2 The Continuum Equations

2.1 The conservation of mass

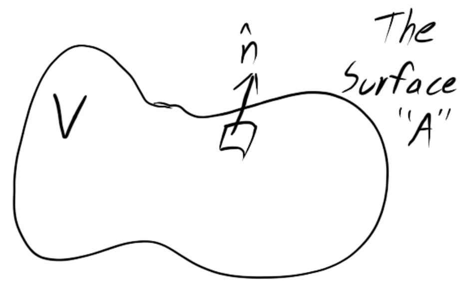
In solid mechanics, a mass of interest is usually trivial to define. For example, if looking at simple orbital motion, the mass of the moon is fairly unambiguous .

This is not so when describing a continuum!

First, we must consider how to write a statement of mass conservation for a continuous fluid that is analogous to how we treat mass conservation in the description of the motion of a single solid body.

Outline of a derivation of conservation of mass:

- Consider a fixed, closed, imaginary (arbitrary) surface A drawn in the fluid. It encloses a fixed volume V (called the *control volume*), whose outward normal at each point on the surface is the unit vector $\hat{n} = (\hat{n}_1, \hat{n}_2, \hat{n}_3)$.



- The surface of our arbitrary volume is infinitely permeable. That is, fluid flows through the surface with the fluid velocity at the interface.

- At each time t, the mass enclosed in V by the surface A is:

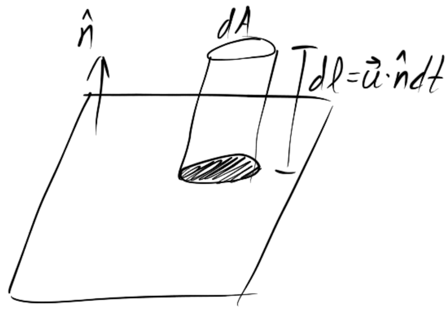
$$M(t) = \int_V \rho \, \mathrm{d}V \tag{2.1}$$

  Where $\int_V$ is an integral over the volume of V and ρ is the fluid density – generally a time-dependent function of position within V ($\rho = \rho(\vec{x}, t)$).

- The mass of a fluid flowing out of V across its boundary A (the **mass flux**) is:

$$\text{Mass flux} = \int_A \rho \vec{u} \cdot \hat{n} \, \mathrm{d}A \tag{2.2}$$

This is illustrated by a *pillbox* crossing a surface element of A in a time $\mathrm{d}t$.



- In each interval of time $\mathrm{d}t$, a small pillbox of mass whose cross-sectional area is $\mathrm{d}A$ and whose height $\mathrm{d}l = \vec{u} \cdot \hat{n}\ \mathrm{d}t$ yields a volume $\vec{u} \cdot \hat{n}\ \mathrm{d}t\ \mathrm{d}A$ leaving V in $\mathrm{d}t$.

- Equation 2.2 gives the rate (volume per unit time) at which the fluid mass leaves the volume.

- To conserve mass, the rate of change of the mass in the fixed volume must be equal to the mass entering or leaving:

$$\frac{\partial}{\partial t} \int_V \rho\ \mathrm{d}V = -\int_A \rho \vec{u} \cdot \hat{n}\ \mathrm{d}A \tag{2.3}$$

Since the volume is fixed in space, V has no dependence on time:

$$\int_V \frac{\partial \rho}{\partial t}\ \mathrm{d}V = -\int_A \rho \vec{u} \cdot \hat{n}\ \mathrm{d}A \tag{2.4}$$

The divergence theory states that for any well behaved vector field $\vec{Q}$,

$$\int_A \vec{Q} \cdot \vec{n}\ \mathrm{d}A = \int_V \nabla \cdot \vec{Q}\ \mathrm{d}V \tag{2.5}$$

So equation 2.4 can be rewritten,

$$\int_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{u} \right]\ \mathrm{d}V = 0 \tag{2.6}$$

- Recall that the volume V was selected arbitrarily. It could have been any volume within the fluid. Thus, for equation 2.6 to always be true, the integrand must vanish everywhere. If there existed any sub-domain in which it did not vanish, we could choose V to correspond to that sub-domain and would obtain a violation of mass conservation.

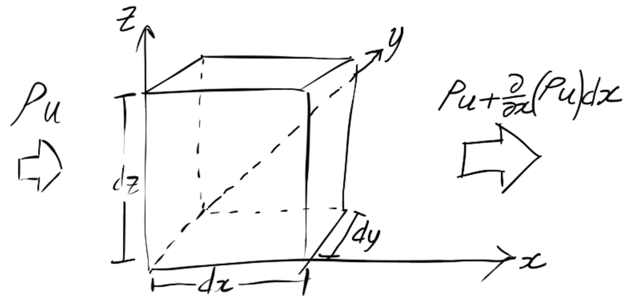
This leaves us with the differential statement of mass conservation, often called the continuity equation:

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \vec{u} = 0} \tag{2.7}$$

(Hint: Boxed equations are very important. You will be using this throughout the rest of this course.)

Example 2.1. We are going to repeat this derivation in a more elementary form to emphasize the physical nature of the result.

Consider an elementary cube with sides $\mathrm{d}x$, $\mathrm{d}y$, and $\mathrm{d}z$,



The net mass flux **leaving** the cube through the face perpendicular to the x-axis with area $\mathrm{d}y\,\mathrm{d}z$ is

$$\left(\rho u + \frac{\partial}{\partial x}(\rho u)\,\mathrm{d}x\right)\,\mathrm{d}y\,\mathrm{d}z - \rho u\,\mathrm{d}y\,\mathrm{d}z = \frac{\partial}{\partial x}(\rho u)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \tag{2.8}$$

A similar calculation for the other four faces of the cube yields the net mass flux leaving the cube as:

$$\left(\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = (\nabla\cdot\rho\vec{u})\,\mathrm{d}V \tag{2.9}$$

This is the definition of the divergence of the vector field $\rho\vec{u}$ and $\mathrm{d}V = \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$.

Any net mass flux leaving the cube must be balance by a decrease in the mass within the cube. Since our volume is fixed in space, $\frac{\partial}{\partial t}(\rho\,\mathrm{d}V) = \frac{\partial\rho}{\partial t}\,\mathrm{d}V$. Equating these two terms,

$$\frac{\partial\rho}{\partial t} = -\nabla\cdot\rho\vec{u} \tag{2.10}$$

This elementary derivation using an elementary cube is merely the basis of the proof of the divergence theorem. **Physically, it says that at each point, the local decrease of density compensates for the local divergence of mass flux.**

Example 2.2 (Alternative forms of the mass conservation equation)**.** In vector notation,

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho u_j)}{\partial x_j} \tag{2.11}$$

Or, expanding the derivative using the product rule,

$$\frac{\partial\rho}{\partial t} + u_j\frac{\partial\rho}{\partial x_j} + \rho\frac{\partial u_j}{\partial x_j} = 0 \tag{2.12}$$

Note that the first two terms are the total derivative (recall equation **??**)). This gives a vector form of the continuity equation using a total derivative of density.

$$\boxed{\frac{D\rho}{Dt} + \rho\nabla\cdot\vec{u} = 0} \tag{2.13}$$

Equations 2.7 and 2.13 are equivalent, but suggest a slightly different interpretation. In this form, the equation describes the rate of change of density following a fluid particle and relates it to the local divergence of velocity.

To understand these interpretations:

Let's again think about our arbitrary control volume. It is still perfectly permeable, but instead of assuming it is fixed in space, we will assume it is fixed to the fluid so that it deforms and stretches as the fluid composing its surface moves. At each point on the surface, the outward movement of the fluid leads to a local volume increase. So the volume increase of the control volume as a whole is:

$$\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}t} &= \int_A \vec{u}\cdot\hat{n}\,\mathrm{d}A \quad \text{Using the divergence theorm} \\ &= \int_V \nabla\cdot\vec{u}\,\mathrm{d}V \end{aligned} \tag{2.14}$$

Now consider the limit as the volume under consideration gets very small ($V \to \delta V \to 0$),

$$\frac{\mathrm{d}}{\mathrm{d}t} = \delta V\nabla\cdot\vec{u} \tag{2.15}$$

Substituting $\nabla\cdot\vec{u}$ into equation 2.13,

$$\begin{aligned} \frac{\mathrm{d}\rho}{\mathrm{d}t} + \frac{\rho}{\delta V}\frac{\mathrm{d}}{\mathrm{d}t} &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}t} &= 0 \end{aligned} \tag{2.16}$$

This states that the total mass ($\rho\delta V$) in the moving volume fixed to the fluid is conserved. As the volume of the fluid element increases (or decreases) the density must decrease (or increase) to compensate in order to conserve total mass.

Example 2.3 (Insights from scaling of the mass conservation equation, specifically for conservation of volume/incompressibility)**.** It can be insightful to do a scale analysis (as if often the case) of the various terms in the mass conservation equation to consider the implications.

First, we start with the statement of conservation of mass from equation 2.13. We will label the terms of this equation $\textcircled{1} = \frac{\mathrm{d}\rho}{\mathrm{d}t}$ and $\textcircled{2} = \rho\nabla\cdot\vec{u}$.

We assume:

- $\delta\rho$: Characteristic scale for the density variation or density anomaly
- ρ: Characteristic value of the density
- L: Lengthscale over which the density anomaly changes
- T: Timescale over which the density anomaly changes
- U: Characteristic value of velocity **and** its variations.

The scales of each term is then:

$$\mathcal{O}\left(\textcircled{1}\right) = \left[\frac{\delta\rho}{T}\right] \qquad \mathcal{O}\left(\textcircled{2}\right) = \left[\frac{\rho U}{L}\right]$$

If, as in many oceanographic and atmospheric situations, the time scale is given by the so-called "advective time" $T = L/U$ (The time it takes a disturbance to move a distance L moving with the fluid at a rate U). These scales then become:

$$\mathcal{O}\left(\textcircled{1}\right) = \left[\frac{\delta\rho U}{L}\right] \qquad \mathcal{O}\left(\textcircled{2}\right) = \left[\frac{\rho U}{L}\right]$$

The relative sizes of term ① to term ② is:

$$\frac{\left[\frac{D\rho}{Dt}\right]}{[\rho\nabla\cdot\vec{u}]} = \mathcal{O}\left(\frac{U\delta\rho/L}{\rho U/L}\right) = \mathcal{O}\left(\frac{\delta\rho}{\rho}\right)$$

If $\frac{\delta\rho}{\rho}$ is small (it is of the order 10^{-3} in the ocean) then the first ① is small compared to each of the three velocity divergence terms in ②.

Thus, to conserve mass, it is necessary (to order $\mathcal{O}\left(\frac{\delta\rho}{\rho}\right)$) to also conserve volume.

Further, equation 2.15 implies (given $\frac{\mathrm{d}}{\mathrm{d}t} = 0$) that:

$$\boxed{\nabla\cdot\vec{u} = 0} \tag{2.17}$$

We define an **incompressible fluid** as one that satisfies equation 2.17.

Some implications of this:

- In such cases, the fluid needs to keep the volume of every fluid parcel constant even though the volume will generally become distorted by the motion. This requirement results in the above condition on the divergence of the velocity vector field.

- It is very important to realize that if equation 2.17 is a valid approximation to the full conservation of mass, it **does not** follow that $\frac{\mathrm{d}\rho}{\mathrm{d}t} = 0$. That is, equation 2.17 does not allow us to extract a second equation constraining the density (you can't get two equations from one!). Instead, equation 2.17 implies that the variation of density, $\frac{\mathrm{d}\rho}{\mathrm{d}t}$, is too small to be a player in the mass budget.

- Another subtle point is that even if $\frac{\delta\rho}{\rho}$ is small, there are situations where equation 2.17 is not true.

- For example, if the time scale of the motion is not the advective time scale, but instead some period of oscillation that is very short compared to the advective scale, the local rate of change of density can be the same order as the velocity divergence.

- If this is the case, the fluid will not act incompressibly. A good example of this is an underwater acoustic wave.

- In general, we think of the flow of water as a very nearly incompressible in its dynamics, but underwater acoustic waves depend on the compressibility of the water and the high frequency of the sound wave.

Incompressibility is therefore an approximation that for any fluid depends on the process and not just the fluid and must be determined by the nature of the dynamics being considered!

2.2 The momentum equation (Newton's second law of motion)

The momentum equation is an expression of Newton's second law of motion, $\vec{F} = m\vec{a}$ (force equals mass times acceleration). Let's start by considering the representation of the force acting on each fluid element.

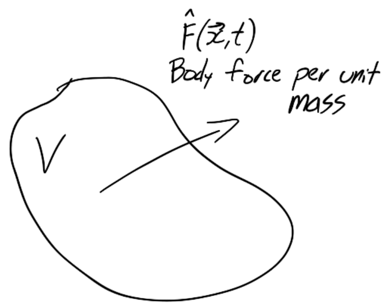
Forces on a fluid element

In continuum mechanics, we first suppose that the forces acting on each fluid element can be separated into two types.

Body forces:

Long range forces that act directly on the mass of the fluid element. These include gravitational and electromagnetic forces (if the fluid is conducting).

These forces are distributed over the volume of the fluid element. The body force is often given as a force per unit mass $F(\vec{x}, t)$.



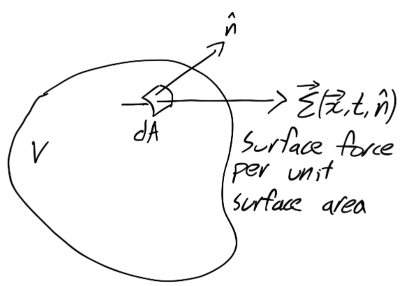
The total volume force on a fixed mass of fluid enclosed by the volume V is:

$$\text{total body force} = \int_V \rho \vec{F}\, \mathrm{d}V \tag{2.18}$$

Surface forces:

Short range forces that act only on the surface of the fluid element. Pressure is a common example of a surface force.

The surface force is often given as a force per unit surface area, $\vec{\Sigma}$ (not a summation over anything! Yet another reason to prefer summation notation.



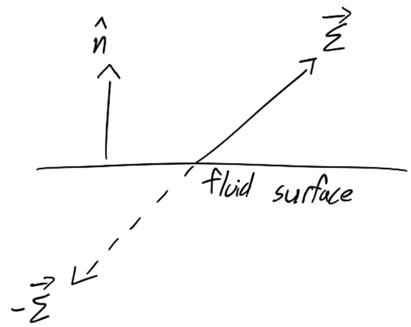
$$\text{total surface force} = \int_A \vec{\Sigma}\, \mathrm{d}A \tag{2.19}$$

This can be interpreted as the "stress" exerted by the fluid external to V on the fluid inside V. We call the surface force per unit area $\vec{\Sigma}$ the stress. **Note!** $\vec{\Sigma}$ is not necessarily perpendicular to the surface.

$\vec{\Sigma}$ is a function of the orientation of the elemental surface $\mathrm{d}A$ (the direction of the normal to the surface of the fluid element at each point $\hat{n}$).

It is defined so that $\vec{\Sigma}(\vec{x}, t, \hat{n})$ is the stress exerted **by** the fluid that is on the side of the area into which the normal points **on** the fluid from which the normal points. By Newton's law of action and reaction, it follows that:

$$\vec{\Sigma}(\vec{x}, t, \hat{n}) = -\vec{\Sigma}(\vec{x}, t, -\hat{n}) \tag{2.20}$$



Therefore, if we write Newton's second law of motion ($\vec{F} = m\vec{a}$) for a volume with a fixed mass of fluid, we have:

$$\underbrace{\int_V \rho \vec{F}\, \mathrm{d}V + \int_A \vec{\Sigma}\, \mathrm{d}A}_{\text{“}\vec{F}\text{”}} = \underbrace{\frac{\mathrm{d}}{\mathrm{d}t} \int_V \rho \vec{u}\, \mathrm{d}V}_{\substack{\text{“}m\vec{a}\text{”} \\ \text{for a volume with a} \\ \text{fixed mass of fluid}}} \tag{2.21}$$

This is an integral statement of momentum conservation!

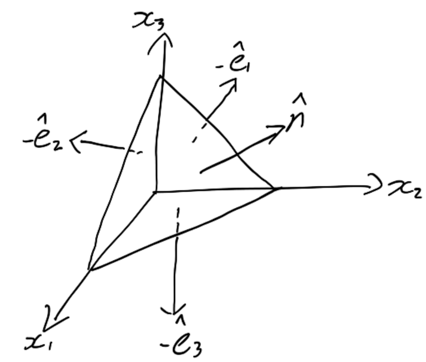
A couple of observations:

- The obvious next question is whether the second term of the left hand side of this equation can be written as a volume integral so that we can extract (as we did for mass conservation) a differential statement for momentum conservation from the integral relation in equation 2.21. (i.e. a statement that applies for an infinitesimally small parcel of fluid)
- This turns out to be a rather subtle issue. The key difficult in proceeding directly is that as we make the control volume smaller and smaller, it appears that the surface term would dominate all of the volume terms. The volume terms would be of order $\mathcal{O}\left(l^3\right)$ while the surface term be of order $\mathcal{O}\left(l^2\right)$, and would thus dominate in the limit $l \to 0$.
- Thankfully, this is not the case! However, it does impose an important constraint on the basic structure of $\vec{\Sigma}$. So before proceeding, we must take a momentary diversion to discuss the stress tensor.

2.3 A momentary diversion: The stress tensor

To calculate the force balance due to surface stresses, consider the tetrahedron shown below.

The outward normal from the slanting face is $\hat{n}$, while the outward normals from the other three faces are $-\hat{\imath}_j$ for



$j = 1, 2, 3$ (note the minus sign!). These are the (negative) unit vectors in the x_1, x_2, and x_3 directions.

We will assume that the tetrahedron is small, so each linear dimension is $\mathcal{O}(l)$ and we will examine the limit $l \to 0$.

The total surface force on the tetrahedron, $\vec{S}$, is:

$$\vec{S} = \underbrace{\vec{\Sigma}(\hat{n})\ \mathrm{d}A}_{\substack{\text{Surface force per unit surface} \\ \text{area acting on the slanted face}}} + \underbrace{\vec{\Sigma}\left(-\hat{i}_1\right)\ \mathrm{d}A_1 + \vec{\Sigma}\left(-\hat{i}_2\right)\ \mathrm{d}A_2 + \vec{\Sigma}\left(-\hat{i}_3\right)\ \mathrm{d}A_3}_{\substack{\text{surface forces acting on the three faces} \\ \text{aligned with the } x_1 - x_2,\ x_2 - x_3, \text{ and } x_3 - x_1 \text{ planes}}} \tag{2.22}$$

A little geometry and trigonometry shows that

$$\mathrm{d}A_j = \hat{n} \cdot \hat{i}_j\ \mathrm{d}A \tag{2.23}$$

Where $\mathrm{d}A_j$ is the area of the triangle perpendicular to the j^{th} coordinate axis. We also know, from Newton's second law,

$$\underbrace{\vec{\Sigma}\left(-\hat{i}_j\right)}_{\substack{\text{Stress exerted by the fluid} \\ \text{on the outside of the volume} \\ \text{on the fluid inside the volume}}} = \underbrace{-\vec{\Sigma}\left(\hat{i}_j\right)}_{-\left(\substack{\text{Stress exerted by the fluid} \\ \text{on the inside of the volume} \\ \text{on the fluid outside}}\right)} \tag{2.24}$$

Using these, equation 2.24 becomes:

$$\mathrm{d}A\left(\vec{\Sigma}(\hat{n}) - \vec{\Sigma}\left(\hat{i}_1\right)\hat{n}_1 - \vec{\Sigma}\left(\hat{i}_2\right)\hat{n}_2 - \vec{\Sigma}\left(\hat{i}_3\right)\hat{n}_3\right) = \vec{S} \tag{2.25}$$

Where $\vec{n} = (n_1, n_2, n_3) = n_j$ where $j = 1, 2, 3$. Now, as we discussed, as $l \to 0$, the size of the surface force is of order $\mathrm{d}A$ ($\mathcal{O}(l^2)$), while all the body forces are of order $\mathrm{d}V$ ($\mathcal{O}(l^3)$). So in the limit of small l, volume forces are negligible compared to surface force. Thus, **to lowest order**, to preserve the force balance for each fluid element, $\vec{S}$ must vanish! Setting equation 2.25 equal to zero and rearranging,

$$\vec{\Sigma}(\hat{n}) = \vec{\Sigma}\left(\hat{i}_1\right)\hat{n}_1 + \vec{\Sigma}\left(\hat{i}_2\right)\hat{n}_2 + \vec{\Sigma}\left(\hat{i}_3\right)\hat{n}_3 \tag{2.26}$$

Or, in component form:

$$\Sigma_i(\hat{n}) = \Sigma_i\left(\hat{i}_1\right)\hat{n}_1 + \Sigma_i\left(\hat{i}_2\right)\hat{n}_2 + \Sigma_i\left(\hat{i}_3\right)\hat{n}_3 = \Sigma_{ij}\hat{n}_j \tag{2.27}$$

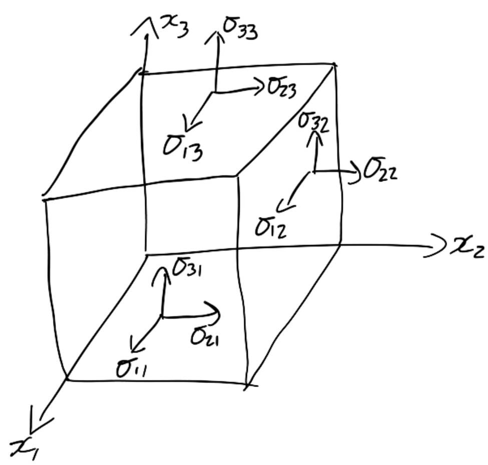
Here, $\Sigma_{ij} \equiv \Sigma_i(\hat{i}_j)$ is the stress (force per unit area) in the i^{th} direction on the surface perpendicular to the j^{th} axis.

Equation 2.27 shows us that we can write the stress on the surface with any orientation (with arbitrary outward normal $\vec{\Sigma}(\hat{n})$) in terms of the stress tensor Σ_{ij} and the normal vector to that surface $(\hat{n}_1, \hat{n}_2, \hat{n}_3)$.

In continuum mechanics, the stress tensor is usually written with the notation

$$\sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix} \tag{2.28}$$

In the notation of the derivation, $\sigma_{ij} = \Sigma_{ij} = \Sigma_i(\hat{i}_j)$, so that $\Sigma_i(\hat{n}) = \sigma_{ij}\hat{n}_j$. A helpful geometrical picture to keep in mind is shown below.



This picture should always be in your head when thinking of elements of the stress tensor. σ_{ij} is the force per unit area in the i^{th} direction on the face perpendicular to the j^{th} axis.

Example 2.4 (Two dimensional stress tensor)**.** To get a feeling for the relationship between the elements of the stress tensors and the forces on fluid elements, consider a simple 2D example. Consider the following stress tensor:

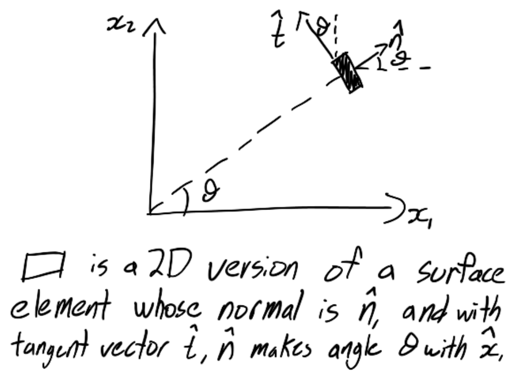
$$\sigma_{ij} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$$

and the surface projected in the $x - y$ plane as shown below.

The components of the unit vectors $\hat{n}$ and $\hat{t}$ are:

$$\hat{n} = (\cos\theta, \sin\theta)$$

$$\hat{t} = (-\sin\theta, \cos\theta)$$



The stress on a surface with arbitrary outward normal $\hat{n}$ is:

$$\Sigma_i(\hat{n}) = \sigma_{ij} n_j$$

$$= \sigma_{i1}\cos\theta + \sigma_{i2}\sin\theta$$

Thus, the stress in the direction of the normal is (*note:* You can read $\Sigma_i(\hat{n}) \cdot \hat{n}$ as "stress on a surface with arbitrary outward normal $\hat{n}$ dotted with the unit vector defining the direction you want to know the stress in initially):

$$\Sigma_i(\hat{n}) \cdot \hat{n} = (\sigma_{i1}\cos\theta + \sigma_{i2}\sin\theta) \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$= (\sigma_{11}\cos\theta + \sigma_{12}\sin\theta, \sigma_{21}\cos\theta + \sigma_{22}\sin\theta) \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$= \sigma_{11}\cos^2\theta + \sigma_{12}\sin\theta\cos\theta + \sigma_{21}\cos\theta\sin\theta + \sigma_{22}\sin^2\theta \qquad \text{remembering that } \sin 2x = 2\sin x\cos x$$

$$= \sigma_{11}\cos^2\theta + \sigma_{22}\sin^2\theta + \frac{\sigma_{12} + \sigma_{21}}{2}\sin 2\theta$$

Following a similar procedure, you can show that the stress in the direction of the tangent vector $\hat{t}$ is:

$$\Sigma_i(\hat{n}) \cdot \hat{n} = \frac{\sigma_{22} - \sigma_{11}}{2}\sin 2\theta + \sigma_{21}\cos^2\theta - \sigma_{12}\sin^2\theta$$

2.4 The momentum equation in differential form

We are now (almost!) in a position to turn the integral statement of momentum conservation into a more useful differential statement (one that applies to an infinitesimally small parcel of fluid).

First, we note that if we consider any integral of the form:

$$I = \frac{\mathrm{d}}{\mathrm{d}t} \int\limits_V \rho\varphi \, \mathrm{d}V \tag{2.29}$$

where V is a volume enclosing a **fixed** mass of fluid. and φ is any scalar (for example, could be a component of velocity). We can think of V as consisting of an infinite number of small, fixed mass-volumes, each with mass $\rho \, \mathrm{d}V$. When mass is conserved following the fluid motion (remember that $\frac{\mathrm{d}}{\mathrm{d}t} = 0$), it follows that we can rewrite the integral as:

$$I = \int\limits_V \rho \frac{\mathrm{d}\varphi}{\mathrm{d}t} \, \mathrm{d}V \tag{2.30}$$

We can then write Newton's second law in integral form. Taking equation 2.21 and using equation 2.30 to rewrite $\int\limits_V \rho\vec{F} \, \mathrm{d}V$ and writing the surface stress in terms of the stress tensor ($\vec{\Sigma} = \sigma_{ij}n_j$):

$$\int\limits_V \rho F_i \, \mathrm{d}V + \int\limits_A \sigma_{ij}n_j \, \mathrm{d}A = \int\limits_V \rho \frac{Du_i}{Dt} \, \mathrm{d}V \tag{2.31}$$

Where σ_{ij} applies for each velocity component u_i, $i = 1, 2, 3$.

Noting that the area integral, $\int\limits_A \sigma_{ij}n_j \, \mathrm{d}A$ can be written as a volume integral for the divergence of the vector (for each i), $\sigma_{(i)j}$ dotted with the normal vector n_j, we can use the usual trick of applying the divergence theorem to write the area integral in the form of a volume integral, $\int\limits_A \sigma_{ij}n_j \, \mathrm{d}A = \int\limits_V \frac{\partial}{\partial x_j}\sigma_{ij} \, \mathrm{d}V$.

Newton's second law becomes:

$$\int\limits_V \left[\rho \frac{Du_i}{Dt} - \rho F_i - \frac{\partial}{\partial x_j}\sigma_{ij}\right] \mathrm{d}V = 0 \tag{2.32}$$

The fact that the surface integral of the stress can be written as a volume integral is a direct consequence of the fact that we required $\vec{S}$ (The total surface force on our tetrahedron as $l \to 0$) to vanish so the force balance for each fluid element was preserved. This requirement allowed us to write $\Sigma_i(\hat{n}) = \sigma_{ij}n_j$ (That is, the surface force per unit surface area acting on a surface with outward normal $\hat{n}$ is a function of the stress tensor σ_{ij}). Thus, our ability to rewrite equation 2.31 entirely as a volume integral is not a coincidence, but rather a result of our basic physical formulation of the dynamics.

Now, we use the usual argument about integral statements: the volume chosen in equation 2.32 is entirely arbitrary, so for the integral to vanish for any arbitrarily chose V, it must be true that the integrand vanishes. Thus,

$$\boxed{\rho \frac{Du_i}{Dt} = \rho F_i + \frac{\partial}{\partial x_j}\sigma_{ij}} \tag{2.33}$$

This is the differential statement of momentum conservation. Note that this equation is valid for any continuum – fluid or solid.

Example 2.5 (An alternate form to give different physical insight)**.** If we expand the total derivative on the LHS of equation 2.33,

$$\rho \frac{\partial u_i}{\partial t} + \rho u_j \frac{\partial u_i}{\partial x_j} = \rho F_i + \frac{\partial}{\partial x_j}\sigma_{ij} \tag{2.34}$$

Combining this with mass conservation, $\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j) = 0$, we can write this as (exercise: convince yourself that this is true!)

$$\frac{\partial}{\partial t}(\rho u_i) + \frac{\partial}{\partial x_j}(\rho u_j u_i) = \rho F_i + \frac{\partial}{\partial x_j}\sigma_{ij} \tag{2.35}$$

(The somewhat unwieldy $\frac{\partial}{\partial x_j}(\rho u_j u_i)$ is the divergence of momentum flux. Discussed very shortly!) Rearranging,

$$\frac{\partial}{\partial t}(\rho u_i) = \rho F_i + \frac{\partial}{\partial x_j}(\sigma_{ij} - \rho u_j u_i) \tag{2.36}$$

We will now integrate this equation over a **fixed**, stationary, and perfectly permeable volume. Again, we will use the divergence theorem – this time to write the volume integral of the divergence of the vector field $(\sigma_{ij} - \rho u_j u_i)$ as an area integral of this vector dotted with $\hat{n}$.

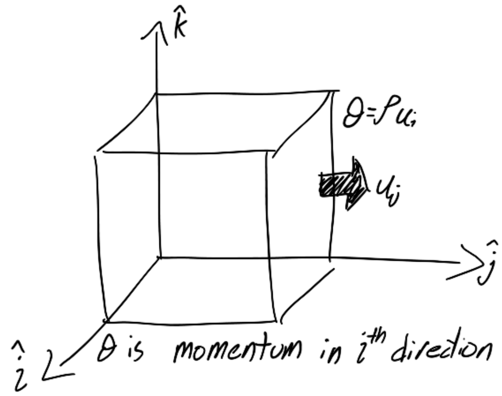
$$\frac{\partial}{\partial t}\int\limits_V \rho u_i \, \mathrm{d}V = \int\limits_V \rho F_i \, \mathrm{d}V + \int\limits_A (\sigma_{ij} - \rho u_i u_j) n_j \, \mathrm{d}A \tag{2.37}$$

Aside: What is $\rho u_i u_j$???

Interpret as the flux of the i^{th} component of momentum ρu_i across the face of the volume element perpendicular to the j^{th} axis. The velocity component u_j carries a flux of momentum ρu_i in the i^{th} direction across the face perpendicular to the j^{th} axis. A divergence of the momentum flux (more momentum leaving that entering) will reduce the momentum within the volume.

A couple of observations:

- Physically, equation 2.37 says that the rate of change of the momentum ρu_i in the fixed volume must be equal to [the total body force applied to the mass of the fluid enclosed by the volume V] **plus** [the total surface force on the fluid element where the stress giving rise to this surface force is given by the applied stress $\Sigma_i = \sigma_{ij} n_j$]



plus [the divergence of the momentum flux that appears as if it were equivalent to a stress acting on the fluid within a volume.]

- This equivalence between the momentum flux and the stress tensor is fundamental.
- If you have studied the kinetic theory of gases, you may remember that the viscosity of a gas is due to this kind of momentum flux on the molecular level. Here, we see the macroscopic analogue.
- In fact, when people try to find representations of the turbulent stresses (which is nothing more than the momentum flux by motions on smaller space scales and faster time scales than we can hope to calculate directly) an appeal by analogy is often made to represent the turbulent stresses in terms of the large-scale flow in the same way that molecular stresses are related to the macroscopic flow. More on this later in the course...