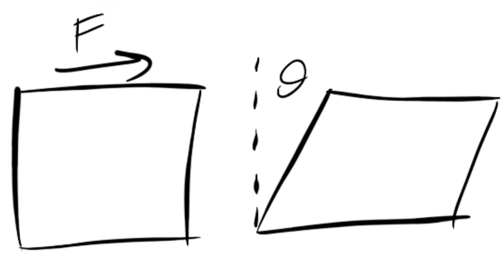
1 Some Basics

1.1 What is a fluid?

It is surprisingly difficult to be rigorous about what constitutes a fluid, but fortunately for the materials we are interested in – air & water – the definition is fairly straightforward.

Working definition of a fluid: A material which suffers an unbounded displacement when acted upon by a small force. This is in contrast to a perfectly elastic solid, which is a material which suffers a proportionally small and reversible displacement when acted upon by a small force.



If the solid is elastic, the removal of the force leads to a restoration of the original situation. In a fluid, the rate of increase of θ (rate of deformation) is proportional to the force applied. For a fluid, tangential forces lead to continuous deformations whose rate is proportional to the applied force.

Useful definition of a fluid:

A material that cannot remain motionless under the action of forces that leave its volume unchanged but otherwise act to deform it.

One of the principal challenges for fluid dynamics is to obtain a useful mathematical representation of the relation between surface forces acting on pieces of fluid and the resulting deformations.

However, there exists a bit of a grey area. It is sometimes unclear whether a material is always either a fluid or a solid. Some materials can act as elastic solids when they are forced on short timescales – but deform like a liquid when the force operates of a long time (A material's reaction to force is not always the same as its reaction to *impulse*). Silly putty and Earth's mantle both fit these criteria.

1.2 The Continuum Hypothesis

One of the pleasures of watching a moving fluid lies in the sinuous character of its motion. e.g.

- The continuity of water in a waterfall
- The slow rise of smoke that becomes more complex but remains continuous as it rises
- The rippling of the sea surface

We intuitively think of a fluid as a continuum even though we are aware that the material is composed of atoms with large open spaces between. However, we are able to validate the continuum perspective by the fact that:

$$\text{Macroscopic scale of interest} \gg \text{inter-atomic distances}$$

Typically, we will be concerned with macroscopic scales from fractions of centimeters to thousands of kilometers, but inter-molecular distances in air are on the order of nanometers (and less in water). Even on scales of $\mathcal{O}(10^{-3}\text{cm})$, a volume $\mathcal{O}(10^{-9}\text{cm}^3)$ would contain $\mathcal{O}(10^{10})$ air molecules in the atmosphere, which is more than enough to establish a thermodynamically stable definition of temperature.

We can therefore average over scales that are large compared to inter-atomic distances but still small compared to the scales of interest. This allows us to define a continuum with properties that vary continuously over out scales of interest. This allows us to consider fields like the temperature field as continuous functions of space and time, $T = T(\vec{x}, t)$ (instead of a discrete assignment of temperature to each molecule within the fluid). Here, the arrow over x indicates that $\vec{x}$ is a vector quantity defining position in three dimensions, $\vec{x} = (x, y, z)$. More on this later!

Some implications of this assumption:

1. This allows us to assume that such functions are smooth (of class C^∞), which implies that they are continuously differentiable to an arbitrarily high order.
2. If there are discontinuities in our description of a fluid, we consider them to be idealizations and simplifications of rapid variations in the continuum (occurring on scales $\ll$ scale of interest, but still very large relative to

the molecular scale).

3. It is a significant oversimplification to ignore the atomic nature of matter and treat the fluid and a continuous medium. Thus, certain fluid properties (viscosity, thermal conductivity, etc.) cannot be predicted by theory, but must instead be specified as given properties of the fluid. These can be complicated thermodynamic functions of temperature and pressure, and their behaviour must also be specified external to the theory.

1.3 The Fluid Element

In spite of our assertion of the continuum hypothesis, we will often speak of a *fluid element* or *particle*. This refers to an arbitrary and arbitrarily small "piece" of the continuum. We conceptually (although not physically) isolate this "piece" of fluid from its surroundings to facilitate in our understanding of how the fluid behaves.

We assume that:

1. Its volume is so small that its properties are uniform
2. It has a fixed mass (usually)
3. It moves under the influence of the surrounding fluid

This concept is useful in practice because any application of Newton's laws of motion are most easily written in terms of force balances on a "particle" of fixed mass (although strictly it is not necessary to formulate it in this way). This allows us to picture the dynamics of an isolated element which is often an intuitive way to consider the nature of fluid phenomena. Like all simplificiations and conceptualizations, we have to be prudent with its usage.

1.4 Kinematics

Before discussing the dynamics of fluids, we must first consider and agree upon how to describe a fluid's motion (hence, kinematics!)

For a small solid object like a stone, the description is usually quite simple. We can describe the trajectory (of the centre of mass) as a function $\vec{x} = \vec{x}(t)$ in three spatial dimensions and relate the forces to the velocity and acceleration determined from our knowledge of $\vec{x}(t)$.

For a fluid, it is not so simple. A fluid is "everywhere" (Or at least extends over some finite 3-dimensional space). We are no longer interested in the motion of the centre of gravity, but instead want to describe the moving continuum composed of infinitude of fluid elements. In general, there are two methods of description that are useful and illuminating: The Eulerian method and the Lagrangian method (In practice, both are from Euler, but the latter was exploited by Lagrange).

1.4.1 The Eulerian Description of Fluid Motion

Consider a fluid that extends over some 3-dimensional space and exists in time. We can describe a fluid property P as a function of a position vector $\vec{x} = (x_1, x_2, x_3)$ ($\vec{x} = (x, y, z)$ in Cartesian coordinates) and time, $P = P(\vec{x}, t)$. This gives us a description of fluid properties at each location and each time without specifying **which** fluid element occupies a given position at a given time. In this description, we do not keep track of individual fluid elements. Instead, it provides a **field description of the fluid**. This can be a key advantage, as we need not concern ourselves with the fate of particular fluid elements.

Fixed anemometers (wind meters) or ocean current meters measure fluid flows in a Eulerian context. Check out `nvs.nanoos.org` and select "Fixed Platforms" for examples of local Eulerian timeseries!

1.4.2 The Lagrangian Description of Fluid Motion

The Lagrangian approach keeps track of indiviual fluid elements as they move and describes fluid properties (P) as a function of which particle it refers to and at each time. In this description, each particle is a given a name and label which serves to identify the fluid particle at all subsequent times. A convenient and commonly used label is the position $\vec{x} = (x_1, x_2, x_3)$ that the particle had at some "initial" time $t = t_0$. We'll call that position $\vec{X}$ so that $\vec{X} = \vec{x}$ at $t = t_0$.

It is common to describe a fluid element's trajectory when working in the Lagrangian frame. A simple way to do this is to specify:

$$\vec{x} = \vec{x}(t, \vec{X}_j) \tag{1.1}$$

Where $\vec{x}(0, \vec{X}_j) = \vec{X}$ where j refers to a label and not an index of a vector. You could read this equation as *A description of the 3D position of a particle with label X_j as a function of time.* This is the trajectory equation for a particular element that tells us where, at time t, the fluid element that initially (at $t = 0$) was at position $\vec{X}$ is now located.

For all $t > 0$, there is an inverse to this relation:

$$\vec{X} = \vec{X}(t, \vec{x}) \tag{1.2}$$

This tells us which particle is at position $\vec{x}$ at time t. We can equally describe any fluid property P in terms of Lagration variables, $P = P(\vec{X}, t)$. Using equations 1.1 and 1.2 we can go back and forth between Eulerian and Lagrangian descriptions of P.

Floats (ocean observing tool) and radiosondes (atmospheric observing tool) are approximately Lagrangian measuring tools. Note that in real-world observational situations, the description of any property field is usually incompletely sampled because there are typically a limited number of floats available and current meters are sparsely distributed. This means that in practice, it is often challenging to switch between kinematic descriptions with observational data.

See `www.argo.ucsd.edu` and `www.aoml.noaa.gov/phod/dac` for cool ocean examples of Lagrangian measurements!

1.5 Rates of change

Consider a property $P(\vec{x}, t)$ in the Eulerian description. Its rate of change with respect to time is central to the description of the fluid's dynamics. However, there are several different natural rates of change and we need to distinguish carefully between them. For example, if you are travelling from Vancouver to Vancouver Island, you may detect a change in the temperature as you go along. The change may occur because the temperature is changing w.r.t. time at each location **OR** because the temperature is changing in space and you are travelling along a temperature gradient. We provide three different notions of rate of change:

1. **Local time derivative (rate of change at a fixed point)**

   We define the rate of change (w.r.t. time) at a fixed point as:

   $$\text{Local Time Derivative} = \left.\frac{\partial P}{\partial t}\right|_{\vec{x}} \tag{1.3}$$

   We usually suppress the subscript $\vec{x}$ as it is typically implied. You can visualize this as the change in P as a series of fluid elements move through the point $\vec{x}$. It is therefore possible for $\left.\frac{\partial P}{\partial t}\right|_{\vec{x}} \neq 0$ even if the property P

is constant for each fluid element along its trajectory.

2. **Rate of change for a moving observer**

Now consider the rate of change of P seen by a moving observer (perhaps on a moving ferry between Vancouver and Vancouver Island). The observer moves with a velocity $\vec{v} = (v_1, v_2, v_3)$ (**Not** the same as the velocity of the fluid) such that for the observer, $\vec{v} = \frac{\mathrm{d}\vec{x}}{\mathrm{d}t}$. Thus, the total rate of change of P as seen by the observer is:

$$\begin{aligned}\frac{\mathrm{d}P}{\mathrm{d}t} &= \frac{\partial P}{\partial t} + \sum_{i=1}^{3} \frac{\partial P}{\partial x_i}\frac{\mathrm{d}x_1}{\mathrm{d}t} \\ &= \underbrace{\frac{\partial P}{\partial t}}_{\text{Local time derivative}} + \underbrace{\vec{v}\cdot\nabla P}_{\substack{\text{Rate of change of P} \\ \text{seen by the observer} \\ \text{due to the observer's motion} \\ \text{in the gradient of } P}}\end{aligned} \tag{1.4}$$

Aside: We will be using Einstein summation convention to simplify our sums over repeated indices. Instead of writing, say, $\sum_{i=1}^{3} \frac{\partial P}{\partial x_i}$, we drop the summation sign and it becomes an implicit sum over our repeated indices i (this is made feasible since we are only working in three spatial dimensions, so there is no ambiguity over how many indices to use in our sum). This greatly simplifies the amount we have to write. For example,

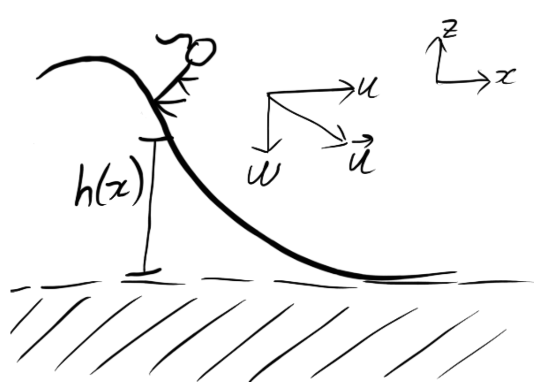
$$\frac{\partial P}{\partial x_i} = \frac{\partial P}{\partial x_1} + \frac{\partial P}{\partial x_2} + \frac{\partial P}{\partial x_3}$$

This can (and will!) be used with multiple indices as well. For example,

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_2} + \frac{\partial u_1}{\partial x_3} + \frac{\partial u_2}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_1} + \frac{\partial u_3}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$$

Since we have to implicitly sum over indices i **and** j.

Example 1.1. The rate at which a child on a slide changes his/her elevation above the ground:



$$
\begin{aligned}
\frac{Dh}{Dt} &= \frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} + w\frac{\partial h}{\partial z} \\
&= 0 + u\frac{\partial h}{\partial x} + 0 + 0
\end{aligned}
\tag{1.5}
$$

3. **Rate of change of a fluid property following a fluid element**

   Of particular interest is the rate of change of a fluid property when the observer is the fluid itself (The rate of change of a fluid property following a fluid element is simply the rate of change of the property for the fluid element).

$$
\frac{\mathrm{d}x_i}{\mathrm{d}t} = u_i(x_i, t) \tag{1.6}
$$

   (Note that an integration in time of this equation yields the trajectory of the fluid element). Then the rate of change of a proprty P following each fluid element is:

$$
\begin{aligned}
\frac{DP}{Dt} &= \frac{\partial P}{\partial t} + u_i\frac{\partial P}{\partial x_i} \\
&= \frac{\partial P}{\partial t} + \vec{u}\cdot\nabla P
\end{aligned}
\tag{1.7}
$$

   This is identical to equation 1.4, except now the observer is moving at velocity $\vec{u}$ along with the fluid instead of a different velocity $\vec{v}$. This is the Eulerian representation of the derivative, and is often called the **total derivative** or material derivative, denoted with a capital D (some sources use d, but will will try to use $\frac{D}{Dt}$ throughout). The total derivative consists of two parts:

$$
\frac{DP}{Dt} = \underbrace{\frac{\partial P}{\partial t}}_{\substack{\text{Local time derivative} \\ \text{(rate of change w.r.t time} \\ \text{of the property at a fixed point)}}} + \underbrace{\vec{u}\cdot\nabla P}_{\substack{\text{Advective time derivative} \\ \text{(Rate of change `seen' at a fixed point} \\ \text{due to the motion of the} \\ \text{fluid in the gradient of } P)}} \tag{1.8}
$$

   Note that we need to know the fluid velocity as well as the space and time behaviour of P to calculate the total derivative of P. If both P and u_i are unknown (as they usually are in the atmosphere and ocean – at least on small-ish scales) the total derivative involves the product of two unknowns, which leads to a non-linearity when solving these equations. The fundamental non-linearity of fluid mechanics and all of the associated phenomena like turbulence, weather, eddies, chaos, etc. arise from this term!

Example 1.2. Consider a wave-like fluid property distribution given by:

$$
P = P_0 \cos\omega t \sin kx \tag{1.9}
$$

where ω is the wave frequency ($\omega = 2\pi/T$, where T is the wave period) and k is the zonal (x-component) wavenumber ($k = 2\pi/\lambda_x$, where λ_x is the zonal wavelength). Suppose the fluid velocity is in the x-direction only, with magnitude

U. The total derivative of P in the Eulerian framework is:

$$\begin{aligned}\frac{DP}{Dt} &= \frac{\partial P}{\partial t} + \vec{u}\cdot\nabla P \\ &= -P_0\omega\sin(\omega t)\sin(kx) + ukP_0\cos(\omega t)\cos(kx)\omega \quad \text{Under the guidance of trig identities} \\ &= \frac{P_0}{2}\left[(Uk+\omega)\cos(kx+\omega t) + (Uk-\omega)\cos(kx-\omega t)\right]\end{aligned}$$

Often, we want to examine the relative order of terms in equations to establish dominant balances. In this example, we can find the ratio of the local portion of the time derivative to the advective part by:

$$\begin{aligned}\frac{\frac{\partial P}{\partial t}}{\vec{u}\cdot\nabla P} &= \mathcal{O}\left(\frac{\omega}{Uk}\right) \\ &= \mathcal{O}\left(\frac{c}{U}\right)\end{aligned}$$

Where $c = \omega/k$ is the phase speed of the wave. This ratio (describing the relative important of the local time derivative to the advective derivative) is given by the size of the phase speed of the wave relative to the phase speed of the fluid.

If $\frac{c}{U} \gg 1$: The advective derivative can be ignored to lowest order, implying that the system will be (roughly) linear

If $\frac{c}{U} \sim \mathcal{O}(1)$ (or smaller): The advective derivative is leading order, and the system will be non-linear.

Note: If the Eulerian velocity field is given, the trajectories of each fluid element can be found by integrating $\frac{\mathrm{d}x_i}{\mathrm{d}t} = u_i(x_i, t)$ with $x_i(0) = X_i$ as the initial condition. Even for simple velocity fields $\frac{\mathrm{d}x_i}{\mathrm{d}t} = u_i(x_i, t)$ is usually highly non-linear, and if the motion is time-dependent, the resulting trajectories can be unexpectedly complex and even chaotic.

1.6 Streamlines and Streaklines

A streamline is a line in the fluid that, at each instant, is parallel to the velocity.

If $d\vec{x}$ is a vector element along the streamline,

$$d\vec{x} = \lambda\vec{u} \tag{1.10}$$

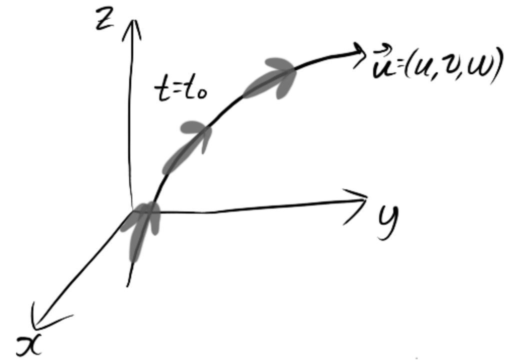
Where lambda is an arbitrary constant. In component form, it follows that

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} = \lambda \tag{1.11}$$

By dividing this equations, we can produce two differential equations:

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{w}{u}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{v}{u} \tag{1.12}$$

When provided with initial conditions, this yields a curve in space. This is the streamline.



The velocity vector is always tangent to this curve! The streamlines evolve in time as the velocity vector field changes.

An equivalent representation can be used in which the arbitrary scalar λ is replaced by the differential parameter ds which yields 3 ordinary differential equations (and alleviated the apparent singularity when $u = 0$):

$$\frac{\mathrm{d}x}{\mathrm{d}s} = u$$
$$\frac{\mathrm{d}y}{\mathrm{d}s} = v \tag{1.13}$$
$$\frac{\mathrm{d}z}{\mathrm{d}s} = w$$

Where s is the distance parameter along the streamline. Regardless of the chosen representation, the streamline is constructed at an arbitrary fixed time and will change its geometry if the velocity field is time-dependent. It provides a direction a fluid element would move **at that instant in time** if the velocity field were steady. You can view this as a highway for the flow provided a steady velocity field. If the flow is not steady, streamlines give a picture of the direction of the flow at a given instant in time but not the trajectory of path of the fluid element along it (the highway can keep changing direction with time so the path of the particle will diverge from the streamline.

Streamlines and fluid particle trajectories are different if the velocity field is time-dependent

Example 1.3. Consider a 2D flow ($w = 0$) for which:

$$u = U_0$$
$$v = V_0 \cos(k(x - ct)) \tag{1.14}$$

With constant U_0 and V_0. The streamline passing through the point $(x, y) = (X, 0)$ at $t = 0$ is determined from the streamline equation 1.12.

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{v}{u} \\
&= \frac{V_0}{U_0}\cos(k(x-ct)) \quad \text{Integrating w.r.t. } x \text{ to define streamlines as } y = fn(x) \\
y &= \frac{V_0}{kU_0}(\sin(k(x-ct)) - \sin(kX))
\end{aligned}$$

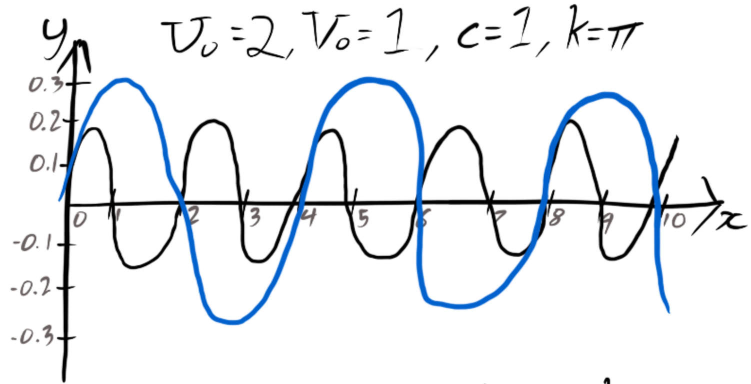
The fluid element trajectory is found by integrating the Lagrangian trajectory Equation 1.6.

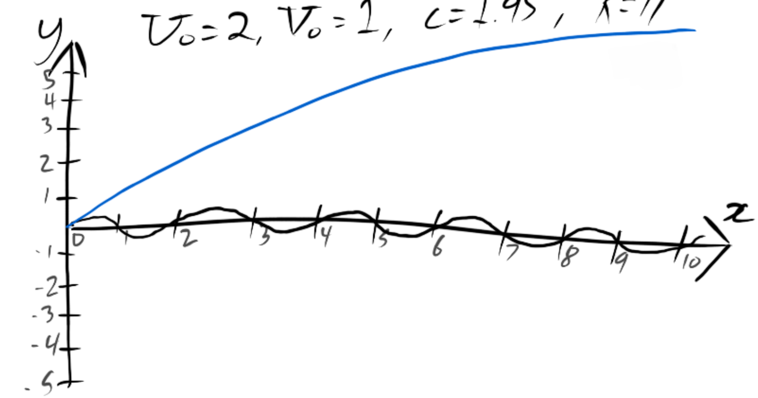
$$\begin{aligned}
\frac{\mathrm{d}x_i}{\mathrm{d}t} &= u_i(x_i, t) \\
\frac{\mathrm{d}x}{\mathrm{d}t} &= U_0 \quad \text{Integrating w.r.t. } x \text{ to define streamlines as } y = fn(x) \\
x(t) &= U_0 t + X \\
\frac{\mathrm{d}y}{\mathrm{d}t} &= V_0\cos(k(x-ct)) \quad \text{Substitute } x(t) \text{ from above so } y' = fn(t) \\
&= V_0\cos(k(U_0 - c)t + kX) \quad \text{Integrating w.r.t. } t \\
y(t) = y(t) &= \frac{V_0}{k(U_0-c)}\left[\sin\left(k(U_0-c)\frac{x-X}{U_0} + kX\right) - \sin(kX)\right]
\end{aligned}$$

What does this look like? The black curve is the streamline at $t = 0$ that passes through the same initial point. The blue curve is the trajectory for the particle that is at $(x, y) = (0, 0)$ at $t = 0$. Both the wavelength of the trajectory and its amplitude are different. These differences become more extreme as $U_0 - c \to 0$.

How do we make sense of these results?

- As c approaches U_0, the fluid element – which moves in x with velocity U_0 – remains fixed to the same phase of the wave moving with phase speed c. It therefore always sees the same value of v.
- This leads to an increasing divergence of the trajectory from the streamline.
- In the limit $c \to U_0$, the trajectory becomes a straight line whose slope y/x becomes V_0/U_0 (See if you can show this!)



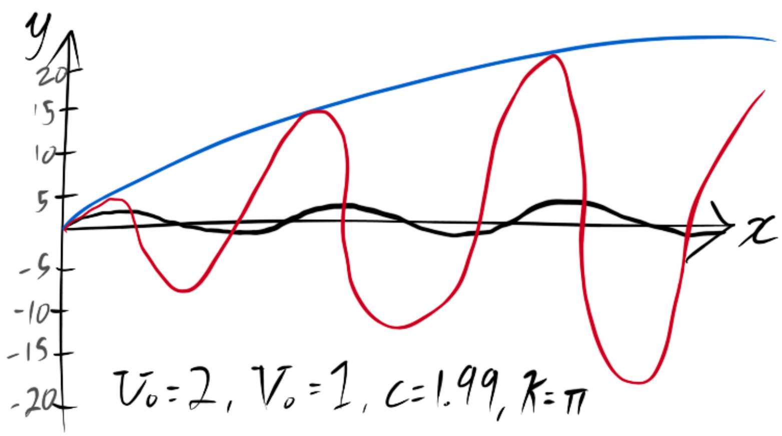


- This has implications for the motion of air and water particles in the generally time-dependent, turbulent atmosphere and oceans. Very orderly streamlines, usually coincident with orderly isobars (lines of constant pressure) in the atmosphere and isopycnals (lines of constant density) in the ocean, often mask significantly more extreme particle transports over greater distances than is evident from the streamline pattern.

A streakline is the line traced out in time by all fluid particles that pass a particular point as time elapses. Steaklines can be viewed as the line of dye released from a particular position as observed at some later time downstream.

Example 1.4. (For you!) Calculate the streakline for the flow given by Equation 1.14 that passes through $(x, y) = (0, 0)$. If you select $U_0 = 2, V_0 = 1, c = 1.99, k = \pi$, you should get the following (the red curve is the streakline):

It looks like there is some kind of instability, but the generating velocity dismisses this idea. Can you explain what is going on? (Note: all three curves coincide if $c = 0$)


y
20
15
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x
U₀=2, V₀=1, c=1.99, k=π