6 Fundamental Theorems: Vorticity and Circulation

In GFD, and especially the study of the large-scale motions of the atmosphere and ocean, we are particularly interested in the rotation of the fluid. As a consequence, again assuming the motions are of large enough scale to feel the effects of (in particular, the differential) rotation of the outer shell of rotating, spherical planets: the effects of rotation play a central role in the general dynamics of the fluid flow.

This means that vorticity (rotation or spin of fluid elements) and circulation (a conserved related quantity) play an important role in governing the behaviour of large-scale atmospheric and oceanic motions. This can give us important insight into fluid behaviour that is deeper than what is derived from solving the equations of motion (which is challenging enough in the first place).

In this section, we will develop two theorems and principles related to the conservation of vorticity and circulation that are particularly useful in gaining insight into ocean and atmospheric flows.

6.1 Review: What is vorticity?

Vorticity was previously defined as

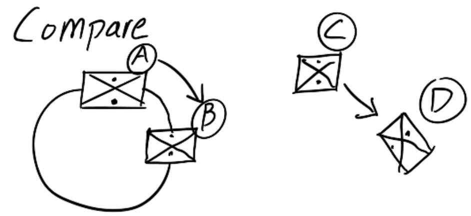
$$\omega_i = \epsilon_{ijk} \frac{\partial u_k}{\partial x_j} \tag{6.1}$$

or, in vector notation,

$$\begin{aligned} \vec{\omega} &= \nabla \times \vec{u} \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} \\ &= \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \right) \hat{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \right) \hat{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \hat{k}. \end{aligned} \tag{6.2}$$

Physically, the vorticity is two times the local rate of rotation (or "spin") of a fluid element.

It is important to distinguish between circular motion and the rotation of the element. This distinction is illustrated in the figure below. Here, the fluid element moving from Ⓐ to Ⓑ on the circular path has no vorticity, while the fluid element moving from Ⓒ to Ⓓ has non-zero vorticity.



In a similar vein, it is important to keep in mind the distinction between vorticity (local rotation of fluid elements) and the curvature of streamlines (motion of the flow in circular-like orbits).

Vorticity in a rotating fluid:

In a rotating fluid, recall:

$$(\vec{u})_{\text{inertial}} = (\vec{u})_{\text{rotating}} + \vec{\Omega} \times \vec{r}. \tag{6.3}$$

The vorticity associated with the velocity in an inertial frame is related to the vorticity in a rotating frame by:

$$\begin{aligned} (\vec{\omega})_{\text{inertial}} &= (\vec{\omega})_{\text{rotating}} + \nabla \times \left(\vec{\Omega} \times \vec{r}\right) \\ &= (\vec{\omega})_{\text{rotating}} + 2\vec{\Omega} \end{aligned} \tag{6.4}$$

Thus, the vorticity in the inertial frame is equal to vorticity seen in the rotating frame (we call this the **relative vorticity**) plus the vorticity of the velocity due to the frame's rotation (sometimes called the **planetary vorticity**).

6.2 Circulation

Circulation is defined for any vector field $\vec{J}$ around some closed curve C as

$$\Gamma_{\vec{J}} = \oint_C \vec{J} \cdot \mathrm{d}\vec{x} \tag{6.5}$$

or, in index notation,

$$\Gamma_i = \oint_C J_i \, \mathrm{d}x_i \tag{6.6}$$

where $\mathrm{d}\vec{x}$ is the differential line element vector along C. By convention, the contour is taken in the counter-clockwise sense. $\vec{J} \cdot \mathrm{d}\vec{x}$ implies that the circulation involves the component of J tangent to the curve.

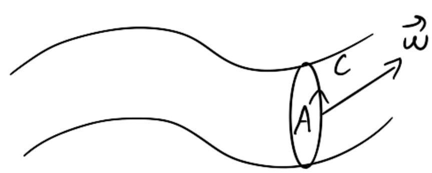
If $\vec{J}$ is taken to be the velocity vector, we call this the circulation and denote $\Gamma_{\vec{u}} = \Gamma$.

$$\Gamma = \oint_C \vec{u} \cdot \mathrm{d}\vec{x} \tag{6.7}$$

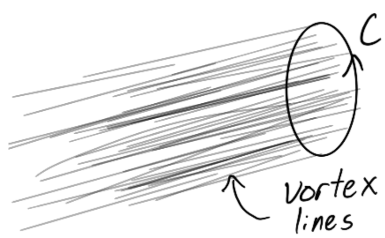
The flow's circulation is closely related to its vorticity in an integral sense. To see this, we can use Stoke's theorem to rewrite this line integral in the form of an area integral involving the curl of the vector field dotted with the area's normal vector:

$$\begin{aligned} \Gamma &= \oint_C \vec{u} \cdot \mathrm{d}\vec{x} \qquad \swarrow \quad \text{applying Stoke's theorem} \\ &= \int_A (\nabla \times \vec{u}) \cdot \hat{n} \, \mathrm{d}A \\ &= \int_A \vec{\omega} \cdot \hat{n} \, \mathrm{d}A \end{aligned} \tag{6.8}$$

The quantity $\int_A \vec{\omega} \cdot \hat{n} \ \mathrm{d}A$ is sometimes referred to as the **vortex strength** of the so-called **vortex tube** with cross-sectional area A, shown in the figure below.

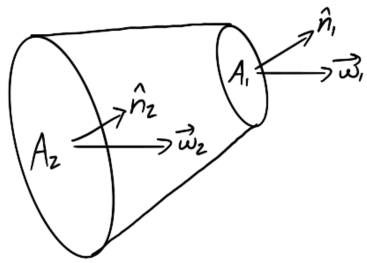


The vortex tube is a cylindrical tube in space whose surface elements are composed of vortex lines passing through the same closed curve C.



Similar to a streamline, a **vortex line** is a line in the fluid that is everywhere tangent to the vorticity vector.

It can be important to note that the strength of the vector vorticity is not constant along a vortex line, in the same way that velocity is not (necessarily) constant along a streamline. *But*, the strength of a vortex tube is constant along the vortex tube, in the same way that the transport between two streamlines is constant in compressible flow. Because of this property, then, considering the following vortex tube, to make the strength constant we must



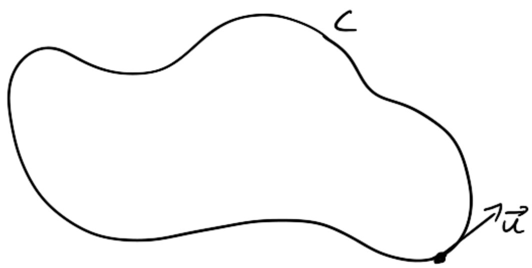
enforce

$$\int_{A_1} \vec{\omega}_1 \cdot \hat{n}_1 \, \mathrm{d}A_1 = \int_{A_2} \vec{\omega}_2 \cdot \hat{n}_2 \, \mathrm{d}A_2 \,. \tag{6.9}$$

6.3 Kelvin's Circulation Theorem

This is a statement of the conservation of circulation under certain conditions.

Consider a closed contour C drawn in the fluid that moves with the fluid such that the motion of the fluid elements on the contour determine its subsequent location and shape.



The velocity vector indicates a fluid element on the contour moving with the fluid velocity at that point. One can think of the contour as being composed of a "pearl necklace" of fluid elements (pearls) that moves and deforms in a way defined by the individual motion of the pearls.

Now consider the time rate of change of the circulation on the contour:

$$\begin{aligned} \frac{\mathrm{D}\Gamma}{\mathrm{D}t} &= \frac{\mathrm{D}}{\mathrm{D}t} \oint_C \vec{u} \cdot \mathrm{d}\vec{x} \\ &= \underbrace{\oint_C \frac{\mathrm{D}\vec{u}}{\mathrm{D}t} \cdot \mathrm{d}\vec{x}}_{\textcircled{1}} + \underbrace{\oint_C \vec{u} \cdot \frac{\mathrm{D}\,\mathrm{d}\vec{x}}{\mathrm{D}t}}_{\textcircled{2}} \end{aligned} \tag{6.10}$$

where ① accounts for change in time of $\vec{u}$ along the contour and ② accounts for change due to changing contour position and shape.

Consider term ②:

If the line element $d\vec{x}$ is moving with the fluid, it stretches and rotates depending on the velocity difference between its endpoints. That is, $\frac{\mathrm{D}d\vec{x}}{\mathrm{D}t} = \delta\vec{u}$. As this distance $d\vec{x} \to 0$, this can be written as:

$$\begin{aligned}
\oint_C \vec{u} \cdot \frac{\mathrm{D}\, d\vec{x}}{\mathrm{D}t} &= \oint_C \vec{u} \cdot d\vec{u} \\
&= \frac{1}{2} \oint_C d|\vec{u}|^2 \\
&= 0
\end{aligned} \tag{6.11}$$

This integral vanishes, as it is the integral of a perfect differential around a closed path!

With term ②=0, our expression for $\frac{\mathrm{D}\Gamma}{\mathrm{D}t}$ reduces to:

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} = \oint_C \frac{\mathrm{D}\vec{u}}{\mathrm{D}t} \cdot d\vec{x}\,. \tag{6.12}$$

Note that we have not yet specified which velocity we are using to define the circulation. It is convenient for our purposes here to use the absolute velocity seen in the inertial frame i.e.

$$\frac{\mathrm{D}\left(\Gamma\right)_{\mathrm{inertial}}}{\mathrm{D}t} = \oint_C \frac{\mathrm{D}\left(\vec{u}\right)_{\mathrm{inertial}}}{\mathrm{D}t} \cdot d\vec{x}\,. \tag{6.13}$$

We can substitute for $\frac{\mathrm{D}(\vec{u})_{\mathrm{inertial}}}{\mathrm{D}t}$ using the momentum equation. Here, for simplicity, we will assume that the viscosity coefficients can be taken to be constants, but you could continue to carry through the terms that depend on the spatial gradients of μ and λ if you wanted or needed it to. We will further assume an incompressible flow. Under these assumptions the momentum equation derived in Chapter 2 is

$$\rho\frac{\mathrm{D}\left(\vec{u}\right)_{\mathrm{inertial}}}{\mathrm{D}t} = -\nabla P + \rho\vec{F} + \mu\nabla^2\vec{u}_{\mathrm{inertial}}. \tag{6.14}$$

We will further assume that the body force per unit mass, F, can be derived from a scalar potential, as in the case of gravity, $\vec{F} = -\nabla\Phi_g$. Note

$$\begin{aligned}\oint_C \vec{F} \cdot \mathrm{d}\vec{x} &= -\oint_C \nabla\Phi_g \cdot \mathrm{d}\vec{x} \\ &= -\oint_C \mathrm{d}\Phi_g \,\mathrm{d}\vec{x} \\ &= 0.\end{aligned} \tag{6.15}$$

The integral vanishes because this is again the integral of a perfect differential around a closed path.

Then, substituting for $\frac{\mathrm{D}(\vec{u})_{\text{inertial}}}{\mathrm{D}t}$ using Eqn 6.14 into the expression for $\frac{\mathrm{D}(\Gamma)_{\text{inertial}}}{\mathrm{D}t}$ (Eqn 6.13) yields:

$$\boxed{\frac{\mathrm{D}\left(\Gamma\right)_{\text{inertial}}}{\mathrm{D}t} = -\oint_C \frac{\nabla P}{\rho} \cdot \mathrm{d}\vec{x} + \nu \oint_C \nabla^2 \vec{u}_{\text{inertial}} \cdot \mathrm{d}\vec{x}} \tag{6.16}$$

This is a statement of Kelvin's circulation theorem!

Now, it is convenient for physical interpretation to write:

- $\nabla P \cdot \mathrm{d}\vec{x}$ as $\mathrm{d}P$ i.e. as an incremental change in pressure along the contour element $\mathrm{d}\vec{x}$
- $\nabla^2 \vec{u}$ as $\nabla^2 \vec{u} = \nabla(\nabla \cdot \vec{u}) - \nabla \times \vec{\omega} = -\nabla \times \vec{\omega}$ if assuming incompressible flow.

With these substitutions, we can write Kelvin's circulation theorem as

$$\boxed{\frac{\mathrm{D}\Gamma_a}{\mathrm{D}t} = -\oint_C \frac{\mathrm{d}P}{\rho} - \nu \oint_C (\nabla \times \vec{\omega}) \cdot \mathrm{d}\vec{x}} \tag{6.17}$$

where we have replaced the label "inertial" with the label 'a', which denotes this is what we call the **absolute circulation**. We will later see that the absolute circulation is composed of the sum of the so-called relative circulation (the circulation seen inside the rotating frame) and the so-called planetary circulation (the circulation arising from the rotating of the coordinate frame) and in this way is equivalent to the circulation seen in the inertial frame.

Equation 6.17 says that for any closed contour C moving with the fluid, circulation can be produced or destroyed in two ways. The first is the baroclinic production term that arises when surfaces of density and pressure do not coincide (more coming on this shortly...). The second is through frictional effects that cause vorticity to diffuse through the walls of the vortex tube.

Before discussing these processes in more detail, let us first consider the **conditions under which circulation is conserved**:

- **If** $\rho = \rho(P)$ only, i.e. density is a function of pressure only, then the surfaces of constant density and pressure coincide. Such a fluid is called **barotropic** and the simplest example is a fluid of constant density. In this case, $\frac{\mathrm{d}P}{\rho(P)}$ is a perfect differential, so the contour integral vanishes.

- **and if** $\nu \approx 0$, i.e. friction can be neglected* (*note that this only needs to be true on the contour!)

then the circulation is conserved. As we will see, this is a powerful constraint that governs certain types of large-scale flows in the ocean and atmosphere.

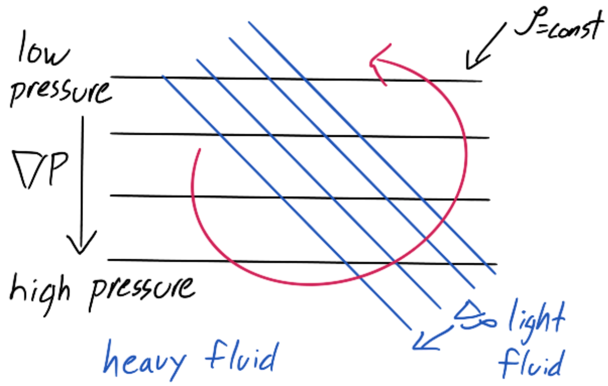
6.3.1 Baroclinic Production

We call the $-\oint_C \frac{\mathrm{d}P}{\rho}$ term on the R.H.S. of Kelvin's circulation theorem the baroclinic production term or sometimes simply the baroclinic term. It will be non-zero whenever the pressure and density surfaces do not coincide (such a fluid is called **baroclinic**).

Mathematically, this term can be written in a few different ways (the latter term is obtained by an application of Stoke's Theorem to the closed contour interval):

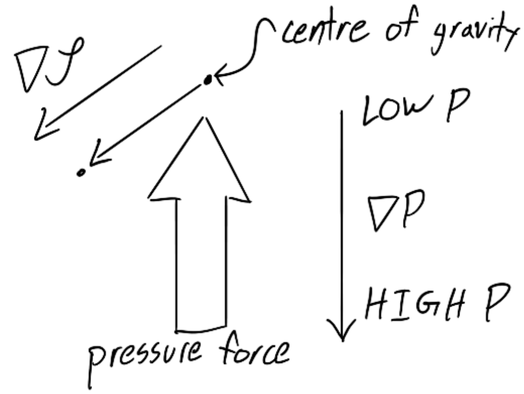
$$-\oint_C \frac{\mathrm{d}P}{\rho} = -\oint_C \frac{\nabla P}{\rho} \cdot \mathrm{d}\vec{x} = \int_A \frac{\nabla \rho \times \nabla P}{\rho^2} \cdot \hat{n}\, \mathrm{d}A \,. \tag{6.18}$$

Physically, this term represents the generation of vorticity and circulation due to a tendency of density surfaces to "slump" in the presence of a pressure gradient when these two surfaces are not aligned. This tendency can be illustrated by the scenario illustrated below. Here, there is a pressure gradient whose direction is being imposed by gravity. The pressure gradient is thus directed purely downward. We then consider a density gradient that involves a component arising from density increasing to the left (say due to uneven heating from left to right).



The gradients of the pressure and density fields produce a tendency for the heavy fluid on the left to sink and the light fluids on the right to rise. This induces a positive (counter-clockwise) circulation of the fluid indicated in red.

Another way to look at this mechanism for the generation of circulation is to examine a fluid parcel whose centre of gravity is displaced to the left by the presence of a density gradient as illustrated below.



Examining torques around the centre of gravity shows that the fluid will start to spin counter-clockwise, producing positive circulation.

6.3.2 Diffusive Destruction

Physically, the diffusive destruction/consumption of vorticity is the representation of the tendency of vorticity to diffuse through the fluid such that it can diffuse across the contour C without regard for the motion of the fluid. The diffusive effect lowers the circulation in the region enclosed by C even though no fluid can cross C (by definition). Mathematically, this term can be written as:

$$\begin{aligned} -\nu \oint\limits_C (\nabla \times \vec{\omega}) \cdot \mathrm{d}\vec{x} &= -\nu \int\limits_A (\nabla \times (\nabla \times \vec{\omega})) \cdot \hat{n}\,\mathrm{d}A \qquad \text{using Stoke's Theorem} \\ &= -\nu \int\limits_A \left(-\nabla^2 \vec{\omega} + \nabla(\nabla \cdot \vec{\omega})\right) \cdot \hat{n}\,\mathrm{d}A \qquad \text{using a vector identity} \\ &= \nu \int\limits_A \nabla^2 \vec{\omega} \cdot \hat{n}\,\mathrm{d}A \qquad \text{assuming the flow is incompressible} \end{aligned} \tag{6.19}$$

This last form best illustrates the diffusive character of this term.

Collectively, these mathematical manipulations let us write Kelvin's circulation theorem either with line or area integrals:

$$\boxed{\frac{\mathrm{D}\Gamma_a}{\mathrm{D}t} = -\oint_C \frac{\nabla P}{\rho} \cdot \mathrm{d}\vec{x} - \nu \oint_C (\nabla \times \omega) \cdot \mathrm{d}\vec{x}} \tag{6.20}$$

$$\boxed{\frac{\mathrm{D}\Gamma_a}{\mathrm{D}t} = \int_A \frac{\nabla\rho \times \nabla P}{\rho^2} \cdot \hat{n}\,\mathrm{d}A + \nu \int_A \nabla^2\vec{\omega} \cdot \hat{n}\,\mathrm{d}A} \tag{6.21}$$

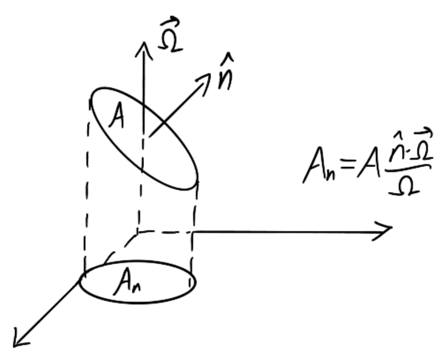
Equations 6.20 and 6.21 are both useful forms expressing the dynamics of absolute circulation in a rotating fluid.

6.4 Kelvin's Theorem in a Rotating Frame

To express Kelvin's theorem in terms of variables observed in the rotating frame, we substitute the relationship between the velocity as seen in an inertial frame and that seen in the rotating frame: $\vec{u}_{\text{inertial}} = \vec{u}_{\text{rotating}} + \vec{\Omega} \times \vec{x}$ into the definition of circulation: $\Gamma = \oint_C \vec{u} \cdot \mathrm{d}\vec{x} = \int_A \vec{\omega} \cdot \hat{n}\;\mathrm{d}A$. It can be shown that this gives:

$$(\Gamma)_{\text{inertial}} = (\Gamma)_{\text{rotating}} + 2\Omega A_n \tag{6.22}$$

where A_n is the projection of the area of the vortex tube A onto a plane perpendicular to $\vec{\Omega}$ as shown:



The first term is the contribution to the circulation from the velocity/vorticity of the flow seen in the rotating frame: the **relative circulation**. The second term represents the contribution to the circulation of the planetary vorticity (the vorticity due to the frame's rotation): the **planetary circulation**. The planetary circulation depends directly on the orientation of the vortex tube with respect to the rotation vector. It is equal to 0 if the tube is perpendicular to the rotation vector and maximized if the vortex tube is parallel to the rotation vector.

Kelvin's circulation theorem rearranged to be a statement of the sources and sinks of relative circulation is thus:

$$\boxed{\frac{\mathrm{D}\left(\Gamma\right)_{\text{rotating}}}{\mathrm{D}t} = -2\Omega\frac{\mathrm{D}A_n}{\mathrm{D}t} - \oint\limits_C \frac{\nabla P}{\rho}\cdot \mathrm{d}\vec{x} - \nu \oint\limits_C \left(\nabla \times \omega\right)\cdot \mathrm{d}\vec{x}} \tag{6.23}$$

or, equivalently, in terms of area integrals:

$$\boxed{\frac{\mathrm{D}\left(\Gamma\right)_{\text{rotating}}}{\mathrm{D}t} = -2\Omega\frac{\mathrm{D}A_n}{\mathrm{D}t} + \int\limits_A \frac{\nabla\rho\times\nabla P}{\rho^2}\cdot \hat{n}\,\mathrm{d}A + \nu \int\limits_A \nabla^2\vec{\omega}\cdot\hat{n}\,\mathrm{d}A} \tag{6.24}$$

Note that these expressions both assume a constant rotation rate.

In the special case where viscosity can be neglected and where the fluid is barotropic, the conservation of absolute circulation implies that changes in relative circulation must be balanced by changes in planetary circulation i.e.

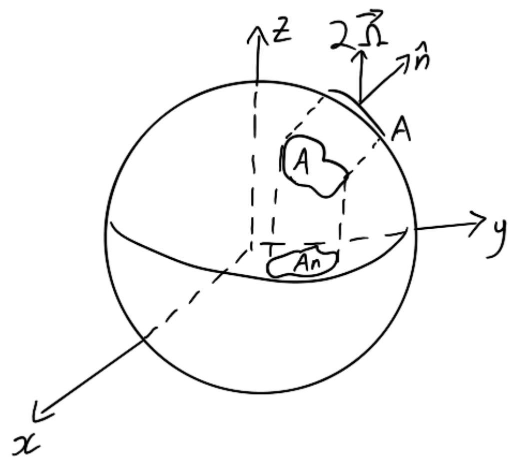
$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} = -2\Omega\frac{\mathrm{D}A_n}{\mathrm{D}t} \tag{6.25}$$

This implies that as the projected area of the vortex tube in the plane perpendicular to the rotation vector changes, the circulation seen in the rotating frame must change to compensate! This dynamic is central to the dynamics of the barotropic Rossby wave discussed next.

6.5 Example Application of Kelvin's Theorem: The Rossby Wave

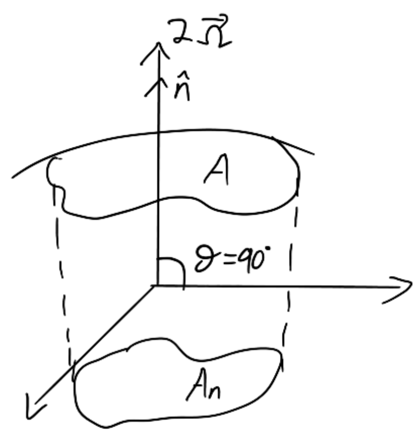
Consider a flow in the atmosphere that we idealize as being two-dimensional and horizontally non-divergent (i.e. $\vec{u} = (u, v, 0)$ and $\nabla \cdot \vec{u}_H = 0$ where $\vec{u}_H = (u, v)$). For such a flow, the horizontal area of any patch of fluid, A, will be constant with time.

If, however, this area slides on the surface of a rotating sphere (like Earth) by moving north or south, then the projected area on the plane perpendicular to the rotation vector, A_n will change.



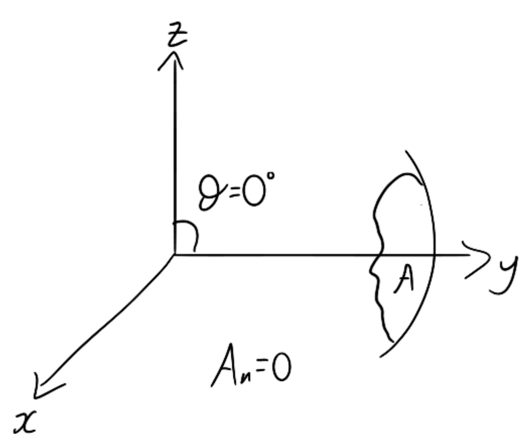
Consider two extremes.

If $\theta = 90^\circ$ as illustrated below:



then $A = A_n$ so the contribution of planetary vorticity to the vortex tube strength is maximum. The penetration of A of the lines of vorticity associated with the planetary vorticity is maximized.

Conversely, if $\theta = 0^\circ$, as illustrated here:



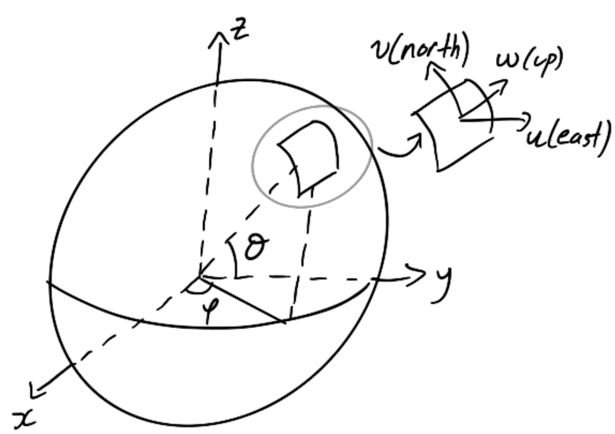
then $A_n = 0$ and the contribution of planetary vorticity to the vortex tube strength is zero. There is no penetration of A of the lines of vorticity associated with the planetary vorticity.

The projected area satisfies the relation $A_n = A\sin\theta$ where θ represents the angle of the radial of the area to the plane perpendicular to the rotation vector. For a barotropic, inviscid fluid, Kelvin's circulation theorem thus becomes:

$$\begin{aligned}\frac{\mathrm{D}\Gamma}{\mathrm{D}t} &= -2\Omega\frac{\mathrm{D}A_n}{\mathrm{D}t}\\ &= -2\Omega A\cos\theta\frac{\mathrm{D}\theta}{\mathrm{D}t}\end{aligned} \tag{6.26}$$

noting that the area A is constant in time owing to the assumption of a two-dimensional, horizontally non-divergent flow.

Now we will write $\frac{\mathrm{D}\theta}{\mathrm{D}t}$ in a standard so-called Earth-centered, Earth-fixed coordinate system. In this coordinate system, the velocity components are defined as u = the velocity to the east, v = the veloicty to the north, and w = the local upwards velocity/the velocity away from the surface as illustrated below. θ is the latitude, ϕ is the longitude, and r is Earth's radius. In this system, we assume that we can use a locally-defined Cartesian co-ordinate to describe motions on a patch of the spherical surface centred at the latitude θ. This assumption is valid if the scales of motion are large enough for rotation to be important, but small enough that a local Cartesian coordinate system is a reasonable approximation to spherical coordinates. The use of a Cartesian coordinate system is a significant geometrical convenience!



Now in this Earth-centered, Earth-fixed coordinate system:

$$\frac{\mathrm{D}\theta}{\mathrm{D}t} = \frac{u}{r}\frac{\partial\theta}{\partial\phi} + \frac{v}{r}\frac{\partial\theta}{\partial\theta} = \frac{v}{r} \tag{6.27}$$

This allows us to rewrite Kelvin's circulation theorem as:

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} = -2\Omega A\cos\theta\frac{v}{r}. \tag{6.28}$$

At the same time, we can rewrite $\frac{\mathrm{D}\Gamma}{\mathrm{D}t}$ as the area integral of the vertical component of vorticity, ξ (a consequence of the fact that the normal to the area is aligned with the vertical axis of our local coordinate system). We use the fact that the area is constant in time to simply this further:

$$\frac{\mathrm{D}\Gamma}{\mathrm{D}t} = \frac{\mathrm{D}}{\mathrm{D}t}\int_A \xi\,\mathrm{d}A = A\frac{\mathrm{D}\bar{\xi}}{\mathrm{D}t} \tag{6.29}$$

where $\bar{\xi}$ is the mean value of ξ over the area A. If we consider a sufficiently small area, then the mean becomes the value itself, so $\bar{\xi} \approx \xi$.

Equating Eqns 6.28 and 6.29 yields

$$\frac{\mathrm{D}\xi}{\mathrm{D}t} = -\frac{2v\Omega\cos\theta}{r}. \tag{6.30}$$

This is a differential statement of the vorticity induction effect that arises due to meridional (north-south) motion on a rotating sphere. A fluid element moving **northward** in this scenario will induce a **decrease** in the vertical component of its relative vorticity (relative to the rotating Earth) to conserve its absolute circulation and vice versa: the trading of relative and planetary components in action!

We can make further progress by noting that 2-dimensional, horizontally non-divergent flow can be conveniently represented in terms of a stream function, ψ, where

$$\begin{aligned} u &= -\frac{\partial\psi}{\partial y} \\ v &= \frac{\partial\psi}{\partial x}. \end{aligned} \tag{6.31}$$

The vertical component of vorticity in terms of the stream function is

$$\xi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial\psi}{\partial y}\right) = \nabla^2\psi. \tag{6.32}$$

We can now rewrite the differential statement of Kelvin's circulation theorem, Eqn 6.30, as a function of ψ:

$$\frac{\mathrm{D}}{\mathrm{D}t}\nabla^2\psi + \beta\frac{\partial\psi}{\partial x} = 0 \tag{6.33}$$

where $\beta = -\frac{2\Omega\cos\theta}{r}$. Expanding the material derivative in the above yields yields:

$$\frac{\partial}{\partial t}\nabla^2\psi + \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}\nabla^2\psi - \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}\nabla^2\psi + \beta\frac{\partial \psi}{\partial x} = 0. \tag{6.34}$$

Equation 6.34 is a statement of the conservation of circulation for this scenario and is equivalent to the conservation of vorticity for a 2-dimensional, barotropic, horizontally non-divergent flow. This is the equation governing the dynamics of the barotropic Rossby wave.

Aside: The β-plane approximation:

We have made the substitution:

$$\beta = \frac{2\Omega\cos\theta}{r} = \frac{1}{r}\frac{\partial}{\partial \theta}\left(2\Omega\sin\theta\right) = \frac{\mathrm{D}f}{\mathrm{D}y} \tag{6.35}$$

where $f = 2\Omega\sin\theta$ is the local normal component of the planetary vorticity, called the Coriolis parameter. f varies from -2Ω at the south pole to 2Ω at the north pole. Its variation is important, but relatively slow compared to the length scale over which atmospheric and oceanic motions vary, so it is often possible to assume that it is nearly constant locally.

The northward gradient of f is given by β. The presence of $\beta\frac{\partial\psi}{\partial x}$ in Equations 6.33 and 6.34 is called the **β-effect**. The β-effect is name of the vorticity induction effect on a rotating planet i.e. the phenomenon that motion of the fluid in the direction of the gradient of the planetary vorticity produces relative vorticity.

The manifestation of this aspect of the sphericity of the Earth in an otherwise flat, Cartesian geometry is the **β-plane approximation**. This approximation was introduced by Rossby in 1939 when he derived the voritcity wave that now bears his name.

6.5.1 The Rossby wave solution

The wave solution that satisfies Equations 6.33 and 6.34 is referred to as a barotropic Rossby wave. This is an important large-scale wave motion in the atmosphere and ocean that exists due to a restoring force supplied by the background meridional planetary vorticity gradient.

To gain some insight into Rossby wave dynamics, we look (using the power of hindsight) for plane-wave solutions to the nonlinear Equation 6.33 of the form:

$$\psi = A\cos(kx + ly - \sigma t) \tag{6.36}$$

where A is the wave amplitude, k is the zonal wave number, l is the meridional wave number, and σ is the wave frequency. With this solution, the nonlinear advection terms of the vorticity of the fluid by its own motion ($\frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\nabla^2\psi$ and $-\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x}\nabla^2\psi$) vanish. Together, they can be written as $J(\psi, \nabla^2\psi) = J(\psi, -(k^2+l^2)\psi) = 0$, where J is the Jacobian.

Equation 6.33 is now simplified and the trial solution can be substituted:

$$\begin{aligned} \frac{\partial}{\partial t}\nabla^2\psi + \beta\frac{\partial\psi}{\partial x} &= 0 \\ A\sin(kx + ly - \sigma t)\left(-\sigma(k^2+l^2) + \beta k\right) &= 0. \end{aligned} \tag{6.37}$$

The only non-trivial ($A \neq 0$) solution requires $-\sigma(k^2+l^2) + \beta k = 0$, which produces the Rossby wave dispersion relation:

$$\sigma = -\frac{\beta k}{k^2+l^2}. \tag{6.38}$$

Examining the phase speed of the wave in the x direction, c_x:

$$\begin{aligned} c_x &= \frac{-\partial(\text{phase})/\partial t}{\partial(\text{phase})/\partial x} = \frac{-\partial(kx+ly-\sigma t)/\partial t}{\partial(kx+ly-\sigma t)/\partial x} = \frac{\sigma}{k} \\ &= -\frac{\beta}{k^2+l^2}. \end{aligned} \tag{6.39}$$

This is always negative! Thus Rossy waves are unusual in the sense that the crests and troughs in the wave always move westward regardless of the orientation of the group velocity.

6.6 The Vorticity Equation

Recall that although vorticity is a vector, Kelvin's theorem and the general equation for the rate of change of circulation are only scalar equations. Hence, much of the vectoral character of the vorticity dynamics is not revealed in this conservation law (although is also the reason that the results is so simple and elegant). To consider this more fully, we will now develop the equation for the **conservation of vorticity**.

We start by exploiting the vector identity:

$$\vec{u}\cdot\nabla\vec{u} = \vec{\omega}\times\vec{u} + \nabla\frac{|\vec{u}|^2}{2} \tag{6.40}$$

We can now expand the material derivative of the Navier-Stokes equation and make this substitution for the non-linear term. Here, we are assuming constant viscosity coefficients, but you don't have to...

$$\frac{\partial \vec{u}}{\partial t} + \vec{\omega}_a \times \vec{u} = -\nabla \frac{|\vec{u}|^2}{2} + \vec{g} - \frac{\nabla P}{\rho} + \nu \nabla^2 \vec{u} + \left(\nu + \frac{\lambda}{\rho}\right) \nabla \left(\nabla \cdot \vec{u}\right) \tag{6.41}$$

Where we have defined $\vec{\omega}_a = 2\vec{\Omega} + \vec{\omega}$ as the absolute vorticity. Taking the curl of this term will give us an equation for vorticity. It will simplify further by nothing that the curl of a gradient is zero. Furthermore, as $\vec{g}$ is linear, its curl will also vanish.

$$\frac{\partial \vec{\omega}}{\partial t} + \nabla \times \left(\vec{\omega}_a \times \vec{u}\right) = \frac{1}{\rho^2}(\nabla \rho \times \nabla P) + \nu \nabla^2 \vec{\omega} \tag{6.42}$$

We can use vector identities to rewrite the curl of a cross product,

$$\nabla \times \left(\vec{\omega}_a \times \vec{u}\right) = \vec{u} \cdot \nabla \vec{\omega}_a - \left(\vec{\omega} \cdot \nabla\right) \vec{u} + \vec{\omega}_a \left(\nabla \cdot \vec{u}\right) - \vec{u} \left(\nabla \cdot \vec{\omega}\right) \tag{6.43}$$

Where the last term vanishes because vorticity is always non-divergent (divergence of a curl is zero). Assuming that planetary rotation is constant, $\frac{\partial \vec{\omega}_a}{\partial t} = \frac{\partial \vec{\omega}}{\partial t}$ and the vorticity equation becomes:

$$\frac{\mathrm{D}\vec{\omega}_a}{\mathrm{D}t} = \underbrace{\left(\vec{\omega}_a \cdot \nabla\right) \vec{u}}_{①} - \underbrace{\vec{\omega}_a \left(\nabla \cdot \vec{u}\right)}_{②} + \underbrace{\frac{1}{\rho^2}(\nabla \rho \times \nabla P)}_{③} + \underbrace{\nu \nabla^2 \vec{\omega}}_{④} \tag{6.44}$$

This is the vorticity equation expresion the conservation of absolute vorticity (which, as a remind is a sum of the relative vorticity $\nabla \times \vec{u}$ and the planetary vorticity $2\vec{\Omega}$). There are four sources and sink of absolute vorticity. ③ and ④ are familiar from Kelvin's circulation theorem, but ① and ② require more thought. We will state them now and derive an interpretation immediately after.

① = Vortex tube stretching

② = Vortex tube tilting

③ = Baroclinic production of vorticity

④ = Viscous diffusion of vorticity

To interpret (1) and (2), consider a scenario wherein $\vec{\omega}_a$ is parallel to the z-axis. That is, $\vec{\omega}_a = \hat{k}\omega_a$. As $\vec{\omega}_a$ only has a component in the z direction, the dot product reduces to one term. (1) and (2) become:

$$\begin{aligned}(\vec{\omega}_a \cdot \nabla)\,\vec{u} - \vec{\omega}_a\,(\nabla \cdot \vec{u}) &= \omega_a \frac{\partial}{\partial z}\left(u\hat{i} + v\hat{j} + w\hat{k}\right) - \omega_a \hat{k}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \\ &= \underbrace{\omega_a \frac{\partial u}{\partial z}\hat{i}}_{\text{(a)}} + \underbrace{\omega_a \frac{\partial v}{\partial z}\hat{j}}_{\text{(b)}} - \underbrace{\omega_a\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)\hat{k}}_{\text{(c)}}\end{aligned} \tag{6.45}$$

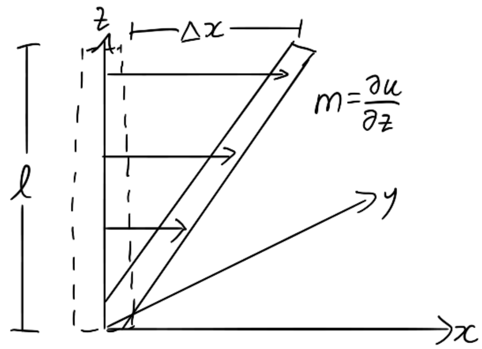
In this way, we can interpret these terms as describing three components that contribute to the rate of change of the absolute vorticity.

- (a) Says $\vec{\omega}_a$ increases in the x direction as the shear $\frac{\partial u}{\partial z}$ tips the vorticity vector in the x direction.

  In an infinitesimal interval Δt, the change in the vorticity vector in the x direction from these terms would be:

$$\begin{aligned}\frac{\Delta \omega_{a,x}}{\Delta t} &= \omega_a \frac{\partial u}{\partial z} \\ \frac{\Delta \omega_{a,x}}{\omega_a} &= \frac{\partial u}{\partial z}\Delta t\end{aligned} \tag{6.46}$$

  For a line element that moves with the fluid and which is originally parallel to the z-axis, this term represents a tilting of the line element by the shear that produces a displacement Δx parallel to the x-axis.



Comparing $\frac{\Delta\omega_{a,x}}{\omega_a} = \frac{\partial u}{\partial z}\Delta t$ to $\Delta a = \frac{\partial u}{\partial z}\Delta t l$ shows that $\frac{\Delta\omega_{a,x}}{\omega_a} = \frac{\Delta x}{l}$.So the production of vorticity parallel to the x-axis can be interpreted as a simple tilting of the vorticity vector (originally parallel to the z-axis) in the x-direction by the shear.

- (b) is interpreted similarly – the production of vorticity parallel to the y-axis due to the tilting of the vorticity vector in the y-direction by the shear.

- (c). We note that since $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$ is the horizontal divergence of of $\vec{u}$, it can be interpreted as $\frac{1}{A}\frac{\mathrm{D}A}{\mathrm{D}t}$, where A is the area perpendicular to tge vortex line associated with $\vec{\omega}_a = \omega_a\hat{k}$, so:

$$-\omega_a\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = -\frac{\omega_a}{A}\frac{\mathrm{D}A}{\mathrm{D}t} \tag{6.47}$$

If this were the only effect operating, the the z component of the vorticity equation would reduce to:

$$\frac{\mathrm{D}\omega_a}{\mathrm{D}t} = -\frac{\omega_a}{A}\frac{\mathrm{D}A}{\mathrm{D}t} \implies \frac{\mathrm{D}}{\mathrm{D}t}(\omega_a A) = 0 \tag{6.48}$$

Fo the vortex tube strength would be conserved. In this way, (c) can be interpreted as a change in vorticity in the direction parallel to the vortex line resulting from an increase or decrease in the area A of the associated vortex tube. A reduction of A will concentrate the vortex lines and increase the vorticity to conserve vortex tube strength. Likewise, an increase in A will disperse vortex lines and decrease vorticity to conserve vortex tube strength.

Thus, these terms represent:

(1) – The vortex change in the direction perpendicular to the vortex line due to vortex line tilting by the shear in the direction perpendicular to the vortex line.

(2) – Vortex change in the direction parallel to the vortex line due to vortex line stretching in the direction parallel to the vortex line.

6.7 Potential Vorticity and Ertel's Theorem

The vorticity equation describes vector dynamics of the vorticity in a clear way. However, it is not technically a conservation statement. This is because the identified sources and sinks are both external (pressure force and viscous stresses) and from the interaction of the vorticity and velocity fields. So it is a function of the vorticity field whose time evolution we are trying to prescribe!

We will work with the vorticity equation further and define a new conserved quantity – the potential vorticity (PV) – and further derive its conservation statement. This statement of PV conservation dates back to Ertel in 1942, although Rossby published a slightly less general derivation in 1940.

We will start with the vorticity equation, divide by ρ, and use mass conservation to eliminate the divergence term on the right-hand-side. We can then write this expression in either component or vector form:

$$\begin{aligned}
\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\omega_{a,i}}{\rho}\right) &= \frac{\omega_{a,j}}{\rho}\frac{\partial u_i}{\partial x_j} + \varepsilon_{ijk}\frac{1}{\rho^3}\frac{\partial \rho}{\partial x_j}\frac{\partial P}{\partial x_k} + \frac{\nu}{\rho}\nabla^2\omega_i \\
\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\vec{\omega}_a}{\rho}\right) &= \underbrace{\left(\frac{\vec{\omega}_a}{\rho}\cdot\nabla\right)\vec{u}}_{\text{Vortex tilting}} + \underbrace{\frac{\nabla\rho\times\nabla P}{\rho^3}}_{\text{baroclinic production}} + \underbrace{\frac{\nu}{\rho}\nabla^2\vec{\omega}}_{\text{viscous diffusion}}
\end{aligned} \tag{6.49}$$

We will now assume that there is a scalar property of the fluid, λ, that satisfies an equation of the form:

$$\frac{\mathrm{D}\lambda}{\mathrm{D}t} = \mathcal{S} \tag{6.50}$$

Where $\mathcal{S}$ is the source term for the scalar field λ.

We will now embark on some mathematical gymnastics. Hold on to your hats.

$$\begin{aligned}
\vec{\omega}_a\cdot\frac{\mathrm{D}}{\mathrm{D}t}\nabla\lambda &= \omega_{a,i}\frac{\mathrm{D}}{\mathrm{D}t}\frac{\partial\lambda}{\partial x_i} && \text{Expanding the material derivative,} \\
&= \omega_{a,i}\left(\frac{\partial}{\partial t} + u_j\frac{\partial}{\partial x_j}\right)\frac{\partial\lambda}{\partial x_i} && \text{Expanding and using the product rule,} \\
&= \omega_{a,i}\frac{\partial}{\partial x_i}\left(\frac{\partial\lambda}{\partial t} + u_j\frac{\partial\lambda}{\partial x_j}\right) - \omega_{a,i}\frac{\partial u_j}{\partial x_i}\frac{\partial\lambda}{\partial x_j} && \text{Combing the total derivative,} \\
&= \omega_{a,i}\frac{\partial}{\partial x_i}\frac{\mathrm{D}\lambda}{\mathrm{D}t} - \omega_{a,i}\frac{\partial u_j}{\partial x_i}\frac{\partial\lambda}{\partial x_j} && \text{Substituting } \mathcal{S}, \\
&= \omega_{a,i}\frac{\partial}{\partial x_i}\mathcal{S} - \omega_{a,i}\frac{\partial u_j}{\partial x_i}\frac{\partial\lambda}{\partial x_j}
\end{aligned} \tag{6.51}$$

Dividing by ρ and rewriting in vector notation,

$$\frac{\vec{\omega}_a}{\rho} \cdot \frac{\mathrm{D}}{\mathrm{D}t} \nabla \lambda = \frac{\vec{\omega}_a}{\rho} \cdot \nabla \mathcal{S} - \left(\left(\frac{\vec{\omega}_a}{\rho} \cdot \nabla \right) \vec{u} \right) \cdot \nabla \lambda \tag{6.52}$$

We notice that the final term of Equation 6.52 looks like the RHS of the vorticity equation dotted with $\nabla\lambda$. So, Equation 6.49 dotted with $\nabla\lambda$ is:

$$\nabla \lambda \cdot \frac{\mathrm{D}}{\mathrm{D}t} \left(\frac{\vec{\omega}_a}{\rho} \right) = \left(\left(\frac{\vec{\omega}_a}{\rho} \cdot \nabla \right) \vec{u} \right) \cdot \nabla \lambda + \frac{\nabla \rho \times \nabla P}{\rho^3} \cdot \nabla \lambda + \frac{\nu}{\rho} \nabla^2 \vec{\omega} \cdot \nabla \lambda \tag{6.53}$$

Adding Equation 6.52 and Equation 6.53,

$$\boxed{\frac{\mathrm{D}}{\mathrm{D}t} \left(\frac{\vec{\omega}_a \cdot \nabla \lambda}{\rho} \right) = \nabla \lambda \cdot \frac{\nabla \rho \times \nabla P}{\rho^3} + \frac{\vec{\omega}_a}{\rho} \cdot \nabla \mathcal{S} + \frac{\nu}{\rho} \nabla \lambda \cdot \nabla^2 \vec{\omega}} \tag{6.54}$$

This is **Ertel's Theorem**.

Specifically, Ertel's theorem recognizes the result of the following conditions placed on Equation 6.54:

- **If** λ is a conservative quantity following the fluid motion so that $\mathcal{S} = 0$,
- **And if** The motion is inviscid (friction can be neglected),
- **And either** the fluid is barotropic ($\nabla \rho \times \nabla P = 0$) **or** λ is a thermodynamic function of P and ρ,

Then:

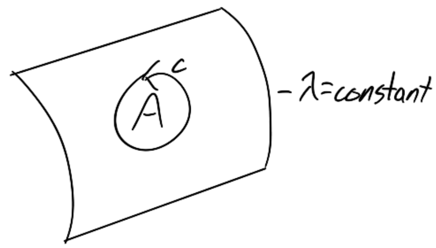
$$\frac{\mathrm{D}}{\mathrm{D}t} \left(\frac{\vec{\omega}_a \cdot \nabla \lambda}{\rho} \right) = \frac{\mathrm{D}q}{\mathrm{D}t} = 0 \tag{6.55}$$

Where $q = \vec{\omega}_a \cdot \nabla \lambda / \rho$ is the potential vorticity, PV. Under these assumptions, potential vorticity is conserved following the fluid motion.

It is hard to exaggerate the importance of this theorem for understanding the large-scale dynamics of both the atmosphere and the ocean. Indeed, in certain limiting and natural approximations we will explore later, it becomes the governing equation of motion. Cyclonic dynamics, synoptic-scale eddies in the ocean, and the very structure of oceanic gyres are all based on potential vorticity dynamics.

6.8 The relation between Ertel's Theorem and Kelvin's Theorem

It is worthwhile spending a little time trying to understand the physical basis for Ertel's theorem and what it means. This is best accomplished by connecting it to Kelvin's theorem, Consider a baroclinic, inviscid fluid for which λ is conserved ($\mathcal{S} = 0$). We Will consider a contour C enclosing an area A on a surface of constant λ:



If C is a contour moving with the fluid and if λ is a conserved quantity (which implies that the surface of constant λ also moves with the fluid). Then **the contour C remains in the same surface as the fluid moves, for all time**. Recall that Kelvin's theorem says that the equation for the conservation of absolute circulation is (assuming an inviscid fluid):

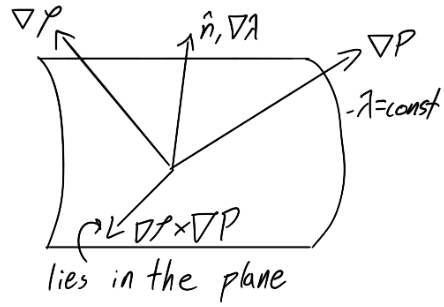
$$\frac{\mathrm{D}\Gamma_a}{\mathrm{D}t} = \int_A \frac{\nabla\rho \times \nabla P}{\rho^2} \cdot \hat{n}\,\mathrm{d}A \tag{6.56}$$

Where $\hat{n}$ is normal to the area A and is hence normal to the surface of constant λ.

If λ is a function of P and ρ, we can write:

$$\nabla\lambda(P,\rho) = \frac{\partial\lambda}{\partial\rho}\nabla\rho + \frac{\partial\lambda}{\partial P}\nabla P \tag{6.57}$$

It follows that $\nabla\lambda$ must lie in the plane of the vectors $\nabla\rho$ and ∇P. $\nabla\rho \times \nabla P$ must be perpendicular to both $\nabla\rho$ and ∇P (and thus $\nabla\lambda$) so must lie in the surface of constant λ, shown on the following page.



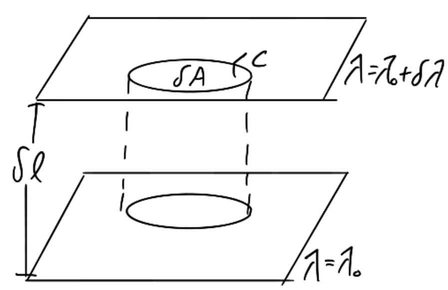
Thus, the integrand $\nabla\rho \times \nabla P \cdot \hat{n}$ is identically 0! We have chosen a contour C for which the baroclinic term has made no contribution even though the fluid is baroclinic! Of course, if the fluid were barotropic, then this term would certainly be zero. Either way, the absolute circulation associated with this contour is conserved,

$$\frac{\mathrm{D}\Gamma_a}{\mathrm{D}t} = 0 \tag{6.58}$$

Now, we will shrink the contour C until the area A is the infinitesimal area δA. The absolute circulation is then:

$$\Gamma_a = \vec{\omega}_a \cdot \hat{n}\delta A \tag{6.59}$$

Now, consider an adjacent λ surface where lambda is also constant, with a value of $\lambda = \lambda_0 + \delta\lambda$. These surfaces are separated by a distance δl:



The mass contained in the infinitesimal cylinder with upper surface δA enclosed by the contour C is

$$\delta m = \rho \delta l \delta A \tag{6.60}$$

Since $\delta\lambda = \nabla\lambda \cdot \hat{n}\delta l$ and $\hat{n} = \nabla\lambda / |\nabla\lambda|$, it follows that $\delta l = \delta\lambda / |\nabla\lambda|$. Combining with the above equation,

$$\delta A = \frac{\delta m}{\rho\delta\lambda}|\nabla\lambda| \tag{6.61}$$

Substituting into the absolute circulation equation,

$$\Gamma_a = \vec{\omega}_a \cdot \hat{n}\frac{\delta m}{\rho\delta\lambda}|\nabla\lambda| = \frac{\vec{\omega}_a \cdot \nabla\lambda}{\rho}\left(\frac{\delta m}{\delta\lambda}\right) = q\left(\frac{\delta m}{\delta\lambda}\right) \tag{6.62}$$

Thus, since the circulation Γ_a, δm, and $\delta\lambda$ are conserved following the fluid motion, the potential vorticity q must also be conserved.

Some observations:

- Ertel's theorem is a differential statement of Kelvin's theorem where the Kelvin contour is chosen to lie in a surface for which the baroclinic vector $\nabla\rho \times \nabla P$ lies in the surface and makes no contribution to the change in circulation.

- From the visualization of an infinitesimal cylinder between two surface of constant λ, if the lambda surfaces are pried apart (decreasing $|\nabla\lambda|$) then the area contained in the contour C must shrink.

- Further, the consequence of this vortex tube stretching is that the absolute vorticity ω_a must increase (in the direction of the normal to the surface) to keep Γ_a constant. *So as the gradient of λ decreases, the part of $\vec{\omega}_a/\rho$ parallel to the gradient of λ must increase.*

- It is in this sense that q is a "potential" vorticity. Vorticity can be extracted/stored in the stretching apart/-compression of the spacing of the λ surfaces as potential vorticity.

- In large-scale flows for which planetary vorticity is ever-present, changes in the spacing of the λ surfaces can produce relative vorticity!

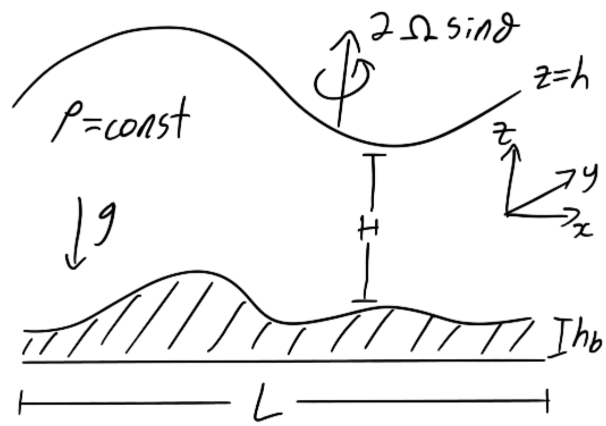
6.9 Conservation of Potential Vorticity: Examples

Example 6.1 (2-Dimensional, barotropic, inviscid motion)**.** Suppose the motion of the fluid is strictly two-dimensional ($w = \frac{\mathrm{D}z}{\mathrm{D}t} = 0$). Furthermore, if the fluid is barotropic, we can choose λ to be the coordinate z ($\frac{\mathrm{D}z}{\mathrm{D}t}$ looks like $\frac{\mathrm{D}\lambda}{\mathrm{D}t} = \mathcal{S}$ with $\mathcal{S} = 0$). Then the potential vorticity is:

$$q = \frac{\vec{\omega}_a}{\rho} \cdot \nabla\lambda = \frac{\vec{\omega}_a}{\rho} \cdot \nabla z = \frac{\vec{\omega}_a}{\rho} \cdot \hat{k} = \frac{\xi_a}{\rho} = \frac{\xi + f}{\rho} \tag{6.63}$$

Where ξ_a is the vertical component of the absolute vorticity, and $\xi + f$ is the vertical component of the relative vorticity plus the vertical component of the planetary vorticity. For the case when $\rho \sim$ constant, as the fluid moves northward (f increases), it spins down and develops negative vorticity (ξ decreases).

Example 6.2 (Motions described by the shallow water model)**.** Consider the motion of a shallow layer of homogeneous fluid with a constant density and negligible viscosity. This model (*the shallow water model*) is frequently used in both atmospheric and oceanic dynamics. A layered version of it can be applicable to stratified fluids as well. The model is illustrated below.



We look for equations that describe the fluid motion in the fluid layer by considering the governing equations and the boundary conditions.

Since ρ is constant, conservation of mass reduces to

$$\boxed{u_x + v_y + w_z = 0} \tag{6.64}$$

where subscripts denote partial differentiation.

On $z = h$ (the upper free surface), the motion of the fluid defines the motion of the surface:

$$\boxed{w = \frac{\mathrm{D}h}{\mathrm{D}t} \qquad @ \quad z = h} \tag{6.65}$$

On $z = h_b$ (the lower solid surface), we can use the boundary condition of no normal flow. We will assume that $\frac{\partial h_b}{\partial t} = 0$ (no earthquakes!) so:

$$\boxed{w = \vec{u} \cdot \nabla h_b \qquad @ \quad z = h_b} \tag{6.66}$$

Now we look to make simplifications that are valid for this specific problem set-up. The first approximation made in the shallow water model is that the vertical scale of the motion, $\mathcal{O}(H)$, is much less than the horizontal scale of the motion, $\mathcal{O}(L)$, i.e. $H \ll L$ (hence the name "shallow water"). This a very common scaling assumption to apply to the description of atmospheric and oceanic flows, which are essentially contained in a very thin shell of fluid on the outside of a very big planet. Under these conditions, we expect the vertical velocity to be small and, from strictly geometrical considerations, we expect:

$$\frac{w}{u} = \mathcal{O}\left(\frac{H}{L}\right) \ll 1. \tag{6.67}$$

Additionally, for for the shallow layer of constant density fluid, we also assume that the horizontal velocity is independent of depth. This turns out to be an excellent approximation if viscous boundary layers are excluded. This assumption makes it useful to consider the integral of Equation 6.64 with respect to z, which yields:

$$\begin{aligned} \int \frac{\partial w}{\partial z}\,\mathrm{d}z &= -\int (u_x + v_y)\,\mathrm{d}z \\ w(z) &= -z(u_x + v_y) + A(x, y, t) \end{aligned} \tag{6.68}$$

where $A(x, y, t)$ is a constant of integration that needs to be determined using boundary conditions.

We can solve for the constant A by applying the lower boundary condition at $z = h_b$ (Equation 6.66):

$$A = \vec{u} \cdot \nabla h_b + h_b(u_x + v_y). \tag{6.69}$$

Thus

$$w(z) = -(z - h_b)(u_x + v_y) + u\frac{\partial h_b}{\partial x} + v\frac{\partial h_b}{\partial y} \tag{6.70}$$

We must still satisfy the boundary condition at the upper free surface (Equation 6.65). At $z = h$:

$$w = \frac{\mathrm{D}h}{\mathrm{D}t} = -(h - h_b)(u_x + v_y) + u\frac{\partial h_b}{\partial x} + v\frac{\partial h_b}{\partial y} \tag{6.71}$$

Rewriting in terms of $(h - h_b)$ and assuming $\frac{\partial h_b}{\partial t} = 0$,

$$\frac{\mathrm{D}}{\mathrm{D}t}(h - h_b) = -(h - h_b)(u_x + v_y) \tag{6.72}$$

But $h - h_b = H$, so:

$$\begin{gathered} \boxed{\frac{\mathrm{D}H}{\mathrm{D}t} + H(u_x + v_y) = 0} \\ \boxed{\frac{\partial H}{\partial t} + (uH)_x + (vH)_y = 0} \end{gathered} \tag{6.73}$$

Equation 6.73 is the equivalent to the continuity equation for the shallow water model and one of the model's key governing equations. It says local changes in thickness, H, must be balanced by a horizontal divergence in the "thickness-weighted" velocity. Note the analogy between H (in the shallow water model) and ρ (in the continuity equation for a compressible fluid).

We can use this expression to eliminate the divergence term in the expression for $w(z)$ (Equation 6.70), yielding,

$$w = \left(\frac{z - h_b}{H}\right)\frac{\mathrm{D}H}{\mathrm{D}t} + \vec{u} \cdot \nabla h_b \tag{6.74}$$

Now, consider the term $(z - h_b)/H$. This is a measure of the relative height of a fluid element with respect to the bottom of the column (its fractional height relative to the total column height). This metric is referred to as *status* of the fluid element in the shallow water system. If this quantity is conserved following the flow, is an interesting candidate for the scalar function in the definition of Ertel's PV.

To evaluate the suitability of the status function as the scalar function in the definition of PV, consider $\lambda = (z - h_b)/H$ and then its time rate of change:

$$\begin{aligned} \frac{\mathrm{D}\lambda}{\mathrm{D}t} &= \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{z - h_b}{H}\right) \\ &= \frac{1}{H}\left(w - \frac{\mathrm{D}h_b}{\mathrm{D}t} - \frac{z - h_b}{H}\frac{\mathrm{D}H}{\mathrm{D}t}\right) \end{aligned} \tag{6.75}$$

But $\frac{\mathrm{D}h_b}{\mathrm{D}t} = \vec{u} \cdot \nabla h_b$ because we are assuming $\frac{\partial h_b}{\partial t} = 0$. Now, substituting w from Equation 6.74, shows

$$\frac{\mathrm{D}\lambda}{\mathrm{D}t} = 0! \tag{6.76}$$

Thus, for a fluid of constant density, the status function is a proper candidate for use in defining the potential vorticity, as $\frac{\mathrm{D}\lambda}{\mathrm{D}t} = 0$.

To formulate an expression for Ertel's PV using the status function as the scalar field λ, first consider the absolute vorticity vector for this set-up:

$$\begin{aligned} \vec{\omega}_a &= (w_y - v_z)\hat{\imath} + (u_z - w_x)\hat{\jmath} + (f + v_x - u_y)\hat{k} \\ &= w_y\hat{\imath} - w_x\hat{\jmath} + (f + v_x - u_y)\hat{k} \end{aligned} \tag{6.77}$$

where the simplifications arise because horizontal velocities are independent of z. The contributions is the $\hat{\imath}$ and $\hat{\jmath}$ directions are proportional to the vertical velocity, but are smaller by $\mathcal{O}(H/L)$ compared to the horizontal velocity terms. In the ocean, where synoptic scale eddies have a horizontal scale of $\mathcal{O}(50\text{ km})$ and a vertical scale of $\mathcal{O}(1\text{ km})$, only the term in the $\hat{k}$ direction is relevant.

Thus, for the shallow water system, the potential vorticity can be defined as:

$$\begin{aligned} q &= \frac{\vec{\omega}_a}{\rho} \cdot \nabla\lambda \\ &\simeq \frac{1}{\rho}\xi_a\hat{k} \cdot \nabla\left(\frac{z - h_b}{H}\right) \\ &=\simeq \frac{1}{\rho}\xi_a\frac{1}{H}. \end{aligned} \tag{6.78}$$

It is common to ignore the factor of constant density in this definition, leaving:

$$q = \frac{\xi_a}{H} = \frac{f + \xi}{H} \tag{6.79}$$

which must be conserved following the motion of fluid columns in the layer. Thus, as the fluid column shrinks – perhaps by being squeezed into shallower water – the total vertical component of vorticity must decrease. This is an intuitive connection with Ertel's theorem!