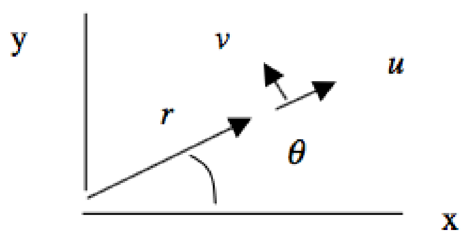
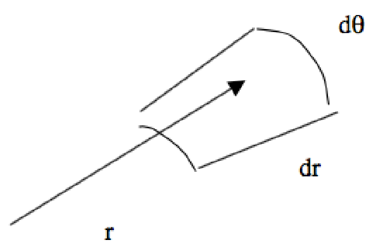
EOSC 512
Advanced Geophysical Fluid Dynamics

Problem Set 2

1. Consider the two-dimensional flow in the x-y plane and use polar coordinates r and θ to describe the motion. Let u by the radial velocity and v the azimuthal velocity (as in the figure below).



Construct a small surface element as shown below:



a) By constructing a mass budget for the element show that the net mass flux out of the element (per unit depth in z) is:

$$\frac{\partial}{\partial r}(\rho r u)\, dr d\theta + \frac{\partial}{\partial \theta}(\rho v)\, dr d\theta = \text{mass out}$$

Then show that the mass conservation equation in polar coordinates in 3 dimensions becomes:

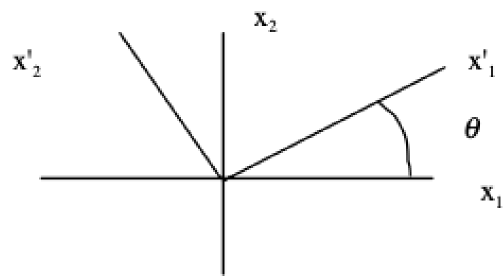
$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r u) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

b) Consider a flow that has cylindrical symmetry (no θ variations and is independent of z). Find the radial dependence of the mass flux, ρu, for a 2-dimensional, steady flow independent of z (this might work for your bathtub drain).

(c) Using similar reasoning, what would be the radial dependence of the mass flux issuing from a localized source if the flow is steady and has *spherical symmetry*. (Here we might be thinking of the solar wind). Note: it is not necessary to derive the full equation. You can just reason from the geometry to find the r dependence where r is the spherical radius.

2. Derive the momentum equation, $\rho \frac{Du_i}{Dt} = \rho F_i + \frac{\partial \sigma_{ij}}{\partial x_j}$, by using the "elemental cube" argument and working out the net stress on the small, cubic fluid element.

3. Consider the coordinate frames in two dimensions as shown in the figure below.



a) In terms of the angle θ, find the four components of the transformation matrix $\mathbf{a_{ij}}$ that maps variables in the un-primed frame to the primed frame.

(b) If the stress tensor in the un-primed frame is given by

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$$

find the four components of the stress tensor in the new, primed frame in terms of the angle θ and the stress tensor components in the old frame.

(c) For this two-dimensional stress tensor, find the angle θ for which the tangential (shear) stress is a maximum assuming only that $\sigma_{12} = \sigma_{21}$.

(d) Describe what happens if $\sigma_{12} = \sigma_{21} = 0$ and $\sigma_{11} = \sigma_{22} =$ - p.