress and velocity solutions under a variety of glacier geometries and sliding conditions. Exercises B, D, and E can be applied to 2-D geometries and are thus considered here. Exercises B and D are conducted using various aspect ratios H/L (where H is the mean ice thickness and L the domain length scale), with B considering ice flow over a rippled bed with no basal slip, and D considering ice stream-type flow (with basal slip) over a flat bed with periodically varying basal traction. Exercise E adopts the Haut Glacier d’Arolla Little Ice Age geometry (Figure 1) under both no-slip and local free-slip boundary conditions. Results of these exercises show our model to fall within the standard deviation generated by other higher-order non-full-Stokes models (Table 2). In many instances the solution from our model remains close to the full Stokes solutions with the departure being greatest for high-aspect ratios [cf. Pattyn et al., 2008]. Our results for exercise E are plotted along with other model results in Figure 2.

#### 3.2. Subspace Iterations

[28] For many applications the use of a higher-order model is prohibitive, because of the additional computational requirements compared to a model that employs the SIA. We therefore focus some attention on computational efficiency by analyzing variations to the subspace solver (section 2.5). The ISMIP-HOM exercise E was performed using four different procedures.

1. Successive substitution (the standard Picard iteration), setting $\gamma = 1$ in (36).
2. The relaxation formula of Hindmarsh and Payne [1996]. This method was designed for solving the nonlinear diffusion equation for ice thickness (equation (29)) but has been used for the velocity field of higher-order models [e.g., Pattyn, 2002, 2003; De Smedt et al., 2010]. The method may apply a relaxation by calculating $\tilde{\gamma}$ (36), depending on the angle between successive correction

### Table 2. Result From the ISMIP-HOM Experiments

<table>
<thead>
<tr>
<th>Exercise</th>
<th>Mean Surface Velocity</th>
<th>Maximum Surface Velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Our Model</td>
<td>ISMIP Mean</td>
</tr>
<tr>
<td>B (L = 160 km)</td>
<td>41.66</td>
<td>41.38</td>
</tr>
<tr>
<td>B (L = 80 km)</td>
<td>40.04</td>
<td>39.76</td>
</tr>
<tr>
<td>B (L = 40 km)</td>
<td>35.91</td>
<td>35.55</td>
</tr>
<tr>
<td>B (L = 20 km)</td>
<td>28.36</td>
<td>27.80</td>
</tr>
<tr>
<td>B (L = 10 km)</td>
<td>19.77</td>
<td>18.24</td>
</tr>
<tr>
<td>B (L = 5 km)</td>
<td>10.85</td>
<td>10.54</td>
</tr>
<tr>
<td>D (L = 160 km)</td>
<td>57.43</td>
<td>58.30</td>
</tr>
<tr>
<td>D (L = 80 km)</td>
<td>44.40</td>
<td>38.46</td>
</tr>
<tr>
<td>D (L = 40 km)</td>
<td>25.43</td>
<td>24.20</td>
</tr>
<tr>
<td>D (L = 20 km)</td>
<td>18.01</td>
<td>18.33</td>
</tr>
<tr>
<td>D (L = 10 km)</td>
<td>16.08</td>
<td>16.28</td>
</tr>
<tr>
<td>D (L = 5 km)</td>
<td>10.69</td>
<td>12.85</td>
</tr>
<tr>
<td>E (no sliding)</td>
<td>31.30</td>
<td>32.00</td>
</tr>
<tr>
<td>E (sliding)</td>
<td>46.17</td>
<td>47.15</td>
</tr>
</tbody>
</table>

The ISMIP mean and ISMIP standard deviation (SD) refer to the ensemble of non-full-Stokes models that participated in ISMIP-HOM (values taken from Pattyn et al. [2008, Tables 4, 5, and 6]). Units are in meters per year. L denotes the length scale used in exercises B and D, and the mean ice thickness was set at 1 km giving aspect ratios in the range 0.006–0.2.
vectors [see Hindmarsh and Payne, 1996; Pattyn, 2002, 2003; De Smedt et al., 2010].

3. A constant underrelaxation, fixing \( \gamma = 0.8 \) in (36).

4. An adaptive underrelaxation, using (36), (37), and (38) with \( \gamma = 0.9 \).

The results for each of these methods clearly illustrate the substantive advantage of using an adaptive underrelaxation scheme in order to optimize the rate of convergence and thereby reduce the computation time (Table 3). For this application a variety of parameter sets produced effective convergence; however, for some applications trial and error may be needed to determine the parameter values that optimize convergence. The usefulness of the relaxation formula of Hindmarsh and Payne [1996] (method 2) for the velocity solution in higher-order models has also come under scrutiny by De Smedt et al. [2010], who instead recommend another Picard relaxation scheme. Further in-depth analysis comparing our own suggested method (method 4) with that of De Smedt et al. [2010] will be needed to establish further gains in efficiency.

3.3. Lateral Drag

Using the Haut Glacier d’Arolla profile (Figure 1) we compare the effects that a variety of glacier flow-band half-widths have on the computed lateral drag, and therefore on the flow line velocities. No sliding and a rectangular cross section are assumed such that \( u_L = 0 \) and \( \gamma \) remains constant with depth. Lateral drag introduces an intuitive reduction in flow line speeds with decreasing glacier width (Figure 3). In this particular example, the lateral drag is shown to have a discernible effect on the velocity even when the geometry is such that glacier width exceeds glacier length (i.e., with half-width \( W = 3000 \) m in Figure 3).

To explore the influence of lateral sliding on the parameterized lateral shear stress we permit lateral slip to occur. The basal speed, \( u_b \), and the wall speed, \( u_L \), are computed using the friction law (equations (22) and (23)) with a steady state water pressure distribution from the subglacial hydrology model (discussed below). We fix \( W = 1000 \) m and compare results with \( (u_L > 0) \) and without \( (u_L = 0) \) lateral sliding (Figure 4). Sliding at the sidewalls reduces the lateral velocity gradient and the magnitude of lateral shearing. Knowledge of the cross-sectional shape of the glacier can be specified by allowing \( W \) to vary with depth. More lateral drag is generated with an elliptical versus rectangular bed shape, because the valley cross-section narrows with depth in the former (Figure 4).

For the Haut Glacier d’Arolla case, allowing lateral sliding and assigning a nonrectangular valley shape have a smaller effect on the flow line speed than variations in the flow-band width (compare Figures 3 and 4). Thus, the additional time required to compute the lateral sliding speed may not be warranted for some situations. However, the inclusion of lateral resistance in the force-balance is clearly important for many glacier geometries (Figure 3) and its

![Figure 2. Surface speed and basal shear stress in the direction of ice flow for ISMIP-HOM exercise E (Haut Glacier d’Arolla) in the (top) no-sliding and (bottom) local free-slip cases. All ISMIP-HOM participating models are included [Pattyn et al., 2008].](image-url)
absence from flow line models will result in biases toward higher centerline velocities. Our calculation of valley sidewall friction is easy to implement in the force-balance equations, and follows a similar approach to the use of shape factors [Paterson, 1994] or the lateral drag component of a one-dimensional ice stream model [e.g., van der Veen and Whillans, 1996; Dupont, 2004].

[37] Taking account of basin shape and effective-pressure-dependent sliding along the sidewalls and the bed, we contrast the computed ice discharge of several idealized glacier basins in simulations analogous to those of Schoof [2006, Figure 3]. An infinitely long channel, uniform in the flow line direction, is modeled by using periodic boundary conditions in the upstream and downstream direction and a

Figure 3. Effect of glacier width on (a) vertically averaged horizontal flow line speed and (b) vertically averaged lateral drag for Haut Glacier d’Arolla profile.

Figure 4. Comparison of lateral sliding $u_L > 0$ (grey lines) with no lateral sliding $u_L = 0$ (black lines) and a rectangular bed shape (solid lines) with an elliptical bed shape (dashed lines). Influence on (left) vertically averaged horizontal flow line speed (zero lateral drag case shown as thick solid line for reference) and (right) vertically averaged lateral drag for Haut Glacier d’Arolla and $W = 1000$ m.
constant centerline ice thickness of 200 m. We use a triangular cross-sectional shape with a half-width \( W \) set at the surface to be 100 m, 200 m, and 400 m.

[38] Ice discharge (equation (30)) as a function of prescribed basal effective pressure for the various channel shapes and surface slopes is presented in Figure 5. Ice discharge is shown to be an increasing function of slope (steeper gradients increase the driving stress) and a decreasing function of effective pressure. For a given slope, as the effective pressure \( N \) is lowered, ice discharge begins to increase as a result of sliding below the water table elevation \( z_s \); however, on smaller slopes with sufficiently large \( N \) discharge remains unaffected by \( N \) (in these cases \( N \) is large enough to prevent the initiation of slip). For sufficiently steep and wide glaciers, the effective pressure cannot be lowered below a critical value \( N_c \).

3.4. Basal Sliding

[39] A distinguishing feature of our model is the ability to explicitly treat spatial and temporal variations in water pressure that are coupled to the ice dynamics by means of the friction law. The Haut Glacier d’Arolla Little Ice Age geometry (Figure 1) is used as a basis for examining modeled glacier response to water inputs, as well as to changes in key model parameters. We first highlight the distinction between shallow-ice and higher-order solutions when basal slip occurs.

3.4.1. Shallow-Ice Versus Higher-Order Model Results

[40] The Coulomb friction law (equation (22)) cannot be used in conjunction with the SIA (equation (35)). The friction law predicts bounded basal drag (see section 2.2 and Schoof [2005] and Gagliardini et al. [2007]), whereas basal drag is equated to the driving stress in a shallow-ice model. Equating basal drag and driving stress is possible with the more traditional sliding law (equation (21)), which allows unbounded basal shear stress, but not for the Coulomb friction law where the driving stress may exceed the maximum basal drag.

[41] Under the SIA, longitudinal stresses do not play a role in balancing the driving stress, and deviatoric normal stresses are omitted from the stress balance. To illustrate this point we compare velocity solutions under a local free-slip condition on the Haut Glacier d’Arolla profile (Figure 1) with the SIA and our higher-order model (Figure 6). Even without slip, the shallow-ice and higher-order solutions are substantially different (Figure 6a). With the introduction of a narrow zone of moderate slip (\( \tau_D = \beta^2 u_b \), with \( \beta = 1 \times 10^{11} \) Pa s m\(^{-1} \) for 2.2–2.5 km downglacier and \( u_b = 0 \) elsewhere) the surface speed is equal to the sum of the basal speed and the deformational speed for the shallow-ice case (Figure 6c); in the higher-order model, basal slip changes the mechanics of the flow over an area much larger than where slip is prescribed (Figure 6d). Furthermore, the driving stress is not equal to the basal shear stress and the higher-order case allows us to more properly use \( \tau_b = \beta^2 u_b \) which again changes the force-balance (Figure 6d) [see also Truffer et al., 2001]. Large amounts of slip have a leading-order effect on ice-flow mechanics that can be captured with a higher-order model, and motivates a more considered treatment of sliding within these models.

3.4.2. Response to Spatially Variable Water Pressure

[42] We first examine the basal friction law under a spatially variable steady state water pressure distribution. A constant water source rate, \( b^s \) in equation (25), is applied at the glacier bed and a steady state water pressure distribution is obtained by numerically integrating equation (25) until equilibrium is achieved. Throughout this section, lateral drag is calculated with \( u_s = 0 \) and \( W = 500 \) m.

[43] Initially, we set the water input rates to be homogeneous along the flow line (Figure 7a). Increasing the water input rate \( b^s \) from 0 to 0.3 m yr\(^{-1} \) results in a significant and nonlinear increase in the basal water pressure (Figure 7b). Note that setting \( b^s = 0 \) does not eliminate the subglacial water pressure, due to other source terms on the right hand side of equation (25). The link between \( b^s \) and \( P_w^s \) is governed by the transmissivity of the subglacial system \( K b^s \) which is determined, in part, by our choice of parameters (see section 3.4.3); for example, a higher conductivity \( K \) will give a lower basal water pressure \( P_w^s \) for a given source rate \( b^s \). Even with a spatially homogenous and temporally constant \( b^s \) the basal water pressure distribution deviates substantially from the pattern set by the ice overburden pressure.
For instance, a localized minimum in water pressure develops in response to a bump in the basal topography (Figure 1). Thus, a uniform flotation fraction would be a poor representation of the modeled basal water pressure distribution, especially for low water pressures.

The departure of modeled basal water pressure from the ice overburden pressure is further exposed when we adopt a more realistic, spatially varying water input rate. We explore three linearly increasing profiles of basal water input rate along the flow line that decrease to zero at prescribed surface elevations of 2800 m, 3000 m, and 3200 m (Figure 7c), representing surface water input at increasing values of the equilibrium line altitude. Subglacial water pressures now have a propensity toward higher values at the lower elevations of the glacier where basal water pressures are high. Although the water input gradient influences the pressure distribution, it is the integrated amount of water that is paramount in setting the overall magnitude of the water pressure. In Figures 7b and 7d, for the cases of greatest water input, the modeled basal water pressure exceeds the local ice overburden pressure over some fraction of the glacier bed. While basal water pressures in excess of overburden have been recorded in the field, such water pressures would not be sustained over a large area of the glacier bed without a significant dynamic response (e.g., a glacier surge). In most cases, such high water pressures would be reduced through a combination of hydraulic uplift of the ice [e.g., Jóhannesson, 2002], hydrofracturing, or the formation of an efficient drainage system involving channelized flow [e.g., Raymond et al., 1995]. Such processes have been neglected in this model but would need to be addressed in more physically realistic future developments.

We now contrast the velocity and stress responses to two water-pressure distributions (Figure 8). In the first instance we prescribe basal water pressure to be a fixed flotation fraction, homogenous along the flow line (55% of ice overburden in Figure 8a). We then apply a uniform water input rate to model the basal water pressure distribution (basal water pressure = 0.11 m yr\(^{-1}\) in Figure 8a). The mean subglacial water pressure along the flow line is identical in both cases. However, the ice dynamic response in our coupled system differs considerably. Glacier speed is similar for the two cases over the first two kilometers of the flow line, but deviates appreciably thereafter (Figure 8b). Higher modeled basal water pressure in the region 3 km to 4.5 km downglacier results in a reduction in basal shear stress (Figure 8c), and up to a doubling of lateral drag caused by the increase in shearing (Figure 8d). Inhomogeneities in the basal effective pressure (and flotation...
Figure 7. (b and d) Response of the modeled basal water pressure to (a and c) variations in water input rate. The thin lines in Figures 7b and 7d delineate profiles of 50% and 100% of the ice overburden pressure for reference.

Figure 8. (a) Basal water pressure prescribed as a fixed flotation fraction (thick line) and modeled by applying a constant water input rate (thin line). The response of (b) surface (dashed lines) and basal (solid lines) flow line speed, (c) basal shear stress (driving stress shown by dashed line for reference), and (d) vertically integrated lateral shear (solid lines) and longitudinal (dashed lines) stress to these two subglacial water pressure profiles (thick and thin lines, respectively).
fraction) clearly influence the stress and velocity solutions through the friction law, and hence the stress redistribution of the higher-order force-balance solution. The comparison in Figure 8 is intended to contrast the effects of a prescribed versus modeled basal water pressure profile. The modeled profile is applicable to a distributed drainage system such as might be in place at the onset of a spring speed-up event [e.g., Mair et al., 2003].

### 3.4.3. Sensitivity to Model Parameters

To illustrate the model sensitivity to key parameters that are taken as fixed in the simulations of sections 3.4.2 and 3.4.4, we independently vary hydrological parameters $K$ and $h_c^s$ (the hydraulic conductivity and the critical thickness of the subglacial water sheet, respectively) and friction-law parameters $m_{\text{max}}$ and $\lambda_{\text{max}}$ (the maximum slope and dominant wavelength of bedrock obstacles, respectively).

[47] For this set of experiments, steady state simulations employ a constant water source rate $b^s = 0.1 \text{ m yr}^{-1}$ (Figure 9). The critical sheet thickness $h_c^s$ and the hydraulic conductivity $K$ collectively control the transmissivity of the drainage system, with $h_c^s$ altering its capacity directly and $K$ the flow speed for a given fluid potential gradient.

[48] To simplify these tests we use a homogeneous hydraulic conductivity $K = K_{\text{max}} = K_{\text{min}}$. Larger values of $K$, such as might be more representative of a well-connected drainage system, effectively increase the throughput of the system resulting in lower water pressures (Figure 9a) and lower sliding velocities (Figure 9c).

[49] One physical interpretation of $h_c^s$ is as the areally averaged volume of a cavity system. Another interpretation is as the product of the undilated layer thickness of a macroporous horizon and its porosity. Considering the latter, $h_c^s$ can be adjusted to emulate typical hard- and soft-bedded conditions in the model; larger values would represent thick unconsolidated layers of high porosity with a higher drainage capacity, and thus lower water pressures (Figure 9b) and sliding velocities (Figure 9d) for a given water input rate. As $h_c^s$ decreases, hard-bedded conditions are emulated where high basal water pressures are achieved with less water volume; this conceptual model holds provided a channelized drainage system does not develop over the hard bed. In practice, the effective transmissivity of a real hydrological system could be determined, for example, by borehole slug tests [e.g., Stone et al., 1997; Meierbachtol et al., 2008].

[50] For a given effective pressure the flow line velocities can be regulated, to some extent, by the friction law parameters $m_{\text{max}}$ and $\lambda_{\text{max}}$. Decreasing $m_{\text{max}}$ (the maximum slope parameter) results in higher sliding speeds (Figure 9e).

---

**Figure 9.** Model sensitivity to hydrology and friction-law parameters. (a) Basal water pressure response to subglacial hydraulic conductivity: $K = 0.01 \text{ m s}^{-1}$ (solid line), $K = 0.0025 \text{ m s}^{-1}$ (dashed line), and $K = 0.001 \text{ m s}^{-1}$ (dotted line). (b) Basal water pressure response to critical water-sheet thickness: $h_c^s = 1 \text{ m}$ (solid line), $h_c^s = 0.15 \text{ m}$ (dashed line), and $h_c^s = 0.075 \text{ m}$ (dotted line). Thin lines in Figures 9a and 9b represent the ice overburden pressure. (c) Basal sliding speed corresponding to the water pressure profiles in Figure 9a. (d) Basal sliding speed corresponding to the water pressure profiles in Figure 9b. (e) Response of basal (thick lines) and surface (thin lines) speeds to friction law parameter $m_{\text{max}}$: $m_{\text{max}} = 1$ (dotted line), $m_{\text{max}} = 0.75$ (dashed line), and $m_{\text{max}} = 0.5$ (solid line). (f) Response of basal (thick lines) and surface (thin lines) speeds to friction law parameter $\lambda_{\text{max}}$: $\lambda_{\text{max}} = 0.5 \text{ m}$ (dotted line), $\lambda_{\text{max}} = 2 \text{ m}$ (dashed line), and $\lambda_{\text{max}} = 4 \text{ m}$ (solid line).
of bedrock obstacles) reduces friction and generates faster sliding velocities (Figure 9e), whereas decreasing $\lambda_{\text{max}}$ (dominant wavelength of bedrock obstacles) reduces flow line velocities because friction at the glacier bed is increased (Figure 9f). Measurements of real bedrock profiles, for example from the deglaciated forefield of a glacier, could potentially be used to provide estimates of friction law parameters for particular glaciers. While the detailed mapping data required, such as the measurements made by Hubbard et al. [2000], are rare and difficult to obtain, Gagliardini et al. [2007] propose a modeling strategy to determine friction law parameters at specific sites where bedrock roughness is known.

3.4.4. Response to Temporally Varying Water Input

Using the same fixed geometry, we explore modeled temporal variations in basal water pressure and the ice-flow regime in response to seasonally varying water input prescribed as follows:

$$\dot{b}^s = \dot{b}_0^s + \dot{\beta}_0^s \sin \left( \frac{2\pi t}{T} \right),$$  \hspace{1cm} (40)

where $t$ is time, $T = 0.5$ years, and $\dot{b}_0^s = 0.3$ m yr\(^{-1}\) (Figure 10a). This sinusoidal forcing is intended to mimic a seasonal cycle without replicating fine-scale details such as the diurnal variations characteristic of summer. Note that no surface or englacial drainage system is included here, so the water source is applied directly to the subglacial system. For simplicity of interpretation, this forcing is applied uniformly along the flow line. In order to allow some dynamic adjustment of the basal drainage system, we set $K_{\text{max}} = 10^{-2}$ m s\(^{-1}\) and $K_{\text{min}} = 10^{-3}$ m s\(^{-1}\) in this example (equation (27)). The modeled basal effective pressure is responsive to the changing water source rate (Figure 10b) and produces seasonal variation in modeled ice dynamics (Figures 10c and 10d). In the lower section of the glacier (3.3 km shown for illustration) seasonal cycles in the partitioning of the stress components are clearly discernible, with basal shear stress being reduced and lateral and longitudinal stresses increasing during the peak melt period (Figure 10c). A 100% seasonal increase in the sliding speed of the glacier is achieved at 3.3 km downglacier (Figure 10d) for these conditions. Observations linking basal water pressure to enhanced basal sliding during summer peak meltwater production have been reported extensively [e.g., Bindschadler, 1983] and our numerical model does a reasonable job at qualitatively replicating this behavior.

While the model captures key features of a seasonal transition, its simplified representation of drainage system evolution presents a fundamental limitation. The current implementation of the hydrology model accommodates changes in drainage system transmissivity only through changes in conductivity, rather than in system capacity or volume. Furthermore, the model system is fundamentally diffusive and does not capture the essential physics of a conduit-based drainage system typical of temperate alpine glaciers during the melt season. Such a system would provide the mechanism for large changes in effective trans-

---

**Figure 10.** (a) Prescribed rate of water input $\dot{b}^s$; (b) position-time diagram of effective pressure at the bed; (c) seasonal evolution of basal shear stress (dotted line), vertically integrated lateral shear stress (dashed line), and basal longitudinal stress (solid line) at 3.3 km along the flow line, and (d) seasonal evolution of the surface (dashed line) and basal (solid line) speed at 3.3 km along the flow line.
missivity and allow for a more cogent representation of seasonal variability, including the possibility of strong diurnal fluctuations in water pressure. In addition to lacking conduits, the model neglects creep closure of the roof of the subglacial drainage system which limits our ability to capture the high water pressures characteristic of winter. This is a result of the hydrology–ice-dynamics feedback loop remaining open in our model: while hydrology affects ice dynamics through the basal friction law, ice dynamics only affects hydrology indirectly through changes in the ice pressure and basal meltwater production. Although we cannot reproduce the seasonal cycle in basal water pressure accurately, this does not detract entirely from our ability to represent the influence of hydrology on glacier dynamics; observations attest to low winter sliding speeds (glacier surges excepted) due to the low volume of water at the bed, despite the high basal water pressures recorded [e.g., Raymond et al., 1995; Fountain and Walder, 1998].

Implementation of coexisting channelized and distributed drainage systems [e.g., Flowers, 2008] is currently being pursued in this higher-order model [Pimentel and Flowers, 2010]. Future work should address creep closure of the distributed system [Crevets and Schoof, 2009], among other processes, toward a more complete description of coupling between hydrology and dynamics.

4. Summary and Conclusions

Higher-order models are needed in regions of transitional flow where all components of the stress tensor become important, especially when basal sliding is significant. Sliding velocity and basal drag depend critically on basal water pressure. Coupling between ice dynamics and subglacial hydrology is an essential feature of the glacier environment that is neglected in most numerical models. The model presented here is designed to simulate realistic basal boundary conditions capable of hydrologically-driven acceleration. Higher-order models have previously been constructed with evolving geometry and thermomechanical coupling; the principal contribution of this study is the coupling of dynamic subglacial water pressure through a physically realistic basal friction law.

The Little Ice Age geometry of Glacier Haut d’Arolla is used to test the response of the model to a range of spatial and temporal forcings, as well as model sensitivity to key hydrologic and friction-law parameters. An adaptive underrelaxation subspace solver for the nonlinear problem is shown to reduce computation time. Another novel feature of our model is a new lateral drag parameterization, designed for inclusion in the force-balance of flow line models. We illustrate the influence of valley wall friction effects on computed glacier velocities. A Coulomb friction law describing glacier sliding with cavitation is demonstrated within the higher-order model framework. This friction law differs from standard treatments of sliding as it exhibits bounded basal drag. Spatial and temporal evolution of subglacial water pressure is shown to regulate basal sliding speed through the Coulomb friction sliding law. The mechanism of increased water input to the bed increasing basal water pressures and resulting in enhanced basal sliding is clearly manifested in our model, with spatially variable water pressures that would be difficult to diagnose a priori. While seasonal variations in flow speed and stress balance in response to meltwater input to the subglacial system are qualitatively captured, the model lacks a physically correct description of conduit-based (fast) drainage, limiting our ability to replicate details of the seasonal cycle. Fast drainage of water through a channelized subglacial network has a strong theoretical and observational basis [Rothlisberger, 1972; Raymond et al., 1995], and future deployments of the model will be augmented with equations [Flowers, 2008] to describe this type of flow [Pimentel and Flowers, 2010].

Appendix A: Coordinate Transform

We apply the following coordinate transformation which allows for coordinate stretching at moving boundaries,

\[(x, z, t) \rightarrow (\xi, \zeta, \tau) = \left( \frac{x}{x_1} \frac{s - z}{h}, t \right). \tag{A1} \]

In the vertical dimension, following Jensen [1977] and many others, equation (A1) maps the local ice thickness onto unity such that \( \zeta = 0 \) at the surface and \( \zeta = 1 \) at the base. Similarly, in the horizontal, we allow the grid to follow the evolution of the terminus position, \( x_1 \), by mapping this position onto unity [Hindmarsh, 1996]. Coordinate stretching in the horizontal has been shown to be superior to a fixed grid approach in simulating grounding line migration [Goldberg et al., 2009]. These transforms are desirable numerically as all ice boundaries now coincide with grid points. The functional derivatives transform as follows

\[
\frac{\partial f}{\partial x} = \frac{\partial f}{\partial \xi} + \frac{\partial f}{\partial \zeta} + \frac{\partial f}{\partial \tau} = \frac{1}{x_1} \left( \frac{\partial f}{\partial \xi} + \alpha \frac{\partial f}{\partial \zeta} \right), \tag{A2} \]

\[
\frac{\partial f}{\partial z} = \frac{\partial f}{\partial \xi} + \frac{\partial f}{\partial \zeta} + \frac{\partial f}{\partial t} = -\frac{1}{h} \frac{\partial f}{\partial \zeta}, \tag{A3} \]

\[
\frac{\partial^2 f}{\partial x^2} = \frac{1}{x_1} \left( \frac{\partial^2 f}{\partial \zeta^2} + \beta \frac{\partial f}{\partial \zeta} + \alpha^2 \frac{\partial^2 f}{\partial \xi^2} + 2 \alpha \frac{\partial^2 f}{\partial \xi \partial \zeta} \right), \tag{A4} \]

\[
\frac{\partial^2 f}{\partial \zeta^2} = \frac{1}{h^2} \frac{\partial^2 f}{\partial \zeta^2}, \tag{A5} \]

\[
\frac{\partial f}{\partial \tau} = \frac{\partial f}{\partial \tau} + \left( \gamma - \frac{\xi \partial x_T}{x_1} \frac{\partial f}{\partial \zeta} - \frac{\xi}{x_1} \frac{\partial f}{\partial \zeta} \right), \tag{A6} \]

where

\[
\alpha = \frac{1}{h} \left( \frac{\partial s}{\partial \zeta} - \zeta \frac{\partial h}{\partial \zeta} \right), \tag{A7} \]

\[
\beta = \frac{\partial \alpha}{\partial \zeta} + \alpha \frac{\partial s}{\partial \zeta} = \frac{1}{h} \left( \frac{\partial^2 s}{\partial \zeta^2} - \zeta \frac{\partial^2 h}{\partial \zeta^2} - 2 \alpha \frac{\partial h}{\partial \zeta} \right), \tag{A8} \]

\[
\gamma = \frac{1}{h} \left( \frac{\partial s}{\partial \tau} - \zeta \frac{\partial h}{\partial \tau} \right). \tag{A9} \]
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