5 Boundary Conditions and Frictional Boundary Layers

The boundary conditions to apply to these equations of motion are just as important as the equations themselves! It is important to keep in mind that the formulation of a problem in fluid mechanics requires the specification of the pertinent equations and the relevant boundary conditions with equal care and attention.

In GFD in particular, there are many examples of important physical problems, like the physics of surface waves on water, in which the fundamental dynamics are contained entirely within the boundary conditions. Likewise, the theory for cyclogensis (spontaneous appearance of weather waves in the atmosphere) is similar.

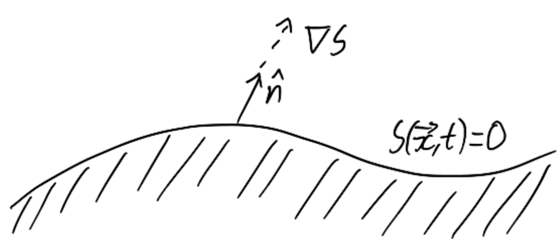
In these problems, the boundary conditions are dynamical in nature and their correct formulation is critical to understanding the relevant physical phenomena.

5.1 Boundary conditions at a solid surface

The most straightfoward conditions apply when the fluid is in contact with a solid surface. For the following arguments, consider a solid surface described by the equation $S(\vec{x}, t) = 0$. It has a normal vector $\hat{n}$ that is oriented in the direction of the gradient of S, ∇S. Note that this implies:

$$\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{\nabla S}{\frac{\partial S}{\partial n}} \tag{5.1}$$

Where n is the distance coordinate normal to the surface.



1. **Condition on the normal velocity of a fluid element on the surface:**

   If the surface is solid, we must impose the condition that there is no fluid flow through the surface. Additionally, if the surface is moving, this implies that the velocity of the fluid normal to the surface must equal the velocity of the surface normal itself.

If fluid cannot go through the surface, a condition of no normal-flow must be imposed. This implies:

$$
\begin{aligned}
\frac{\mathrm{D}S}{\mathrm{D}t} &= 0 \qquad \text{on } S = 0 \\
\frac{\partial S}{\partial t} + \vec{u} \cdot \nabla S &= 0 \qquad \text{substituting } \hat{n} \text{ as above,} \\
\frac{\partial S}{\partial t} + \vec{u} \cdot \frac{\partial S}{\partial n}\hat{n} &= 0 \\
\vec{u} \cdot \hat{n} = -\frac{\frac{\partial S}{\partial t}}{\frac{\partial S}{\partial n}} &= \left.\frac{\partial n}{\partial t}\right|_S \equiv u_n
\end{aligned}
\tag{5.2}
$$

Where u_n is the velocity of the surface normal. Of course, if the surface is not moving, $u_n = 0$. This condition is called the kinematic boundary condition.

2. **Condition on the tangential velocity of a fluid element on the surface:**

In the presence of friction (whenever $\mu \neq 0$), we observe that the fluid at the boundary sticks to the boundary. This is often called the "no-slip" boundary condition. In this case, the tangential velocity of the fluid is also equal to the tangential velocity of the boundary.

The microscopic explanation (easily verified for a gas) is that as a gas molecule strikes the surface, it is captured by the surface potential of the molecules constituting the boundary. The gas molecules are captured long enough to have their average motion annulled and their eventual escape velocity is random (no macroscopic mean velocity is imparted by this process)

Except for very rare gases with extremely low density (for which this randomization of the escape velocity – called thermalization – may be incomplete) the appropriate boundary condition is:

$$\vec{u} \cdot \hat{t} = u_t \tag{5.3}$$

Where u_t is the tangential velocity of the surface.

It is important to note that this expression is independent of the magnitude of the viscosity coefficient even though the condition is physically due to the presence of frictional stresses in the fluid.

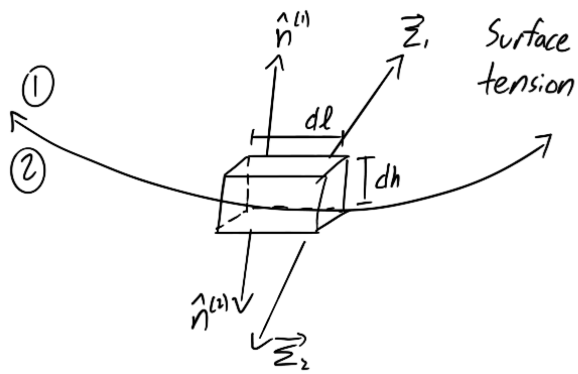
One might imagine that for small enough values of μ, the viscous terms in the Navier-Stokes equations could be ignored. **However, eliminating the viscous terms lowers the order of the differential equations so that they are no longer able to satisfy all boundary conditions!** This is one of the factors that puzzled those studying fluid mechanics in its enfancy. The resolution of this apparent paradox forms an important part of the dynamics which we shall explore in future examples.

5.2 Boundary conditions at a fluid interface

The boundary conditions at the interface of two immiscible fluids is more interesting – and highly relevant to the boundary condition between the ocean and the atmosphere.

Consider the fluid interface between two immiscible fluids $\textcircled{1}$ and $\textcircled{2}$ as illustrated below. We will label properties in the upper fluid with a 1 and those in the lower layer of a fluid with a 2. We will also draw a small pillbox constructed at the fluid interface that we will use to balance the surface forces, shown on the following page. $\hat{n}^{(1)} = \left((n_1^{(1)}, n_2^{(1)}, n_3^{(1)}\right)$ and $\hat{n}^{(2)} = \left((n_1^{(2)}, n_2^{(2)}, n_3^{(2)}\right)$ are the unit vectors normal to the surface of fluid $\textcircled{1}$ and $\textcircled{2}$, respectively. Additionally, $\hat{n}^{(1)} = -\hat{n}^{(2)}$.

There are two types of boundary conditions to consider: kinematic and dynamic.



The **kinematic** conditions are:

1. The velocity normal to the interface is continuous across the interface. This assures that the fluids remain as continua with no holes.

2. The velocity tangent to the interface is continuous at the interface. This is a consequence of a non-zero viscosity, as a discontinuous tangential velocity would imply an infinite shear and hence an infinite stress at the interface. This would immediately expunge the velocity discontinuity. Note that an idealization of a fluid that completely ignores viscosity can allow such discontinuities and again the relationship between a fluid with a small viscosity and one with $\mu = 0$ is a singular one that needs special examination.

Dynamic conditions:

Consider the small pillbox constructed at the fluid interface shown in the previous figure. If we balance the forces on the mass in the box and then take the limit $\mathrm{d}h \to 0$, the volume forces will go to zero faster than the surface forces. As in our argument for the symmetry of the stress tensor, we can argue that in this limit, the surfaces forces must balance – except for the action of surface tension forces.

$$\vec{\Sigma}_1\left(\hat{n}^{(1)}\right) + \vec{\Sigma}_2\left(\hat{n}^{(2)}\right) = -\gamma\left(\frac{1}{R_a} + \frac{1}{R_b}\right)\hat{n} \tag{5.4}$$

Where R_a and R_b are the radii of curvature of the surface in any two orthogonal directions and γ is the surface tension coefficient. Noting that $\hat{n}^{(1)} = -\hat{n}^{(2)} \equiv \hat{n}$ and $\vec{\Sigma}_2(-\hat{n}) = -\vec{\Sigma}_2(\hat{n})$ by Newton's third law, this expression becomes:

$$\vec{\Sigma}_1(\hat{n}) - \vec{\Sigma}_2(\hat{n}) = -\gamma\left(\frac{1}{R_a} + \frac{1}{R_b}\right)\hat{n} \tag{5.5}$$

In terms of the stress tensor (recalling $\Sigma_i(\hat{n}) = \sigma_{ij}n_j$):

$$\sigma_{ij}^{(1)}n_j^{(1)} - \sigma_{ij}^{(2)}n_j^{(2)} + \gamma\left(\frac{1}{R_a} + \frac{1}{R_b}\right)n_i = 0 \tag{5.6}$$

Consider first the dynamic conditions for the continuity of normal stress force across the interface:

The stress in the direction normal to the surface is then (Recalling that this is given by $\Sigma_i(\hat{n}) \cdot \hat{n}$):

$$\sigma_{ij}^{(1)}n_i^{(1)}n_j^{(1)} - \sigma_{ij}^{(2)}n_i^{(2)}n_j^{(2)} + \gamma\left(\frac{1}{R_a} + \frac{1}{R_b}\right) = 0 \tag{5.7}$$

Using our expression for the stress tensor in terms of the contributions arising from the symmetric and antisymmetric parts of the deformation tensor (Recall $\sigma_{ij} = -P\delta_{ij} + 2\mu e_{ij} + \lambda e_{kk}\delta_{ij}$) and noting that $n_i n_i = 1$:

$$-p_1 + 2\mu e_{ij}^{(1)}n_i n_j + \lambda e_{kk}^{(1)}\delta_{ij}n_i n_j + p_2 - 2\mu e_{ij}^{(2)}n_i n_j - \lambda e_{kk}^{(2)}\delta_{ij}n_i n_j + \gamma\left(\frac{1}{R_a} + \frac{1}{R_b}\right) = 0 \tag{5.8}$$

We can rewrite this using $e_{kk}^{(\beta)}\delta_{ij}n_i n_j = \nabla \cdot \vec{u}^{(\beta)}$ with $\beta = 1, 2$ and $\big|_2^1$ to denote the difference of two terms across the interface:

$$p_1 - p_2 = \gamma\left(\frac{1}{R_a} + \frac{1}{R_b}\right) + \left(2\mu e_{ij}n_i n_j + \lambda\nabla \cdot \vec{u}\right)\Big|_2^1 \tag{5.9}$$

Typically, the last term on the RHS is very small and of the order $\mu\frac{\partial u_3}{\partial x_3}$. The surface tension term is also negligible – unless we are at scales where the radii of curvature are small enough to accommodate capillary waves.

Thus, for larger scales, the dynamic condition of the continuity of normal stress force reduces to the continuity of pressure across the interface,

$$p_1 = p_2 \tag{5.10}$$

Now consider the dynamic condition for the continuity of tangential stress force across the interface.

The stress in the direction tangential to the surface, $\Sigma_i(\hat{n}) \cdot \hat{t}$ (Where $\hat{t}$ is a vector tangent to the interface. There will be two such orthogonal vectors for a 3D surface!) is:

$$\sigma_{ij}^{(1)} \hat{n}_j \hat{t}_i - \sigma_{ij}^{(2)} \hat{n}_j \hat{t}_i = 0 \tag{5.11}$$

We will make the same substitution as previously ($\sigma_{ij} = -P\delta_{ij} + 2\mu e_{ij} + \lambda e_{kk}\delta_{ij}$) but this time $-P\delta_{ij}$ and $\lambda e_{kk}\delta_{ij}$ will vanish because the normal and tangent vectors are orthogonal, $\delta_{ij}\hat{n}_j\hat{t}_i = 0$.

$$\mu_1 e_{ij}^{(1)} n_j t_i^{(m)} - \mu_2 e_{ij}^{(2)} n_j t_i^{(m)} = 0 \tag{5.12}$$

where $m = 1, 2$ for two tangential stresses in the horizontal plane.

Example 5.1. For the sake of argument, let's say the surface is flat and perpendicular to the x_3 axis. In this case, the two tangential vectors are $\hat{t}_1 = (1, 0, 0)$ and $\hat{t}_2 = (0, 1, 0)$, and $\hat{n} = (0, 0, 1)$. Equation 5.12 tells us that:

$$\begin{aligned} &(\text{for } m = 1) \qquad \mu\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \qquad \text{Is continuous across the interface} \\ &(\text{for } m = 2) \qquad \mu\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) \qquad \text{Is continuous across the interface} \end{aligned} \tag{5.13}$$

Typically, the velocities in the plane of the interface (u_1 and u_2) vary rapidly in the direction normal to the interface ($\hat{x}_3$) so this usually – although not always – reduces to the shear of the velocity across the interface being continuous:

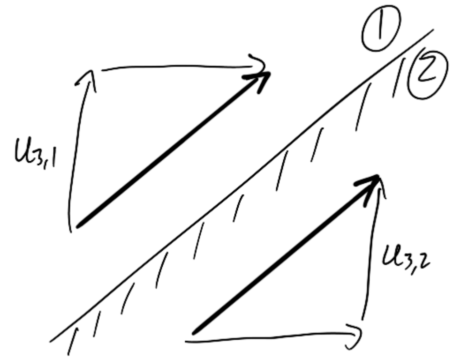
$$\mu\frac{\partial u_1}{\partial x_3} \quad \& \quad \mu\frac{\partial u_2}{\partial x_3} \qquad \text{Are continuous across the interface} \tag{5.14}$$

Some interesting side notes:

- If the viscosity is completely ignored, as is done in some problems for which the effect of viscosity is deemed to be of minor importance, both the tangential velocity and across-interface velocity shear can be discontinuous!
- For this case, We define the interface by: $S(x_i, t) = x_3 - f(x_1, x_2, t) = 0$ (a flat surface perpendicular to the x_3 axis minus a perturbation that varies with x_1, x_2, and time). The condition that fluid element remain on the surface is:

$$\begin{gathered} \frac{\mathrm{D}S}{\mathrm{D}t} = 0 \\ \frac{\partial S}{\partial t} + \vec{u} \cdot \nabla S = 0 \quad \text{substitute } S = x_3 - f(x_1, x_2, t) \\ \frac{\partial f}{\partial t} + u_1^{(1)}\frac{\partial f}{\partial x_1} + u_2^{(1)}\frac{\partial f}{\partial x_2} - u_3^{(1)} = \frac{\partial f}{\partial t} + u_1^{(2)}\frac{\partial f}{\partial x_1} + u_2^{(2)}\frac{\partial f}{\partial x_2} - u_3^{(2)} \end{gathered} \tag{5.15}$$

So if the tangential velocities are not continuous, the vertical velocity u_3 need not be continuous if the interface is sloping... It is only the **velocity normal to the interface** that needs to be continuous.



Here, the tilted interface is stationary and the velocity is parallel to the interface on both sides. The condition that the normal velocity is continuous is trivially satisfied as $\vec{u}_n = 0$ on both sides of the interface. Since the tangential velocity is discontinuous (recall the dynamic condition for the continuity of tangential stress force across the interface requires the **shear** of the velocity to be continuous and not the tangential velocity itself), the velocity in the x_3 direction is not continuous, $w_1 \neq w_2$.

5.3 The Ekman layer problem over a solid surface

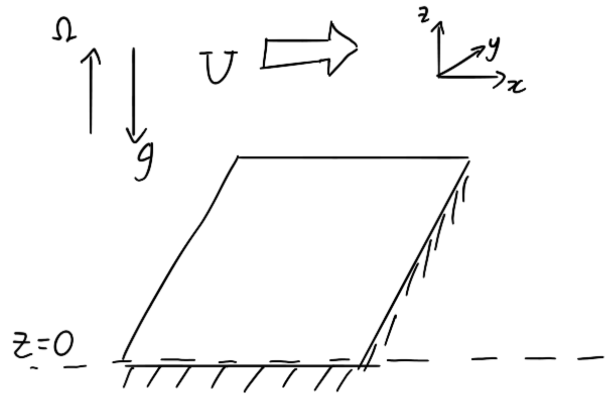
But first, a side note about frictional boundary layers in general.

In the next few sections we will take up the important question of the role of friction, especially in the case when the friction is relatively small (very relevant to the atmospheric and oceanic flows). We will have to find an objective measure of what we mean by small...

As we noted in the last section, the no-slip boundary condition must be satisfied no matter how small friction is. **But** ignoring friction lowers the spatial order of the Navier-Stokes equations and makes the satisfaction of the boundary condition impossible. How do we resolve this perplexity? The examination of this basic fluid mechanical question allows us to simultaneously investigate a physical phenomenon of great importance to both meteorology and oceanography. Enter the fricitonal boundary later in a rotating fluid. The Ekman layer!

The Ekman Layer problem over a solid surface Consider the problem illustrated below.

Although we have not (quite) completed our formulation of the equations of motion in the general case (we have four equations but five unknowns!) we do have a complete set for when we assume constant density.



In this case, this will give us a pretty good picture of the phenomenon of interest and we will return to the problem of the general formulation afterwards to assess the legitimacy of our constant density assumption.

What can we say about the structure of the resulting flow? We will use Cartesian coordinates with axes (x, y, z) and velocities (u, v, w). When density is constant, the statement of conservation of mass simplifies to:

$$\nabla \cdot \vec{u} = 0$$
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{5.16}$$

The momentum equations for this incompressible fluid are:

$$\frac{\mathrm{D}\vec{u}}{\mathrm{D}t} + 2\vec{\Omega} \times \vec{u} = -\frac{1}{\rho}\nabla P - g\hat{k} + \nu \nabla^2 \vec{u} \tag{5.17}$$

Or, in component form:

$$\begin{aligned} \frac{\partial u}{\partial t} + uu_x + vu_y + wu_z - fv &= -\frac{1}{\rho}P_x + \nu(u_{xx} + u_{yy} + u_{zz}) \\ \frac{\partial v}{\partial t} + uv_x + vv_y + wv_z + fu &= -\frac{1}{\rho}P_y + \nu(v_{xx} + v_{yy} + v_{zz}) \\ \frac{\partial w}{\partial t} + uw_x + vw_y + ww_z &= -\frac{1}{\rho}P_z - g + \nu(w_{xx} + w_{yy} + w_{zz}) \end{aligned} \tag{5.18}$$

Here, the sum of the gravitation and centrifugal accelerations is written as $\vec{g}$. This produces a force opposing the rotation. We define the kinematic viscosity $\nu = \mu/\rho$ and Coriolis parameter $f = 2\Omega$, and subscripts denote partial differentiation.

We will be looking for steady solutions to this problem, so we can assume derivatives with respect to time to be equal to zero. Furthermore, We need the velocity at infinity to be indepedent of x and y, so we set derivatives with respect to x and y equal to zero.

From mass conservation, this implies $\frac{\partial w}{\partial z} = 0$ But since $w = 0$ must be imposed as the boundary condition on the lower solid surface to satisfy no normal flow, $w = 0$ must be true everywhere!

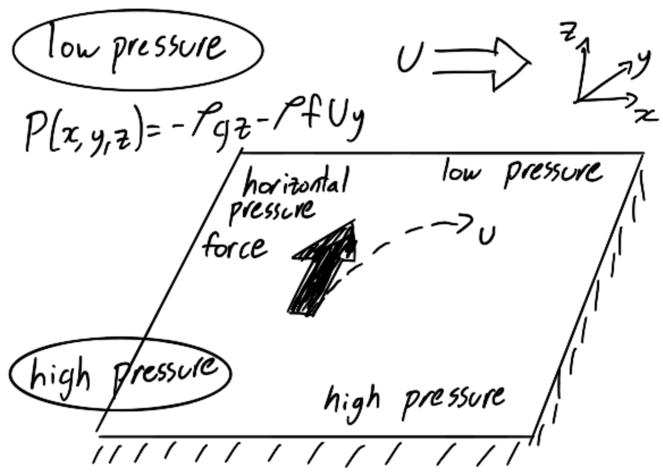
Thus, these equations become:

$$-fv = -\frac{1}{\rho}P_x + \nu u_{zz}$$
$$fu = -\frac{1}{\rho}P_y + \nu v_{zz} \tag{5.19}$$
$$0 = -\frac{1}{\rho}P_z - g$$

Note that these equations are now linear in $\vec{u}$! This makes our life much more pleasant. We now have to see whether we can find solutions satisfying the boundary conditions.

One possible, rather simple solutions is to set $u = U$, $v = 0$, $P = -\rho g z - \rho f U y$. This solution describes a flow that everywhere has a constant velocity in the x direction at large z and in which the pressure field balances the hydrostatic force in the vertical and the Coriolis acceleration in the horizontal.

The principal and inescapable problem with this solution is that it does not satisfy the no-slip boundary condition at $z = 0$ where both u and v should be 0. Thus, although the governing equations are satisfied, **the solution is wrong because it does not satisfy the boundary conditions**. And it is wrong no matter how small μ might be.



So how do we solve these equations in a way that satisfies the boundary conditions?

Note that the presence of rotation implies that although flow is in the x direction far from the wall, the frictional term will force a flow in the y direction under the influence of rotation.

First, we will exploit the z component momentum balance to extract more information on the horizontal pressure gradients. This equality ($g = -\frac{1}{\rho}P_z$) is known as the hydrostatic balance. Taking an x or y derivative of this expression immediately shows

$$\frac{\partial}{\partial z}\left(\frac{\partial P}{\partial x}\right) = \frac{\partial}{\partial z}\left(\frac{\partial P}{\partial y}\right) = 0 \tag{5.20}$$

So the horizontal pressure gradients are independent of z!

Next, we exploit the fact that we know the horizontal pressure gradients in the far field (as we know $(u, v) = (U, 0)$ as $z \to \infty$). The far field limit of the x and y equations is thus:

$$\begin{aligned} 0 &= -\frac{1}{\rho} P_x \implies P_x = 0 \\ fU &= -\frac{1}{\rho} P_y \implies P_y = -\rho f U \end{aligned} \tag{5.21}$$

However, as the pressure gradients are independent of height, these pressure gradients must be present everywhere, even down to the lower surface. $P_y = -\rho f U$ is a statement that the Coriolis acceleration of the far-field uniform flow U is balanced by the large-scale pressure gradient, P_y. This is called a geostrophic balance, and is a very common force balance for the large-scale, quasi-steady flows of the atmosphere and the ocean. It is also worth noting that we have selected our coordinate system to be oriented such that the far-field wind U is only in the x direction, but this derivation would be applicable even your coordinate system couldn't be rotated due to other constraints.

Substituting these pressure gradients gives two ODEs for the velocity components $u(z)$ an $v(z)$:

$$\begin{aligned} -fv &= \nu u_{zz} \\ fu &= fU + \nu v_{zz} \end{aligned} \tag{5.22}$$

These equations are coupled, so we will solve by substituting one equation into the other. Taking two z derivatives of the second equation and substituting into the first gives:

$$\frac{\mathrm{d}^4 v}{\mathrm{d} z^4} + \frac{4}{\delta^4} v = 0 \tag{5.23}$$

Where $\delta = \sqrt{2\nu / f}$ is the *Ekman Layer thickness.* Note that this dependence is such that for a very small viscosity or for very rapid rotation, the thickness of the frictional boundary layer is small. This equation has a general solution of the form:

$$v = C_1 e^{-z/\delta} \sin\frac{z}{\delta} + C_2 e^{-z/\delta} \cos\frac{z}{\delta} + C_3 e^{z/\delta} \sin\frac{z}{\delta} + C_4 e^{z/\delta} \cos\frac{z}{\delta} \tag{5.24}$$

Substituting back into Equation 5.22,

$$u = U - C_1 e^{-z/\delta} \cos\frac{z}{\delta} + C_2 e^{-z/\delta} \sin\frac{z}{\delta} + C_3 e^{z/\delta} \cos\frac{z}{\delta} - C_4 e^{z/\delta} \sin\frac{z}{\delta} \tag{5.25}$$

We can now use the boundary conditions to solve for C_1, C_2, C_3, and C_4.

We know that as $z \to \infty$, we want our solution to remain bounded (as the far-field solution is $(u, v) \to (U, 0)$) So all terms that are proportional to $e^{z/\delta}$ must vanish. This implies $C_3 = C_4 = 0$.

We also impose the no-slip condition at the low boundary, so $(u, v) = (0, 0)$ at $z = 0$. This implies $C_2 = 0$ and $C_1 = U$, giving a final solution of:

$$\begin{aligned} u &= U\left(1 - e^{-z/\delta}\cos\frac{z}{\delta}\right) \\ v &= Ue^{-z/\delta}\sin\frac{z}{\delta} \\ w &= 0 \end{aligned} \tag{5.26}$$

From the solutions, we can see that when $z \gg \delta$ (far from the Ekman frictional boundary layer), $u \to U$ and $v \to 0$ exponentially rapidly. Thus, outside a region of thickness $\mathcal{O}(\delta)$, the solution approaches Equation 5.19, which is the solution we would obtain completely ignoring friction. Inside the region of thickness δ, this solution satisfies the governing equations and the boundary conditions, for any value amount of friction we choose!

The concept of friction always being important in this region is central to the concept of fricitonal boundary layers. The implication of small friction is manifest in the presence of a very small fricitonal boundary layer.

The resolution of the perplexity of how a fluid with low friction still satisfies the no-slip boundary condition is clear from the form of the solution. As friciton gets smaller, the z derivatives in Equations 5.22 increase at the rate necessary to keep the terms νu_{zz} and νv_{zz} at the same order as the C.0oriolis terms. In the limit as $\nu \to 0$, we can examining the limiting form of the solution in two ways:

1. **Fix any value of $z \geq 0$.** This ensures that for sufficiently small ν (equivalently sufficiently small δ) we will be outside the boundary layer and in a region governed by the non-viscous force balance between Coriolis acceleration and pressure gradient force.

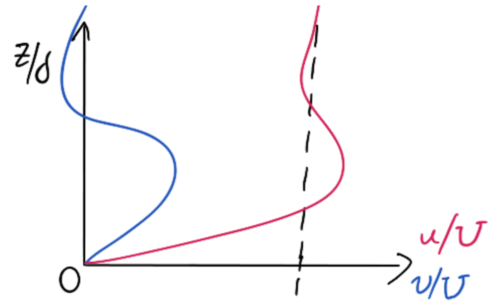
2. **Fix a value of** z/δ (stay within the Ekman layer). Then as the friction decreases, there is always a region near the boundary in which the friction remains important to ensure adherence to the no-slip boundary condition.

5.4 Cool properties of the Ekman frictional boundary layer in a rotating flow

Ekman layers are an important phenomenon for rotating fluids in contact with a boundary and have important relevance to the bottom boundary layer of the atmosphere (against the solid Earth), the bottom boundary of the ocean, and at the boundary layers at the air-sea interface. They have a number of interesting proprties worthy of note.

The Ekman Spiral

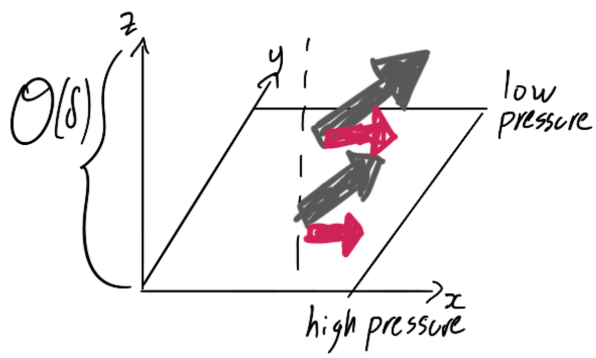
The solutions to the Ekman equations look like:



In this plot of profiles of u and v, we see:

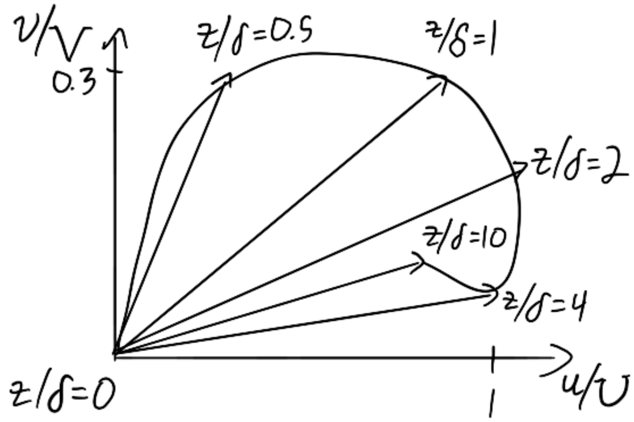
- The significant departure of u from the value U in which the Coriolis acceleration balances the pressure gradient occurs only in a region $\mathcal{O}(\delta)$ (A scale that will decrease as ν decreases.
- For large z/δ, $v \to 0$ and $u \to U$ so the flow is along lines of constant pressure.

But as z decreases, there is a meridional (v component) flow, largely in the positive y direction in this case, so that there is **flow down the pressure gradient** near the wall (From high to low pressure).



Since the pressure is independent of z, as the zonal (x) component of velocity is reduce to satisfy the no-slip condition, the Coriolis acceleration is no longer able to balance the pressure gradient and the fluid begins to flow down the pressure gradient restrained by the friction as in a non-rotating fluid.

If we looked down at the direction of the horizontal velocity from above, we would see:



This elegant figure is called the **Ekman spiral** and shows the turning of the velocity vector with height. For very large heights, the velocity is parallel to the x axis and perpendicular to the pressure gradient toward the y direciton. But as $z \to 0$ the velocity swings in the direction along the pressure force oriented down the pressure gradient until at $z \approx 0$ the velocity makes a $\approx 45^\circ$ angle with the direction of the flow at large z.

Ekman Transport It is illuminating to consider the total mass flux in the Ekman layer due to friciton, $\vec{M}_{\text{EK}}$. This is the vertical integral of the velocities arising from the friction (per unit length of the boundary):

$$\vec{M}_{\text{EK}} = \hat{i} \int_0^\infty (u(z) - U)\, \mathrm{d}z + \hat{j} \int_0^\infty v(z)\, \mathrm{d}z \tag{5.27}$$

You can show that substitution of our solutions for $u(z)$ and $v(z)$ into this expression then yields:

$$\vec{M}_{\text{EK}} = U \frac{\delta}{2} \left(-\hat{i} + \hat{j} \right) \tag{5.28}$$

This can be rewritten:

$$\vec{M}_{\text{EK}} = U \frac{\delta}{2} \hat{k} \times \left(\hat{i} + \hat{j} \right) \tag{5.29}$$

One can also show (another good one to try yourself...):

$$\vec{\tau} = -\rho \nu \frac{U}{\delta} \left(\hat{i} + \hat{j} \right) \tag{5.30}$$

Where $\vec{\tau}$ is the stress exerted on the fluid by its interaction with the wall. This is also the negative of the friction stress exerted by the fluid on the lower boundary. Combining equations 5.29 and 5.30 yields the important results:

$$\vec{M}_{\text{EK}} = -\frac{\hat{k} \times \vec{\tau}}{\rho f} \tag{5.31}$$

So, the total mass flux in the Ekman layer due to friction (frictionally driven mass flux – also called *Ekman transport*) is oriented 90° to the right of the applied stress (if $f > 0$, as it is in the northern hemisphere). This quantity is independent of the magnitude of the friciton and instead on the overall boundary stress applied.

Ekman spindown time: Consider the frictional loss of energy in this problem. Here we are in steady state, which implies that the frictional energy loss is balanced by the work done by the pressure field on the fluid to compensate for the friciton loss. The rate of work done per unit horizontal area, $\dot{W}$, is equal to the force (per unit area) times the velocity in the direction of the force. That is:

$$\begin{aligned}
\dot{W} &= -\int_0^\infty \frac{\partial P}{\partial y} v \, \mathrm{d}z \\
&= -pdvPy \int_0^\infty v \, \mathrm{d}z \qquad \text{From the Ekman solution for } v(z) \\
&= \rho f U^2 \frac{\delta}{2} \qquad \text{Using } \delta = \sqrt{2\nu/f} = \sqrt{\nu/\Omega} \\
&= \rho U^2 \sqrt{\nu\Omega}
\end{aligned} \tag{5.32}$$

Now suppose we have a large cylindrical container filled with fluid whose circulatory velocity is of $\mathcal{O}(U)$. If the container has a depth D, the its kinetic energy per unit horizontal area is:

$$\mathrm{KE} = \frac{\rho U^2}{2} D \tag{5.33}$$

Now, if that energy is not constantly replenished at a rate $\dot{W}$, it will decay due to friction in a time of the order t_E, where:

$$\begin{aligned}
\dot{W} t_E &= \mathrm{KE} \\
t_E &= \frac{\rho U^2 D/2}{\rho U^2 \sqrt{\nu\Omega}} \\
t_E &= \frac{D}{\delta} \frac{1}{2\Omega}
\end{aligned} \tag{5.34}$$

This is the **Ekman spindown time**. If δ is a small fraction of the height of the column of fluid, then the spin down time is large compared to the rotational period. That is, the decay time due to friction is long compared to the rotation rate.

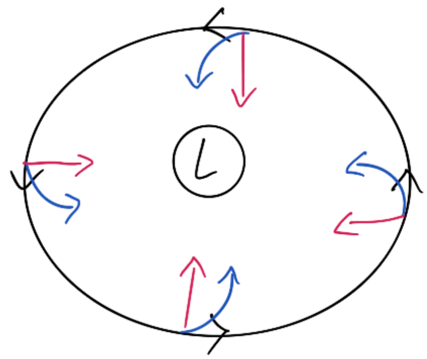
A measure of the importance of friction relative to rotated is more commonly indicated as $\left(\frac{\delta}{D}\right)^2$ So that it is a nice function of ν and Ω.:

$$\mathrm{Ek} = \left(\frac{\delta}{D}\right)^2 = \frac{\nu}{\Omega D^2} \tag{5.35}$$

This is called the Ekman number!

Ekman pumping and Ekman suction associated with atmospheric high and low pressure systems

Consider an atmospheric low pressure system aloft in the northern hemisphere. The view from above (down onto the $x - y$ plane) is:



This figure shows geostrophic flow, (with the coriolis force balancing the horizontal pressure gradient force), which is always along lines of constant pressure. Here we will use the solution to the Ekman layer problem over a solid boundary in its most general form (allowing the far field velocity to be described by $\vec{U}(x, y)$ instead of a constant)

$$\vec{u} = \underbrace{\vec{U}\left(1 - e^{-z/\delta}\cos\frac{z}{\delta}\right)}_{\substack{\text{Flow in the direction of the flow at } \infty \\ \text{along lines of constant pressure (isobars)}}} + \underbrace{\hat{k} \times \vec{U} e^{-z/\delta}\sin\frac{z}{\delta}}_{\substack{\text{flow perpendicular to the isobars} \\ \text{and down the pressure gradient}}} \tag{5.36}$$

A similar generalization for the cross-isobar transport is then:

$$\int_0^\infty u_{\substack{\text{cross}\\\text{isobar}}}\,\mathrm{d}z = \vec{T}_{\substack{\text{cross}\\\text{isobar}}} = \hat{k} \times \vec{u}\frac{\delta}{2} \tag{5.37}$$

Next, to get an expression for w, consider a vertical integral of the continuity equation throughout the Ekman layer – between the lower surface and a height outside the boundary layer.

$$\begin{aligned}
&\int_0^\infty \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\,\mathrm{d}z = 0 \\
&\int_0^\infty \frac{\partial w}{\partial z}\,\mathrm{d}z = -\int_0^\infty \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\,\mathrm{d}z \\
&w(\infty) - w(0) = -\nabla_H\left(\hat{k} \times \vec{u}\frac{\delta}{2}\right) \qquad w(0) = 0 \text{ from no normal flow} \\
&w(\infty) = -\nabla_H\left(\hat{k} \times \vec{u}\frac{\delta}{2}\right)
\end{aligned} \tag{5.38}$$

Where ∇_H is a 2d horizontal divergence. The vertical integral on the RHS is the horizontal divergence of $\vec{u}$, which is the horizontal divergence of the cross-isobar transport. This brings us to an important results:

$$w_{Ek} = \frac{\delta}{2}\hat{k} \cdot \nabla \times \vec{u} \tag{5.39}$$

Where w_{Ek} is the Ekman vertical velocity outside the Ekman layer. It is a consequence of the cross-isobar flow leading to horizontal flow convergence (or divergence) in the Ekman layer below. $\hat{k} \cdot \nabla \times \vec{u}$ is the vertical component of the vorticity of the flow at ∞. As we know that the curl of $\vec{u}$ is the vorticity,

$$w_{Ek} = \frac{\delta}{2} \hat{k} \cdot \vec{\omega} = \frac{\delta}{2} \xi \tag{5.40}$$

Where ξ is the vertical component of vorticity. Thus, the cross-isobar flow driven by down-the-pressure-gradient-flows in the Ekman layer resulting from frictional stresses has nowhere to go but up to conserve mass.

It is this upward-directed vertical velocity associated with low pressure systems aloft that is why cloudy conditions typically prevail in low pressure systems: As the air rises due to Ekman convergence inside the boundary layer, the moisture in the air condenses at higher altitude where the air is colder.

In the case of a high pressure centre, ξ is negative (the motion is anticyclonic) and the reverse occurs – air is drawn down and dries out, leading to clear skies.

Implications of Ekman layer dynamics for large-scale ocean circulation

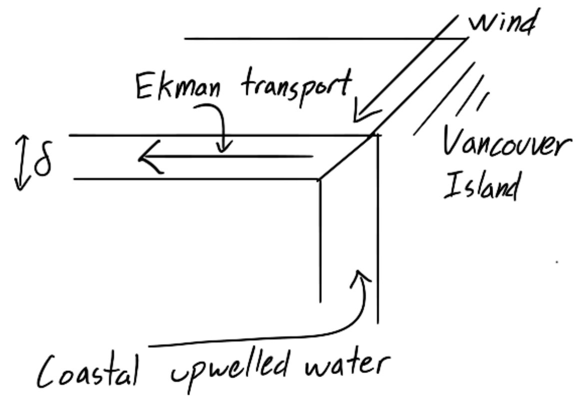
As in the case of the atmosphere, the ocean has a bottom boundary Ekman layer where the ocean floor exerts a stress on the fluid moving above it and similar dynamics result. These dynamics don't tend to be too significant in the deep "open ocean" because δ is a small fraction of the total water column depth and the far field velocities tend to be small. However, bottom Ekman layer dynamics can be significant in coastal flow where δ/D can be quite large.

In the ocean you can (in fact: you will!) also set up an important Ekman layer at the ocean surface in response to the frictional stress applied by the wind. This process has significant implications for a number of ocean phenomena, including:

- **Coastal upwelling**

  Consider the case of a wind blowing parallel to the coast in the northern hemisphere. This is a common scenario observed on the coast of BC in the summer. Associated with this applied wind stress is an Ekman transport in the stress-driven Ekman layer: it is oriented 90° to the right of the applied stress as shown.

  This leads to a horizontal divergence of the flow velocity in the surface layer which is filled by an upwelling of water from below that acts to replenish the mass lost at the surface. This upwelled water is normally cold and rich in nutrients. As the deep ocean is a site of respiration but not photosynthesis, these waters are also

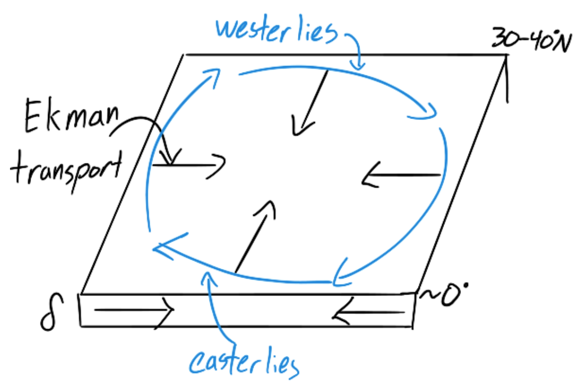


often low in oxygen.

These dynamics are the reason why the BC coasts are so productive and why the major fisheries around the world tend to be found along the coasts that have upwelling-favourable winds. They are also the reason why hypoxia (oxygen concentration in seawater falling below a critical level for marine life) is a threat to BC coastal ecosystems. It is also the reason why the water along the Pacific NW coast is so cold in the summer.

- **Ekman pumping and suction in the ocean gyre circulation**

  In the open ocean, consequences are equally important. Consider the region of the subtropical gyes between the equator and $\sim 40^\circ$ N/S. The large scale wind patterns look like:



The Ekman transport in the northern hemisphere is directed 90° to the right of the applied wind stress, leading to a horizontal convergence of the Ekman flow in the centre of the base. The water has nowhere to go but down, which produces a vertical Ekman velocity with a magnitude proportional to the curl of wind stress (the derivation of which will be left as a mandatory exercise for the reader)

$$w_E = \hat{k} \cdot \nabla \times \left(\frac{\tau_{\text{wind}}}{\rho f} \right) \tag{5.41}$$

This vertical velocity is weak – typically of the order of 10^{-4} cm/s. However, these weak vertical velocities (downward in the subtropic gyres – pumping) & (upward in the subpolar gyres – suction) are responsible for setting up large-scale horizontal pressure gradients which balance the large-scale wind-driven ocean circulation that includes the gulf stream, whose speed is $\mathcal{O}(100)$ cm/s