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-Introduction to Einstein [notation] used [in] Kundu
- vector notation mostly in class

II. CALCULUS OF VECTORS, DYADICS AND TENSORS

A. Introduction & Review

1. scalars & vectors

scalar : magnitude only e.g., mass, temperature

vector : characterized by magnitude & direction; represented geometrically as an arrow ↗

⇒ 2 vectors are equal if they have the same magnitude & direction ; "parallel transport of vectors"

⇒ **A** = **B**

(Nevertheless, it is important to keep in mind that the effect of a given vector may depend upon its location)

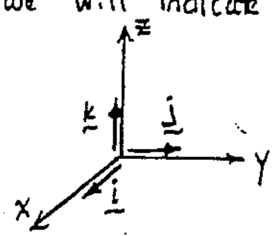
$\boxed{\text{NOTATION}}$: I will typically indicate a vector quantity by an underline, e.g. $\underset{\sim}{a}$ or $\underset{\sim}{b}$.

Another common method is to use arrows, $\vec{a}$, $\vec{b}$.

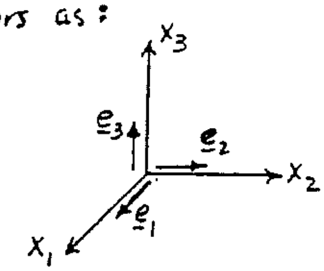
2. Cartesian Coordinate system

$\boxed{\text{unit vector has length} = 1}$

a. We will indicate the unit base vectors as:



or



$\underset{\sim}{e}_3 = (0,0,1)$
$\underset{\sim}{e}_2 = (0,1,0)$
$\underset{\sim}{e}_1 = (1,0,0)$

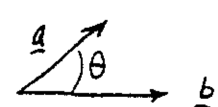
b. In order to describe a vector you must give both the components and the base vectors

e.g., $\underset{\sim}{a} = a_x\,\underline{i} + a_y\,\underline{j} + a_z\,\underset{\sim}{k}$

$\boxed{\text{also called DOT or INNER PRODUCT}}$

c. Recall the definition of the SCALAR PRODUCT of 2 vectors:

(i) $\underset{\sim}{a}\cdot\underset{\sim}{b} = |\underset{\sim}{a}|\,|\underset{\sim}{b}|\cos\theta$



where $|\underline{a}|$, $|\underline{b}|$ are the magnitudes of $\underset{\sim}{a}$ and $\underset{\sim}{b}$

NOTE: If $\underline{a}\cdot\underline{b} = 0$ and $|\underline{a}| \neq 0$, $|\underline{b}| \neq 0$, then $\underline{a} \perp \underline{b}$.

Also, since $\underline{i}\cdot\underline{i} = 1$, $\underline{i}\cdot\underline{j} = 0$, $\underline{i}\cdot\underline{k} = 0$, etc., then

$\underline{a}\cdot\underline{b} = a_x b_x + a_y b_y + a_z b_z$

c. scalar product (continued)

(ii) Clearly, we also have $\underset{\sim}{a} \cdot \underset{\sim}{b} = \underset{\sim}{b} \cdot \underset{\sim}{a}$ and $\underset{\sim}{a} \cdot (\underset{\sim}{b} + \underset{\sim}{c}) = \underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{c}$
and

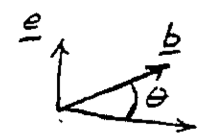
$$|\underset{\sim}{a}|^2 = \underset{\sim}{a} \cdot \underset{\sim}{a} = a^2$$

d. VECTOR PRODUCT (also called CROSS PRODUCT)

(i) The vector product of 2 vectors $\underset{\sim}{a}, \underset{\sim}{b}$ is defined as

NOTICE: My notation for this operation is $\wedge$; many others write $\times$.

$$\underset{\sim}{a} \wedge \underset{\sim}{b} = |\underset{\sim}{a}|\,|\underset{\sim}{b}| \sin\theta \; \underset{\sim}{e}$$



NOTE $\underset{\sim}{a} \wedge \underset{\sim}{a} = 0$.

where $\underset{\sim}{e}$ is a unit vector in the direction perpendicular to the plane formed by $\underset{\sim}{a}$ & $\underset{\sim}{b}$, as given by the RIGHT-HAND RULE.

(ii) From the definition: $\underset{\sim}{a} \wedge \underset{\sim}{b} = - \underset{\sim}{b} \wedge \underset{\sim}{a}$ and $\underset{\sim}{a} \wedge (\underset{\sim}{b} + \underset{\sim}{c}) = \underset{\sim}{a} \wedge \underset{\sim}{b} + \underset{\sim}{a} \wedge \underset{\sim}{c}$

It also follows that $\underset{\sim}{i} \wedge \underset{\sim}{j} = \underset{\sim}{k}$, $\underset{\sim}{i} \wedge \underset{\sim}{k} = -\underset{\sim}{j}$ $\underset{\sim}{j} \wedge \underset{\sim}{k} = \underset{\sim}{i}$ $\underset{\sim}{i} \wedge \underset{\sim}{i} = 0$ etc.

(iii) You may also remember writing something like

$$\underset{\sim}{a} \wedge \underset{\sim}{b} = \det \begin{vmatrix} \underset{\sim}{i} & \underset{\sim}{j} & \underset{\sim}{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \underset{\sim}{i}\,(a_y b_z - a_z b_y) + \underset{\sim}{j}\,(a_z b_x - a_x b_z) + \underset{\sim}{k}\,(a_x b_y - a_y b_x)$$

$\Rightarrow$ Much of the above is cumbersome & frightfully lengthy to write.

We now introduce a special notation which will simplify many manipulations.

B. EINSTEIN INDEX NOTATION AND THE SUMMATION CONVENTION

1. Let us reconsider some of the above. From now on keep in mind that we are representing vectors in a <u>three-dimensional world</u>.

So, we will now label (x, y, z) coordinates by $(1, 2, 3)$

Let the vector $\underset{\sim}{a}$ have components a_i, base vectors $\underset{\sim}{e}_i$.

Then,
$$\underset{\sim}{a} = a_1 \underset{\sim}{e}_1 + a_2 \underset{\sim}{e}_2 + a_3 \underset{\sim}{e}_3 = \sum_{i=1}^{3} a_i \underset{\sim}{e}_i \equiv a_i \underset{\sim}{e}_i \quad (= a_j \underset{\sim}{e}_j)$$
dummy ind

This idea must be clear in your mind before you move on.

$\Rightarrow$ From now on, we will **not** write the summation symbol. Instead we will invoke the <u>SUMMATION CONVENTION</u> — if an index appears <u>twice</u>, we will know that we should do a summation $i = 1, 2, 3$.

$\boxed{\text{IMPORTANT}}$: Use a different index for each vector.

2. scalar product revisited

Consider two vectors $\underset{\sim}{a} = a_i \underset{\sim}{e}_i \qquad \underset{\sim}{b} = b_j \underset{\sim}{e}_j$

Then, $\Rightarrow \underset{\sim}{a} \cdot \underset{\sim}{b} = \sum_{i=1}^{3} a_i \underset{\sim}{e}_i \cdot \sum_{j=1}^{3} b_j \underset{\sim}{e}_j = \sum_{i=1}^{3} a_i b_i = a_i b_i \; (= a_1 b_1 + a_2 b_2 + a_3 b_3)$

(do you understand the notation?) base vectors are orthogonal $\left\{ \begin{array}{ll} \underset{\sim}{e}_i \cdot \underset{\sim}{e}_j = 0 & \text{if } i \neq j \\ \phantom{\underset{\sim}{e}_i \cdot \underset{\sim}{e}_j} = 1 & \text{if } i = j \end{array} \right.$

where we again invoke the summation convention and drop the summation symbol

3. Kronecker delta δ_{ij} $\quad (i = 1,2,3 \quad j = 1,2,3)$

a. Definition :
$$\boxed{\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}}$$

NOTE: If you like, you may think about δ_{ij} as the components of the identity matrix
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

clearly $\quad \underset{\sim}{e}_i \cdot \underset{\sim}{e}_j = \delta_{ij}$

b. With this shorthand we write

δ_{ij} is a 'replacement' operator

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = a_i \underset{\sim}{e}_i \cdot b_j \underset{\sim}{e}_j = a_i b_j \overbrace{\underset{\sim}{e}_i \cdot \underset{\sim}{e}_j}^{\delta_{ij}} = \underbrace{a_i b_j \delta_{ij}}_{\text{implies the double sum } \sum_{i=1}^{3}\sum_{j=1}^{3} a_i b_j \delta_{ij}} = a_i b_i$$

only non-zero if $i = j$

vector operation; only acts on the base vectors not the components

In this eqn, i would be called the summation index. $\rightarrow$ $\quad \therefore \; \boxed{\underset{\sim}{a} \cdot \underset{\sim}{b} = a_i b_i = a_j b_j}$

and we again remark that a different dummy index was used for each vector $(a_i \underset{\sim}{e}_i, \; b_j \underset{\sim}{e}_j)$. $\boxed{\text{NEVER write } a_i \underset{\sim}{e}_i \cdot b_i \underset{\sim}{e}_i \text{ — this is very confusing.}}$

c. Remarks: (i) $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3$ using the summation convention; $\boxed{\delta_{ii} = 3}$

an important idea $\longrightarrow$ (ii) $\delta_{ij} \rightarrow$ the REPLACEMENT OPERATOR: $\boxed{\delta_{ij} c_j = c_i}$

(iii) Very often, one will not write the unit vectors $\underset{\sim}{e}_i$, and will write A_i where it is understood that i may be either 1, 2 or 3. In this case i would be called a free index since it is free to take on the values 1, 2 or 3.

Similarly, the vector eqn $\underset{\sim}{a} = \underset{\sim}{b}$ may be written
$$a_i \underset{\sim}{e}_i = b_i \underset{\sim}{e}_i \quad \text{or} \quad a_i = b_i \quad \text{and}$$
since i only appears once on each side of the eqn, it is free to take on the value 1, 2, or 3 so this stands for 3 seperate equalities: $a_1 = b_1 \quad a_2 = b_2 \quad a_3 = b_3$.

Another example:
$$(\underset{\sim}{a} \cdot \underset{\sim}{b})\, \underset{\sim}{c} = a_i b_i \underset{\sim}{c} = a_i b_i c_j \underset{\sim}{e}_j \quad \text{or} \quad \underbrace{a_i b_i}_{\substack{i \text{ appears twice so} \\ \text{we sum } i = 1 \rightarrow 3}} c_j$$

j is free to take on the values 1, 2 or 3 this symbol will be useful whenever vector products arise.

4. Permutation Symbol ε_{ijk} $\quad i=1,2,3 \quad j=1,2,3 \quad k=1,2,3$

a. Definition: $$\varepsilon_{ijk} = \begin{cases} +1 \text{ or } -1 & \text{if } i,j,k \text{ are all different} \\ 0 & \text{if any two indices are the same} \end{cases}$$

In particular,

$\varepsilon_{ijk} = +1$ if i,j,k are an EVEN permutation of 1,2,3

$\Rightarrow \quad \varepsilon_{123} = 1 \quad \varepsilon_{312} = 1 \quad \varepsilon_{231} = 1$

$\varepsilon_{ijk} = -1$ if i,j,k are an ODD permutation of 1,2,3

$\varepsilon_{213} = -1 \quad \varepsilon_{132} = -1 \quad \varepsilon_{321} = -1$

NOTE: By even permutation we mean that an even # of interchanges of the indices must occur to get back to the order 123; analogous for meaning of odd permutation.

b. This definition has the following cyclic and interchange property:

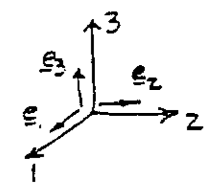
$$\varepsilon_{ijk} = \varepsilon_{kij} = \varepsilon_{jki}$$

and if two indices are simply interchanged, the sign changes,

$$\varepsilon_{ijk} = -\varepsilon_{ikj} \quad \text{or} \quad \varepsilon_{ijk} = -\varepsilon_{jik}$$

Also, since i,j,k can each independently take on the values 1,2,3, then ε_{ijk} represents 27 quantities.

c. We also have $$\boxed{\underline{e}_i \wedge \underline{e}_j = \varepsilon_{ijk}\, \underline{e}_k}$$



∴ Cross-product of base vectors (or any 2 vectors) will always involve the permutation symbol ε_{ijk}.

and by referring to the figure at right, everything is o.k.: $\underline{e}_1 \wedge \underline{e}_2 = +\underline{e}_3 = \underbrace{\varepsilon_{123}}_{+1} \underline{e}_3$ etc.

d. We now have an effective shorthand notation for representing the vector product.

let $\underline{c} = \underline{a} \wedge \underline{b}$; write $\underline{a} = a_i \underline{e}_i$, $\underline{b} = b_j \underline{e}_j$

$\rightarrow \quad = a_i \underline{e}_i \wedge b_j \underline{e}_j = a_i b_j (\underline{e}_i \wedge \underline{e}_j)$

$$\boxed{\underline{a} \wedge \underline{b} = a_i b_j \varepsilon_{ijk}\, \underline{e}_k}$$ ← NOTE CAREFULLY THE ORDER OF THE INDICES

or with $\underline{c} = c_k \underline{e}_k$, we have $c_k = a_i b_j \varepsilon_{ijk}$ which represents 3 eqns for $k = 1, 2$ or 3.

EXERCISE: Verify that this is in agreement

e. *triple scalar product*: $\underset{\sim}{a} \cdot (\underset{\sim}{b} \wedge \underset{\sim}{c})$

Again we are careful to use different dummy indices for each vector so

$$\underset{\sim}{a} \cdot (\underset{\sim}{b} \wedge \underset{\sim}{c}) = a_i\, \underset{\sim}{e}_i \cdot (b_j\, \underset{\sim}{e}_j \wedge c_k\, \underset{\sim}{e}_k) = a_i\, \underset{\sim}{e}_i \cdot (b_j c_k\, \varepsilon_{jkl}\, \underset{\sim}{e}_l)$$

$$= a_i b_j c_k\, \varepsilon_{jkl}\, \underbrace{\underset{\sim}{e}_i \cdot \underset{\sim}{e}_l}_{\delta_{il}}$$

$$= \varepsilon_{jki}\, a_i b_j c_k = \varepsilon_{ijk}\, a_i b_j c_k = (\underset{\sim}{a} \wedge \underset{\sim}{b}) \cdot c = (\underset{\sim}{c} \wedge \underset{\sim}{a}) \cdot b$$

by using cyclic property of ε_{ijk}. Exercise - convince yourself that these last 2 identities follow from index expression.

Recall also that

$$\underset{\sim}{a} \cdot (\underset{\sim}{b} \wedge \underset{\sim}{c}) = \det \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \underbrace{\varepsilon_{ijk}\, a_i b_j c_k}_{\text{index representation of the } 3\times 3 \text{ determinant}}$$

5. Useful identities involving ε and δ

$$\boxed{\varepsilon_{ijk}\, \varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}}$$

i, j, l, m can each independently take on values 1, 2, 3. Hence, this eqn corresponds to 81 quantities.

<u>Proof</u>: verify by brute force for each of the 81 eqns!
However, it is best to make your life easier by noticing that both sides change sign if either $i \neq j$ or $l \neq m$ are interchanged. Also, both sides vanishes if $i = j$ or $l = m$. Then, consider remaining terms like:

$$\varepsilon_{12k}\, \varepsilon_{k12} = \cancel{\varepsilon_{121}\, \varepsilon_{112}}^{\,0} + \varepsilon_{122}\, \varepsilon_{212}^{\,0} + \varepsilon_{123}\, \varepsilon_{312} = 1$$

and

$$\delta_{11}\delta_{22} - \cancel{\delta_{12}}^{\,0}\delta_{12} = 1 \qquad \text{so o.k.}$$

Likewise

$$\varepsilon_{12k}\, \varepsilon_{k13} = \varepsilon_{121}\, \varepsilon_{113} + \varepsilon_{122}\, \varepsilon_{213} + \varepsilon_{123}\, \varepsilon_{313} = 0\text{; also } \delta_{11}\delta_{23} - \delta_{13}\delta_{23} = 0 \text{ so o.k.}$$

etc. //

$\boxed{\text{Example 1}}$: show that $\underset{\sim}{e}_i = \frac{1}{2}\, \varepsilon_{mni}\, \underset{\sim}{e}_m \wedge \underset{\sim}{e}_n$

Well,

$$\varepsilon_{mni}\, \underset{\sim}{e}_m \wedge \underset{\sim}{e}_n = \varepsilon_{mni}\, \varepsilon_{mnj}\, \underset{\sim}{e}_j = \overbrace{\varepsilon_{nim}\, \varepsilon_{mnj}}^{\delta_{nn}\delta_{ij} - \delta_{nj}\delta_{ni}}\, \underset{\sim}{e}_j$$

$$= (3\,\delta_{ij} - \delta_{ij})\, \underset{\sim}{e}_j = 2\, \underset{\sim}{e}_i$$

$\boxed{\text{Example 2}}$: show that $\underset{\sim}{a} \wedge (\underset{\sim}{b} \wedge \underset{\sim}{c}) = \underset{\sim}{b}(\underset{\sim}{a} \cdot \underset{\sim}{c}) - \underset{\sim}{c}(\underset{\sim}{a} \cdot \underset{\sim}{b})$

$$\underset{\sim}{a} \wedge (\underset{\sim}{b} \wedge \underset{\sim}{c}) = a_i\, \underset{\sim}{e}_i \wedge (b_j\, \underset{\sim}{e}_j \wedge c_k\, \underset{\sim}{e}_k) = a_i\, \underset{\sim}{e}_i \wedge (b_j c_k\, \varepsilon_{jkl}\, \underset{\sim}{e}_l) = a_i b_j c_k\, \varepsilon_{jkl} (\underset{\sim}{e}_i \wedge \underset{\sim}{e}_l)$$

$$= a_i b_j c_k\, \varepsilon_{jkl}\, \varepsilon_{ilm}\, \underset{\sim}{e}_m = a_i b_j c_k\, \varepsilon_{jkl}\, \varepsilon_{lmi}\, \underset{\sim}{e}_m = a_i b_j c_k (\delta_{jm}\delta_{ki} - \delta_{ji}\delta_{km})\, \underset{\sim}{e}_m$$

$$= a_i b_m c_i\, \underset{\sim}{e}_m - a_i b_i c_m\, \underset{\sim}{e}_m = (a_i c_i)\, b_m \underset{\sim}{e}_m - (a_i b_i)\, c_m \underset{\sim}{e}_m = (\underset{\sim}{a} \cdot \underset{\sim}{c})\, \underset{\sim}{b} - (\underset{\sim}{a} \cdot \underset{\sim}{b})\, \underset{\sim}{c}$$

Some additional examples of the use of index notation

First, a brief summary of the important ideas

(i) $\underline{e}_i \cdot \underline{e}_j = \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

(ii) $\underline{e}_i \wedge \underline{e}_j = \varepsilon_{ijk}\, \underline{e}_k$ $\qquad \varepsilon_{ijk} = \begin{cases} +1 & i,j,k \text{ an even permutation of } 1,2,3 \\ -1 & i,j,k \text{ an odd permutation of } 1,2,3 \\ 0 & \text{any two indices the same} \end{cases}$

(iii) summation convention: whenever a subscript appears twice, a summation from $1 \rightarrow 3$ is implied.

Examples:

(i) $\delta_{ik}\,\delta_{jk} = \delta_{ij}$ since δ_{jk} is only nonzero when $j=k$ so the k in δ_{ik} may be replaced by j.

(ii) $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 1 + 1 + 1 = 3$ note: since i was a dummy index, $\delta_{ii} = \delta_{kk} = \delta_{mm}$ etc.

(iii) $\delta_{ij}\,\varepsilon_{ijk} = \varepsilon_{iik} = 0$ since two of the indices are the same.

(iv) $\varepsilon_{ijk}\,\varepsilon_{njk} = \varepsilon_{ijk}\,\varepsilon_{knj}$ by first rotating the indices on the second ε.

Next, use the identity: $\varepsilon_{ijk}\,\varepsilon_{klm} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}$

$$\Rightarrow \varepsilon_{ijk}\,\varepsilon_{knj} = \delta_{in}\,\underbrace{\delta_{jj}}_{3} - \delta_{ij}\,\delta_{nj} = 3\,\delta_{in} - \delta_{in} = \underline{2\,\delta_{in}}$$

(v) $a_m b_n \varepsilon_{mnq} - a_n b_m \varepsilon_{mnq} = ?$

$\Rightarrow$ m & n appear twice in each term so summation is implied.

But, m & n are simply dummy variables, i.e., we could just as well use another letter.

So, examine the second term $a_n b_m \varepsilon_{mnq}$.

$$\begin{aligned} a_n b_m \varepsilon_{mnq} &= -a_n b_m \varepsilon_{nmq} \quad ; \text{ now let } j = n,\ m = k \\ &= -a_j b_k \varepsilon_{jkq} \quad \text{which is the same as the above since summation on } j,k \text{ is implied.} \\ &= -a_m b_n \varepsilon_{mnq} \quad \text{by letting } j = m\ k = n \end{aligned}$$

So, we see that

$$a_m b_n \varepsilon_{mnq} - a_n b_m \varepsilon_{mnq} = 2\,\underbrace{a_m b_n \varepsilon_{mnq}}_{(\underline{a} \wedge \underline{b})_q}$$

q^{th} component of $\underline{a} \wedge \underline{b}$

F. An Introduction to Tensors & Dyadics

1. Preliminary remarks

a. We now wish to generalize our ideas concerning vectors to objects called tensors. We will try both to describe some of the mathematics of tensors and show why and how they arise in physical situations.

b. The idea: we have previously described scalars & vectors as

scalar – characterized by magnitude

vector – characterized by magnitude & direction

Now ⟶ 2nd order tensor (or a dyadic) – characterized by magnitude & <u>two</u> directions.

dyad = "two units regarded as one"

c. You have actually seen something very similar before.

For example, the vector $\underset{\sim}{a}\,(\underset{\sim}{b}\cdot\underset{\sim}{c})$ could be written

$$\underbrace{\underset{\sim}{a}\,\underset{\sim}{b}}_{\text{dyadic}}\cdot\underset{\sim}{c}$$

which has the property $\underset{\sim}{a}\,\underset{\sim}{b}\cdot\underset{\sim}{c} \equiv \underset{\sim}{a}\,(\underset{\sim}{b}\cdot\underset{\sim}{c})$

NOTE: To eliminate any ambiguity, vector operations will be defined to occur between the nearest two vectors.

This points the following important property: using index notation, the quantity $\underset{\sim}{a}\,\underset{\sim}{b}$ may be written

$$\underset{\sim}{a}\,\underset{\sim}{b} = \underbrace{a_i\, b_j}_{\text{magnitude of the ij-component}}\ \underbrace{\underset{\sim}{e}_i\underset{\sim}{e}_j}_{2\text{ directions}} \qquad (\text{summation convention in use}) \qquad i,j = 1,2,3$$

d. These mathematical objects that require 2 directions (or 2 indices) to be defined often correspond to PHYSICAL situations where physical properties are <u>different</u> in <u>different</u> directions.

(see page 105)

NOTE: I will almost always adopt the notation that lower case letters are vectors and upper case letters are tensors.

2. Definition of a 2^{nd} order dyadic

a. Define the 2^{nd} order dyadic $\underline{\underline{T}}$ by $\underline{\underline{T}} = \underline{a}\,\underline{b}$ and assign it the following operational properties:

(i) $\underline{c} \cdot \underline{\underline{T}} = \underline{c} \cdot \underline{a}\,\underline{b} = (\underline{c} \cdot \underline{a})\,\underline{b}$ Note: $(\text{vector}) \cdot \begin{pmatrix} 2^{nd}\text{ rank} \\ \text{tensor} \end{pmatrix} \rightarrow$ a vector

(ii) $\underline{\underline{T}} \cdot \underline{c} = \underline{a}\,\underline{b} \cdot \underline{c} = \underline{a}\,(\underline{b} \cdot \underline{c})$

(iii) If $\underline{\underline{T}}, \underline{\underline{S}}, \underline{\underline{R}}$ are dyadics then we also define standard linear operations:

$$\underline{c} \cdot (\underline{\underline{T}} + \underline{\underline{S}} + \underline{\underline{R}} + \cdots) = \underline{c} \cdot \underline{\underline{T}} + \underline{c} \cdot \underline{\underline{S}} + \underline{c} \cdot \underline{\underline{R}} + \cdots$$

$$(\underline{\underline{T}} + \underline{\underline{S}} + \underline{\underline{R}} + \cdots) \cdot c = \underline{\underline{T}} \cdot \underline{c} + \underline{\underline{S}} \cdot \underline{c} + \underline{\underline{R}} \cdot \underline{c} + \cdots$$

(as you would expect)

NOTE:
- ⇒ a vector can be thought of as a first order dyadic or tensor and a scalar " " " " " " zero " " " "
- ⇒ The inner product of dyadic and a vector produces a vector
- ⇒ ORDER IS IMPORTANT: $\underline{c} \cdot \underline{\underline{T}} \neq \underline{\underline{T}} \cdot \underline{c}$

b. Using index notation we write (vector operations occur between the nearest two vectors)

$$\underline{a} = a_i\,\underline{e}_i \qquad \underline{b} = b_j\,\underline{e}_j \qquad \Rightarrow \underline{\underline{T}} = a_i b_j\,\underline{e}_i\,\underline{e}_j$$

so

$$\underline{c} \cdot \underline{\underline{T}} = c_k\,\underline{e}_k \cdot a_i b_j\,\underline{e}_i\,\underline{e}_j = (c_i a_i)\,b_j\,\underline{e}_j = (\underline{c} \cdot \underline{a})\,\underline{b} \qquad \delta_{ik}$$

and

$$\underline{\underline{T}} \cdot \underline{c} = a_i b_j\,\underline{e}_i\,\underline{e}_j \cdot c_k\,\underline{e}_k = (a_i\,\underline{e}_i)\,b_j c_j = \underline{a}\,(\underline{b} \cdot \underline{c}) \qquad \delta_{jk}$$

Alternatively, we can discuss $\underline{\underline{T}} = T_{ij}\,\underline{e}_i\,\underline{e}_j$ and speak of the 2^{nd} order tensor $\underline{\underline{T}}$.

requires 2 indices to characterize completely

NOTATION: I denote a 2^{nd} order tensor using 2 underlines (generally, vectors will be lower case and 2^{nd} order tensors uppercase) a simple idea of a 'linear operator' : The work done by a force acting through a displacement $\underline{d}$ is $\underline{f}\cdot\underline{d}$ and so you could choose to think about $\underline{f}$ as a linear operator that, once fed the displacement $\underline{d}$, yields a scalar we call work, $\underline{f}\cdot\underline{d}$.

3. 2^{nd} order tensors : An alternative way of thinking about things

a. A 2^{nd} order tensor can be thought of as a "machine" that has a vector for its input and outputs another vector :

$$\underset{\sim}{a} \longrightarrow \boxed{\text{THE TENSOR MACHINE}} \longrightarrow \underline{b}$$

which we denote

$$\underline{\underline{T}}\cdot\underset{\sim}{a} = \underline{b}$$

$\underline{\underline{T}}$ operating on a vector produces another vector

b. Cartesian components of a 2^{nd} rank tensor

Lets consider an arbitrary vector $\underset{\sim}{c} = c_x \underline{e}_x + c_y \underline{e}_y + c_z \underline{e}_z$

Then, from a formal (operational) viewpoint,

$$\underline{\underline{T}}\cdot\underset{\sim}{c} = c_x\, \underline{\underline{T}}\cdot\underline{e}_x + c_y\, \underline{\underline{T}}\cdot\underline{e}_y + c_z\, \underline{\underline{T}}\cdot\underline{e}_z \qquad (1)$$

$\Rightarrow$ But $\underline{\underline{T}}$ operates on vectors and produces other vectors $\Leftarrow$
So, for example, we choose to write

$$\underline{\underline{T}}\cdot\underline{e}_x = T_{xx}\,\underline{e}_x + T_{yx}\,\underline{e}_y + T_{zx}\,\underline{e}_z$$ which is a vector.

and similarly

$$\underline{\underline{T}}\cdot\underline{e}_y = T_{xy}\,\underline{e}_x + T_{yy}\,\underline{e}_y + T_{zy}\,\underline{e}_z$$

$$\underline{\underline{T}}\cdot\underline{e}_z = T_{xz}\,\underline{e}_x + T_{yz}\,\underline{e}_y + T_{zz}\,\underline{e}_z$$

Thus, since $c_x = \underline{c}\cdot\underline{e}_x$, $c_y = \underline{c}\cdot\underline{e}_y$, $c_z = \underline{c}\cdot\underline{e}_z$
we can write eqn (1) as

$$\begin{aligned}\underline{\underline{T}}\cdot\underline{c} = {} & T_{xx}\,\underline{e}_x(\underline{e}_x\cdot\underline{c}) + T_{yx}\,\underline{e}_y(\underline{e}_x\cdot\underline{c}) + T_{zx}\,\underline{e}_z(\underline{e}_x\cdot\underline{c}) \\ & + T_{xy}\,\underline{e}_x(\underline{e}_y\cdot\underline{c}) + T_{yy}\,\underline{e}_y(\underline{e}_y\cdot\underline{c}) + T_{zy}\,\underline{e}_z(\underline{e}_y\cdot\underline{c}) \\ & + T_{xz}\,\underline{e}_x(\underline{e}_z\cdot\underline{c}) + T_{yz}\,\underline{e}_y(\underline{e}_z\cdot\underline{c}) + T_{zz}\,\underline{e}_z(\underline{e}_z\cdot\underline{c})\end{aligned}$$

slight rearrangement

$$\underline{\underline{T}}\cdot\underline{c} = \underbrace{\begin{pmatrix} T_{xx}\,\underline{e}_x\underline{e}_x + T_{xy}\,\underline{e}_x\underline{e}_y + T_{xz}\,\underline{e}_x\underline{e}_z \\ + T_{yx}\,\underline{e}_y\underline{e}_x + T_{yy}\,\underline{e}_y\underline{e}_y + T_{yz}\,\underline{e}_y\underline{e}_z \\ + T_{zx}\,\underline{e}_z\underline{e}_x + T_{zy}\,\underline{e}_z\underline{e}_y + T_{zz}\,\underline{e}_z\underline{e}_z \end{pmatrix}}_{\text{Cartesian representation of the } 2^{nd} \text{ order tensor } \underline{\underline{T}}}\cdot\underline{c}$$

the set of dyadics $\{\underline{e}_x\underline{e}_x, \underline{e}_x\underline{e}_y, \underline{e}_x\underline{e}_z, \underline{e}_y\underline{e}_x \cdots\}$ is a basis for the 2^{nd} order

The set of 9 quantities T_{ij} are called the Cartesian components of $\underline{\underline{T}}$.

3. 2^{nd} order tensors (continued)

c. Some sample representations using index notation

$$\underline{c}\cdot\underline{\underline{T}} = c_i \overbrace{\underline{e}_i \cdot T_{jk}\,\underline{e}_j}^{\delta_{ij}}\,\underline{e}_k = c_i T_{ik}\,\underline{e}_k \qquad \text{(a vector)}$$

$$\underline{\underline{T}}\cdot\underline{c} = T_{ij}\,\underline{e}_i \underbrace{\underline{e}_j \cdot c_k\,\underline{e}_k}_{\delta_{jk}} = T_{ij}\,c_j\,\underline{e}_i \qquad \text{(a vector)}$$

→ RULES : Nesting convention ⇒ vector operations occur between the closest pair of unit vectors

Example :

$$\underline{\underline{T}}\wedge\underline{c} = T_{ij}\,\underline{e}_i\,\underline{e}_j \wedge c_k\,\underline{e}_k = T_{ij}\,c_k\,\underline{e}_i\,(\underline{e}_j\wedge\underline{e}_k) = T_{ij}\,c_k\,\varepsilon_{jkm}\,\underline{e}_i\,\underline{e}_m$$

→ $\boxed{\text{order of unit vectors is now important.}}$

d. Notice the similarity with matrices : A 3x3 matrix $\begin{bmatrix} x & x & x \\ x & x & x \\ x & x & x \end{bmatrix}$ has 9 entries which we can think of as the components of a 2^{nd} order tensor. Each of the components of a 2^{nd} rank tensor, though, has two directions associated with it.

e. Notice that

$\underline{a}\cdot\underline{b}$ is a scalar

$\underline{a}\cdot\underline{\underline{T}}$, $\underline{\underline{T}}\cdot\underline{a}$ are vectors (In general, $\underline{a}\cdot\underline{\underline{T}} \neq \underline{\underline{T}}\cdot\underline{a}$)

$\underline{a}\cdot\underline{\underline{T}}\cdot\underline{b}$, $\underline{b}\cdot\underline{\underline{T}}\cdot\underline{a}$ are scalars.

Note : In the expression, $\underline{a}\cdot\underline{\underline{T}}\cdot\underline{b}$, the order in which the inner products are taken does not matter.

To see this,

$$(\underline{a}\cdot\underline{\underline{T}})\cdot\underline{b} = (a_i T_{ij}\,\overbrace{\underline{e}_j)\cdot b_k\,\underline{e}_k}^{\delta_{jk}} = a_i T_{ij} b_j \qquad \left(= \sum_{i=1}^{3}\sum_{j=1}^{3} a_i T_{ij} b_j\right)$$

and

$$\underline{a}\cdot(\underline{\underline{T}}\cdot\underline{b}) = a_i\,\overbrace{\underline{e}_i\cdot(T_{jk} b_k\,\underline{e}_j)}^{\delta_{ij}} = a_i T_{ik} b_k$$

$$\therefore\ (\underline{a}\cdot\underline{\underline{T}})\cdot\underline{b} = \underline{a}\cdot(\underline{\underline{T}}\cdot\underline{b}) = \underline{a}\cdot\underline{\underline{T}}\cdot\underline{b}$$

However, in general, $\underline{a}\cdot\underline{\underline{T}}\cdot\underline{b} \neq \underline{b}\cdot\underline{\underline{T}}\cdot\underline{a}$

3. 2^{nd} order tensors (continued)

f. The unit tensor $\underset{=}{I}$ is defined as

Kronecker delta

$$\underset{=}{I} = \delta_{ij}\, \underset{\sim}{e}_i \underset{\sim}{e}_j$$

→ analogue of the identity matrix:
$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and clearly has the property

$$\underset{\sim}{a} \cdot \underset{=}{I} = \underset{\sim}{a}$$
$$\underset{=}{I} \cdot \underset{\sim}{b} = \underset{\sim}{b}$$

4. Higher order tensors – It is straightforward to construct higher order tensors ⇒ Add indices

a. Here we will only mention <u>third order tensors</u> (or third order dyadics)

e.g.,

$$\underset{\equiv}{S} = S_{ijk}\, \underset{\sim}{e}_i \underset{\sim}{e}_j \underset{\sim}{e}_k$$

$i, j, k = 1 \rightarrow 3$ so S_{ijk} represents 27 terms

Notice, given the vector $\underset{\sim}{a} = a_\ell \underset{\sim}{e}_\ell$, then

$$\underset{\sim}{a} \cdot \underset{\equiv}{S} = a_l\, \underset{\sim}{e}_l \cdot S_{ijk}\, \underset{\sim}{e}_i \underset{\sim}{e}_j \underset{\sim}{e}_k = a_l S_{ljk}\, \underset{\sim}{e}_j \underset{\sim}{e}_k$$

(a 2^{nd} order tensor)

Similarly, $\underset{\equiv}{S} \cdot \underset{\sim}{a} = S_{ijk} a_k\, \underset{\sim}{e}_i \underset{\sim}{e}_j$ is a 2^{nd} order tensor

b. Permutation tensor $\equiv \underset{\equiv}{\varepsilon} = \varepsilon_{ijk}\, \underset{\sim}{e}_i \underset{\sim}{e}_j \underset{\sim}{e}_k$

where ε_{ijk} is the permutation symbol introduced earlier when discussing the vector product.

Exercise: Show $\underset{=}{I} \wedge \underset{=}{I} = -\underset{\equiv}{\varepsilon}$ (be careful with order of vector operations and indices)

5. Symmetric & Anti-symmetric tensors

a. Recall the <u>transpose</u> of a matrix

If $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$ then $A^T = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{12} & A_{32} \\ A_{13} & A_{23} & A_{13} \end{bmatrix}$

↑ transpose

Furthermore, a matrix was called SYMMETRIC if $A = A^T$ and was called ANTI-SYMMETRIC if $A = -A^T$

$A = A^T = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{12} & A_{22} & A_{23} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$ $\quad$ $A = -A^T = \begin{bmatrix} 0 & A_{12} & A_{13} \\ -A_{12} & 0 & A_{23} \\ -A_{13} & -A_{23} & 0 \end{bmatrix}$

b. We now generalize these ideas to 2^{nd} rank tensors

(i) A 2^{nd} order tensor $\underline{\underline{T}}$ is <u>symmetric</u> if $\underline{\underline{T}} = \underline{\underline{T}}^T$ or $T_{ij} = T_{ji}$

(ii) A 2^{nd} order tensor $\underline{\underline{T}}$ is <u>anti-symmetric</u> if $\underline{\underline{T}} = -\underline{\underline{T}}^T$ or $T_{ij} = -T_{ji}$

Some important properties (analogous to matrices shown above):

symmetric tensor $\Rightarrow T_{ij} = T_{ji}$ → only $\underline{6}$ independent components

anti-symmetric tensor → $T_{ij} = -T_{ji}$ → only $\underline{\underline{3}}$ independent components

↳ and we will see in the homework that an anti-symmetric tensor can be represented using a vector.

c. Every 2^{nd} order tensor can be expressed as the sum of a symmetric and an anti-symmetric tensor.

For example,

$$\underline{\underline{T}} = \underbrace{\tfrac{1}{2}\left(\underline{\underline{T}} + \underline{\underline{T}}^T\right)}_{\text{symmetric}} + \underbrace{\tfrac{1}{2}\left(\underline{\underline{T}} - \underline{\underline{T}}^T\right)}_{\text{anti-symmetric}}$$

NOTE: $\boxed{(\underline{\underline{A}}+\underline{\underline{B}})^T = \underline{\underline{A}}^T + \underline{\underline{B}}^T}$

(this is analogous to decomposition of a function into even and odd functions, p. 56)

So, now you have seen that a 2^{nd} order tensor is a mathematical object that (linearly) "sends vectors into other vectors."

This idea is particularly useful when describing physical situations where the physical properties are different in different directions, and you are representing quantities characterized by a vector.

Examples:

(i) current generated due to an applied electric field $\underset{\sim}{E}$

$$\underset{\sim}{j} = \underline{\underline{\sigma}} \cdot \underset{\sim}{E}$$

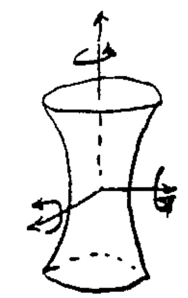
current density (current/(area·time))

electrical conductivity tensor

(ii) Angular momentum $\underset{\sim}{L}$ about a point for a body rotating with angular velocity $\underset{\sim}{\omega}$

$$\underset{\sim}{L} = \underline{\underline{I}} \cdot \underset{\sim}{\omega}$$

moment of inertia tensor



(clearly, for a given rotation speed, angular momentum varies depending on the axis of rotation)

Personally, I find (iii) & (iv) the simplest physical situation which suggest the appearance of tensorial quantities in a mathematical description of the physical world.

(iii) heat flux $\underset{\sim}{q}$ due to a thermal gradient (i.e., a temperature difference)

$$\underset{\sim}{q} = \underline{\underline{K}} \cdot \underset{\sim}{\nabla} T$$

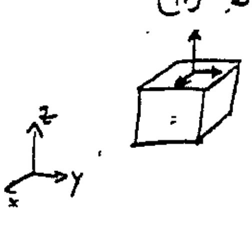
heat flux vector $\left(\frac{\text{energy}}{\text{area time}}\right)$

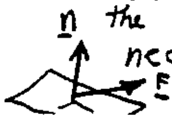
thermal conductivity tensor

⇐ Generalization of Fourier's law of heat conduction

⇒ A practical example is actually a material like wood. For a given temperature difference (or $\underset{\sim}{\nabla} T$) more energy is transferred along the grain than across the grain – the thermal conductivity of the material is different in the different directions.

(iv) STRESS TENSOR – frequently one requires information about stresses (force/area) acting on a material. To describe the state of stress at a point, e.g., $\underset{\sim}{x}$, ...it necessary to specify the ORIENTATION of the surface and the vector Force ⟹ 2 directions are thus necessary. NOTE: surface orientation is given by the unit normal vector.

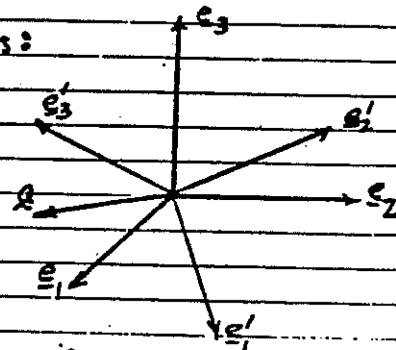




6. Transformation Rules for Tensors

→ We now examine how the components of a tensor change if we change from one set of Cartesian base vectors to another.

a. consider two sets of Cartesian base vectors:



$\underset{\sim}{e}_i \equiv$ "old" set of base vectors

$\underset{\sim}{e}_i' \equiv$ "new" " " " "

o.k. - poor choice of words, since you probably haven't seen this before we'll discuss it now

b. First, recall the transformation rule for the components of a vector.
(remember, that a vector, say $\underset{\sim}{a}$, is invariant with respect to a change of coordinate system. However, the representation of the vector in terms of its components clearly depends on the choice of the coordinate system.)

let $\quad \underset{\sim}{a} = a_i \underset{\sim}{e}_i = a_i' \underset{\sim}{e}_i' \quad (1)$ (the components a_i, a_i' are given by $\underset{\sim}{a} \cdot \underset{\sim}{e}_i = a_i$, $\underset{\sim}{a} \cdot \underset{\sim}{e}_i' = a_i'$)

So,
$$a_i' = (a_j \underset{\sim}{e}_j) \cdot \underset{\sim}{e}_i' = a_j (\underset{\sim}{e}_j \cdot \underset{\sim}{e}_i')$$

c. Definition: the DIRECTION COSINES between the axes are given by

NOTE: $l_{mn} \neq l_{nm}$

$$l_{mn} = \underset{\sim}{e}_m \cdot \underset{\sim}{e}_n'$$
(cosine of the angle between $\underset{\sim}{e}_m$ and $\underset{\sim}{e}_n'$)

"old set" ↑ ↑ "new set"

(2) $\quad \therefore \quad a_i' = a_j l_{ji}$ — TRANSFORMATION RULE FOR THE COMPONENTS OF A VECTOR

Similarly,
$$a_i = a_j' (\underset{\sim}{e}_j' \cdot \underset{\sim}{e}_i) = a_j' \underbrace{(\underset{\sim}{e}_i \cdot \underset{\sim}{e}_j')}_{l_{ij}}$$

so
$$a_i = l_{ij} a_j' \quad (3)$$
Notice how order of indices is reversed.

Notice:
Substituting (3) into (1): $\quad (\underset{\sim}{e}_i \cdot \underset{\sim}{e}_j') a_j' \underset{\sim}{e}_i = a_i' \underset{\sim}{e}_i'$

Rearranging indices allows one to conclude

$$(\underset{\sim}{e}_j \cdot \underset{\sim}{e}_i') \underset{\sim}{e}_j = \underset{\sim}{e}_i' \quad \rightarrow \quad \underset{\sim}{e}_i' = l_{ji} \underset{\sim}{e}_j \quad (4)$$

Similarly,
$$\underset{\sim}{e}_i = l_{ij} \underset{\sim}{e}_j' \quad (5)$$

Exercise: Demonstrate these relations for

d. Properties of the l_{ij}

As shown on the previous page, $\underline{e}'_i = l_{ji}\,\underline{e}_j$, $\underline{e}_i = l_{ij}\,\underline{e}'_j$

So, since the base vectors in each of the coordinate systems are orthogonal,

$$\underline{e}'_n \cdot \underline{e}'_m = \delta_{nm} \Rightarrow (l_{jn}\,\underline{e}_j)\cdot(l_{km}\,\underline{e}_k) = \delta_{nm}$$

($\underline{e}_j \cdot \underline{e}_k = \delta_{jk}$)

$$\therefore \boxed{l_{kn}\, l_{km} = \delta_{nm}}$$

Similarly, $\underline{e}_n \cdot \underline{e}_m = \delta_{nm} \Rightarrow l_{nk}\, l_{mk} = \delta_{nm}$

Notice: one can use the above 2 eqns in useful ways. For example, beginning with eqn (2),

$$a'_i = a_j l_{ji} \rightarrow a'_i l_{ki} = a_j \underbrace{l_{ji} l_{ki}}_{\delta_{jk}}$$ after multiplying both sides by l_{ki}

$$= a_k$$ ← which is the same result as eqn (3) on pg 91

#

⇒ the above illustrates the details of how the components of a vector transform when a different cartesian coordinate system is considered.

e. Now, consider the transformation rule for 2nd order tensors

Begin with $\underline{e}'_i = l_{ji}\,\underline{e}_j$ $\quad \underline{e}_i = l_{ij}\,\underline{e}'_j$ ← we know how vectors transform

The 2nd order tensor $\underline{\underline{T}}$ can then be represented as

$$\underline{\underline{T}} = T_{ij}\,\underline{e}_i\,\underline{e}_j = T_{ij}\,(l_{ik}\,\underline{e}'_k)(l_{jm}\,\underline{e}'_m) = \underbrace{T_{ij}\, l_{ik}\, l_{jm}}_{T'_{km}}\,\underline{e}'_k\,\underline{e}'_m = T'_{km}\,\underline{e}'_k\,\underline{e}'_m$$

$$\therefore \boxed{T'_{km} = T_{ij}\, l_{ik}\, l_{jm}}$$

Transformation rule for the components of $\underline{\underline{T}}$ in the $'$ coordinate system ($\underline{e}'$) relative to the original coordinate system.

Similarly, $\underline{\underline{T}} = T'_{ij}\,\underline{e}'_i\,\underline{e}'_j = T'_{ij}(l_{ki}\,\underline{e}_k)(l_{mj}\,\underline{e}_m) = T'_{ij}\, l_{ki}\, l_{mj}\,\underline{e}_k\,\underline{e}_m = T_{km}\,\underline{e}_k\,\underline{e}_m$

$$\therefore T_{km} = T'_{ij}\, l_{ki}\, l_{mj}$$ ← transformation rule from "new" to "old"

you may wish to try this as an exercise

Again, one can do the same for third order tensors: $\underline{\underline{\underline{S}}} = S_{ijk}\,\underline{e}_i\,\underline{e}_j\,\underline{e}_k = S'_{pqr}\,\underline{e}'_p\,\underline{e}'_q\,\underline{e}'_r$ and one finds $S'_{pqr} = S_{ijk}\, l_{ip}\, l_{jq}\, l_{kr}$ and $S_{ijk} = S'_{pqr}\, l_{ip}\, l_{jq}\, l_{kr}$

7. Remark : The "definition" of 2nd order tensors

On the previous page we presented formulas which show how the components of a tensor are related in different Cartesian coordinate systems.

i.e.,

$$T'_{km} = T_{ij}\, l_{ik}\, l_{jm} \qquad T_{km} = T'_{ij}\, l_{ki}\, l_{mj}$$

where

$$\underline{\underline{T}} = T_{km}\, \underline{e}_k \underline{e}_m = T'_{km}\, \underline{e}'_k \underline{e}'_m$$

Hence, given the components of $\underline{\underline{T}}$, T_{ij}, relative to the $\underline{e}_i$-axes, the above transformation rules specify the components T'_{ij} in any other set of right-hand cartesian orthonormal base vectors, $\underline{e}'_i$.

Classical treatments of tensor often DEFINE a second order tensor as that entity whose components in any, and every, two sets of Cartesian axes transform according to the above rules. The two approaches (the first illustrated on pages 106-107 or the alternative which simply defines a tensor via the above transformation rules) are equivalent and we will call any entity whose components transform according to these rules a second order tensor.

It is nevertheless important to keep in mind that the tensor itself is defined independent of the choice of coordinate system but the representation in terms of components depends on the choice of the coordinate system.

An example of this :

Suppose the "components" T_{ij} are given for all cartesian axes. Also, suppose for all vectors $\underline{a} = a_i \underline{e}_i$ that $a_i T_{ij} \underline{e}_j$ is a vector. Then, $\underline{\underline{T}} = T_{ij}\, \underline{e}_i \underline{e}_j$ (that quantity constructed from the components T_{ij}) must be a second order tensor.

Proof : Use the above transformation which specify how the components of a second order tensor are related.

$a_i T_{ij} \underline{e}_j$ is a vector $\rightarrow$ $a'_i T'_{ij} = (a_k T_{km})\, l_{mj}$ since this is how the components of a vector transform

and since $\underline{a}$ is a vector $\rightarrow$ $a_k = a'_p\, l_{kp}$ $\quad = (a'_p\, l_{kp})\, T_{km}\, l_{mj}$

or $a'_i T'_{ij} = a'_i\, l_{ki}\, T_{km}\, l_{mj}$ after relabelling $p \to i$

or $a'_i \underbrace{(T'_{ij} - T_{km}\, l_{ki}\, l_{mj})}_{0} = 0$ for all vector $\underline{a}$

$\therefore \; T'_{ij} = T_{km}\, l_{ki}\, l_{mj}$ which is the transformation rule for the components of a second order tensor

B. Isotropic tensors

a. Definition : Any tensor which has the same components in all Cartesian axes is called an isotropic tensor.

Now, on the previous two pages, we examined how to represent the components of vectors & tensors for different cartesian coordinate axes.

b. Recall the unit tensor $\underline{\underline{I}} = \delta_{ij}\, \underline{e}_i \underline{e}_j$

What are the components of $\underline{\underline{I}}$, δ_{ij}', relative to the $\underline{e}_i'$ axes.

Well, from pg. 107,

$$\delta_{ij}' = \delta_{mn}\, l_{mi}\, \delta_{nj}$$

$$= l_{ni}\, l_{nj} = \delta_{ij} \quad \text{via the identities on the top of p. 92}$$

$\therefore$ $\underline{\underline{I}}$ has the same components (δ_{ij}) in all Cartesian systems, and hence $\underline{\underline{I}}$ is called an isotropic 2nd order tensor.

$\Rightarrow$ It can be shown that apart from a multiplicative constant, the unit tensor $\underline{\underline{I}}$ is the only isotropic 2nd order tensor.

c. Likewise, consider the permutation tensor $\underline{\underline{\underline{\varepsilon}}} = \varepsilon_{ijk}\, \underline{e}_i \underline{e}_j \underline{e}_k$

$$\underline{\underline{\underline{\varepsilon}}} = \varepsilon_{ijk}\, \underline{e}_i \underline{e}_j \underline{e}_k = \varepsilon'_{ijk}\, \underline{e}_i' \underline{e}_j' \underline{e}_k'$$ in the primed coordinate system

First, a clever observation : $\varepsilon_{ijk} = \underline{e}_i \cdot (\underline{e}_j \wedge \underline{e}_k)$

From the bottom of p.107, we see that the components ε'_{pqr} and ε_{ijk} are related by

$$\varepsilon'_{pqr} = \varepsilon_{ijk}\, l_{ip}\, l_{jq}\, l_{kr} = \underline{e}_i \cdot (\underline{e}_j \wedge \underline{e}_k)\, l_{ip}\, l_{jq}\, l_{kr}$$

$$= (l_{ip}\, \underline{e}_i) \cdot [\, l_{jq}\, \underline{e}_j \wedge l_{kr}\, \underline{e}_k]$$

$$= \underline{e}_p' \cdot (\underline{e}_q' \wedge \underline{e}_r') = \varepsilon_{pqr}$$

simply because $\underline{e}_p'$, $\underline{e}_q'$, $\underline{e}_r'$ form a right-handed orthogonal coordinate system.

$\therefore$ $\underline{\underline{\underline{\varepsilon}}}$ is a third order isotropic tensor.

9. Example: a change of cartesian axes

let $\underset{\sim}{f} = f_i \underline{e}_i = f_i' \underline{e}_i'$ be a vector field. $\rightarrow f_i' = f_m \ell_{mi}$

Consider the derivatives of the f_i and f_i' with respect to position, i.e., consider

$$\frac{\partial f_i}{\partial x_j} \quad \text{and} \quad \frac{\partial f_i'}{\partial x_j'}$$

and we know (p. 91) that the components of the position vector satisfy

$$x_i' = x_j \ell_{ji} \qquad x_i = x_j' \ell_{ij} \rightarrow \frac{\partial x_i}{\partial x_j'} = \ell_{ij}$$

Clearly, via the chain rule,

$$\frac{\partial f_i'}{\partial x_j'} = \frac{\partial (f_m \ell_{mi})}{\partial x_k} \underbrace{\frac{\partial x_k}{\partial x_j'}}_{\ell_{kj}}$$

$$= \frac{\partial f_m}{\partial x_k} \ell_{kj} \ell_{mi}$$

which is the same transformation rule found on p. 107 for the components of a 2nd order tensor

Therefore, $\frac{\partial f_m}{\partial x_k}$ (or $\frac{\partial f_i'}{\partial x_j'}$) represents the components of a 2nd order tensor.

The notation generally used is $\underset{\sim}{\nabla} \underset{\sim}{f} = \frac{\partial f_j}{\partial x_i} \underline{e}_i \underline{e}_j$

9. Scalar Invariants – scalars simply have a magnitude

a. Example: consider the scalar $\underline{a} \cdot \underline{b}$

$\Rightarrow \underline{a} \cdot \underline{b} = a_i b_i = a_i' b_i'$ and this scalar quantity is an INVARIANT, in other words, it has the same value in all Cartesian axes.

To demonstrate this, we the transformation rules discussed earlier.

$$a_i' b_i' = a_j \ell_{ji} b_k \ell_{ki} = a_j b_k \overbrace{\ell_{ji} \ell_{ki}}^{\delta_{jk}} = a_k b_k$$

b. Example: If $\underset{=}{T} = T_{ij} \underline{e}_i \underline{e}_j$ is a second order tensor, then T_{ii} is an invariant.

Proof: $T_{ij}' = T_{km} \ell_{ki} \ell_{mj} \Rightarrow T_{ii}' = T_{km} \overbrace{\ell_{ki} \ell_{mi}}^{\delta_{km}} = T_{kk}$

$\therefore T_{kk} = T_{kk}' \Rightarrow T_{kk}$ is an INVARIANT.

common notation

$T_{kk} = \text{Trace}(\underset{=}{T}) \equiv \text{tr}\, \underset{=}{T}$

NOTE: This does not mean $T_{11} = T_{11}'$; rather, the sums of the diagonal terms are equal
$T_{11} + T_{22} + T_{33} = T_{11}' + T_{22}' + T_{33}'$

c. Invariance of the Divergence of a vector $\underline{\nabla} \cdot \underline{f}$.

Since $\underline{\nabla} \cdot \underline{f} = \frac{\partial f_i}{\partial x_i}$ and since $\frac{\partial f_i}{\partial x_j}$ are the components of a second order tensor, say $\underline{\underline{T}} = T_{ij}\, \underline{e}_i \underline{e}_j$, then it follows that because we just saw that T_{ii} is an invariant, it also must be true that $T_{ii} = \frac{\partial f_i}{\partial x_i}$ is an invariant.

Remark: SOME ADDITIONAL NOTATION

The "Double Dot product":

The following notation is sometimes used;

$$\underline{a}\,\underline{b} : \underline{c}\,\underline{d} \equiv (\underline{a} \cdot \underline{d})(\underline{b} \cdot \underline{c}) \rightarrow \text{a scalar}$$

Also, for two second order tensors, $\underline{\underline{T}}$, $\underline{\underline{S}}$

$$\underline{\underline{T}} : \underline{\underline{S}} = T_{ij}\, \underline{e}_i \underline{e}_j : S_{kl}\, \underline{e}_k \underline{e}_l$$

$$= T_{ij} S_{kl}\, \delta_{il}\, \delta_{jk} = T_{ik} S_{ki}$$

note: nearest two indices are the same

So

$$\underline{\underline{I}} : \underline{\underline{T}} = T_{ii} = \underbrace{tr}(\underline{\underline{T}})$$

trace of the second order tensor; by summation convention, $tr\, \underline{\underline{T}} = T_{11} + T_{22} + T_{33}$. (see bottom of p. 110)

TOPICS 10 & 11 Are basically for your overall education and are meant to try to give you some ideas how the concepts of a tensor arise in different physical situations

10. EXAMPLE : The Stress Tensor

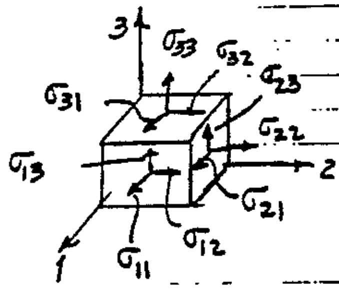
Fundamental to the description and a basic understanding of the deformation of solids and the flow of fluids

a. Why is this concept useful?

Suppose you were interested in the state of equilibrium of a material. Newton's law applied to a small piece of a material says

$$\sum \underset{\sim}{\text{Forces}} = \text{mass} * \text{acceleration}$$

If the material is stationary and at equilibrium (acceleration = 0) then the sum of the forces on a piece of the material must balance. So, now consider the small cube of material shown below.



Of course, we will require that all the forces acting on this body must balance and it is convenient to denote the forces acting on each face. We write

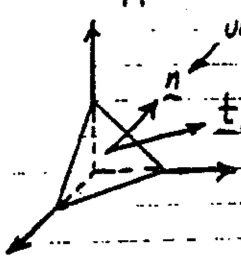
$$\underset{\sim}{t}_{(1)} = \sigma_{11}\, \underline{e}_1 + \sigma_{12}\, \underline{e}_2 + \sigma_{13}\, \underline{e}_3$$

force/area on the face $\perp \underline{e}_1$

the component of the force/area acting in the $\underline{e}_2$ direction on the face $\perp \underline{e}_1$.

In general, $\underset{\sim}{t}_{(i)} = \sigma_{ij}\, \underline{e}_j$ ← "stress vector" ≡ force/area on face $\perp \underline{e}_i$

b. Furthermore, there is a very beautiful result due to Cauchy which considers a tetrahedron-shaped material volume and applies the principle that the sum of the forces must balance.



⇒ CAUCHY'S RESULT: $\underset{\sim}{t}_{(\underline{n})} = \underline{n} \cdot \underline{\underline{\sigma}}$ for all $\underset{\sim}{n}$

STRESS TENSOR

stress vector (force/area) on surface with normal $\underline{n}$

→ $\underline{\underline{\sigma}} = \sigma_{ij}\, \underline{e}_i \underline{e}_j$; by previous "tensor test", $\underline{\underline{\sigma}}$ must be a second order tensor.

c. Derivation of the Equilibrium Equations for a continuous material

At equilibrium (no motion) : net force on a small volume of material → $\displaystyle\int_S \underline{t}_{(\underline{n})}\, dS = \int_S \underline{n} \cdot \underline{\underline{\sigma}}\, dS = 0$

↑ no acceleration (condition for static equilibrium)

$$= \int_S n_i \sigma_{ij}\, \underline{e}_j\, dS$$

C. Equilibrium of a continuous material (continued)

We begin with

$$(1) \quad \int_S \underline{n} \cdot \underline{\underline{\sigma}} \, dS = 0 \qquad \left(= \int_S n_i \sigma_{ij} \, dS \right)$$

We now apply the Divergence Theorem, which holds equally well for tensors, i.e.,

DIVERGENCE THEOREM FOR TENSORS $\Longrightarrow$

$$\int_S \underline{n} \cdot \underset{\substack{\uparrow \\ n^{th} \text{ order tenso}}}{\underline{\underline{T}}} \, dS = \int_V \underline{\nabla} \cdot \underline{\underline{T}} \, dV \qquad \text{or using index notation} \qquad \int_S n_i T_{ijk} \, dS = \int_V \frac{\partial}{\partial x_i} T_{ijk} \, dV$$

So, eqn (1) becomes

$$\int_V \underline{\nabla} \cdot \underline{\underline{\sigma}} \, dV = 0$$

which is true for an arbitrary volume element in a body at equilibrium

and since this must hold for all volume elements V, we conclude

$$\underline{\nabla} \cdot \underline{\underline{\sigma}} = 0 \qquad \left(\text{or} \quad \frac{\partial}{\partial x_i} \sigma_{ij} = 0 \right)$$

If you were to write this out, $\underline{\nabla} \cdot \underline{\underline{\sigma}}$ is a vector so $\underline{\nabla} \cdot \underline{\underline{\sigma}} = 0$ represents 3 eqns, one for each component of the vector. So,

$$\frac{\partial}{\partial x_1} \sigma_{11} + \frac{\partial}{\partial x_2} \sigma_{21} + \frac{\partial}{\partial x_3} \sigma_{31} = 0$$

$$\frac{\partial}{\partial x_1} \sigma_{12} + \frac{\partial}{\partial x_2} \sigma_{22} + \frac{\partial}{\partial x_3} \sigma_{32} = 0$$

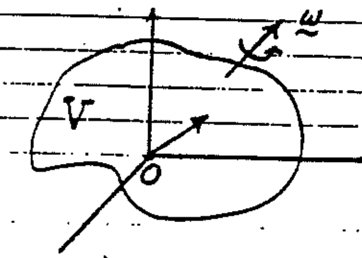
$$\frac{\partial}{\partial x_1} \sigma_{13} + \frac{\partial}{\partial x_2} \sigma_{23} + \frac{\partial}{\partial x_3} \sigma_{33} = 0.$$

Remark: In order to proceed further, you must relate the stress tensor $\underline{\underline{\sigma}}$ to the small displacements that occur in the material.

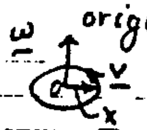
11. The Moment of Inertia Tensor

a. In the study of the mechanics of rotating rigid bodies, it is necessary to know the angular momentum and kinetic energy of the object. We will now see how a 2^{nd} order tensor, the moment of inertia tensor, naturally arises.

b. Consider a rigid body spinning with angular velocity $\underset{\sim}{\omega}$ about a fixed point O.



Recall that for a point mass rotating at angular velocity $\underset{\sim}{\omega}$ about some origin, the actual velocity is



$$\underline{v} = \underset{\sim}{\omega} \wedge \underline{x}$$

and the angular momentum about O is $(\text{mass}) \cdot (\underline{x} \wedge \underline{v})$.

So, the total angular momentum $\underset{\sim}{L}$ of the rigid body is

$$\underset{\sim}{L} = \int_V \underline{x} \wedge (\underset{\substack{\uparrow \\ \text{mass/volume}}}{\rho} \underline{v}) \, dV \overset{\underline{v} = \underline{\omega} \wedge \underline{x}}{=\!=} \int_V \rho \, \underline{x} \wedge (\underline{\omega} \wedge \underline{x}) \, dV$$

where every point of the rigid body rotates with angular velocity $\underset{\sim}{\omega}$.

but

$$\underline{x} \wedge (\underline{\omega} \wedge \underline{x}) = (\underline{x} \cdot \underline{x}) \underset{\sim}{\omega} - \underline{x} (\underline{x} \cdot \underline{\omega})$$

$$= \left[(\underline{x} \cdot \underline{x}) \underline{\underline{I}} - \underline{x} \, \underline{x} \right] \cdot \underset{\sim}{\omega}$$

Exercise: prove this identity

where $r^2 = \underline{x} \cdot \underline{x} = x_j x_j$

So,

$$\underset{\sim}{L} = \int_V \left[r^2 \underline{\underline{I}} - \underline{x} \, \underline{x} \right] \cdot \underset{\sim}{\omega} \, dV$$

or since $\underset{\sim}{\omega} \equiv$ constant throughout V

$$\underset{\sim}{L} = \int_V (r^2 \underline{\underline{I}} - \underline{x} \, \underline{x}) \, \rho \, dV \cdot \underset{\sim}{\omega} = \underset{\approx}{\mathbb{I}} \cdot \underset{\sim}{\omega}$$

where

$$\underset{\approx}{\mathbb{I}}(O) = \int_V \left(r^2 \underline{\underline{I}} - \underline{x} \, \underline{x} \right) \rho \, dV \equiv \text{moment of inertia tensor about O.}$$

Notice: $\underline{\underline{\mathbb{I}}}$ is symmetric