EOSC 512
Advanced Geophysical Fluid Dynamics

Problem Set 5

with hints

1. Consider the Ekman layer over a flat horizontal plate in the presence of a uniform zonal flow, U, at great distances from the lower boundary as we discussed in class. Calculate the frictional stress *on the lower boundary* by the fluid. What is the corresponding stress exerted *on the fluid* by its interaction with the wall?

Hints: This problem presents opportunities to apply your understanding of how the frictional stress applied on the plate by the fluid flow depends on the rate of strain tensor (i.e. local spatial gradients in the velocity field) as we saw in Unit 3, and further to use the solution for the velocity field in the canonical Ekman layer problem over a solid boundary that we derived in class to quantify the stress applied on the boundary by the fluid flow in this case.

Recall the frictional stress (=frictional force per unit surface area) on a boundary with unit normal $\hat{n} = (n_1, n_2, n_3)$ *is:*

$$\Sigma_i(\hat{n}) = \sigma_{ij}\hat{n}_j$$

$$\Sigma_i(\hat{n}) = [-P\delta_{ij} + 2\mu e_{ij} + \lambda e_{kk}\delta_{ij}]\hat{n}_j$$

In this case, the surface normal is $\hat{n} = (0, 0, 1)$*. Given this, use the expression above to compute the stress in the* x *and* y *directions. The stress exerted on the wall by the fluid will have components in each of these direction. Substitute the Ekman layer solutions for* $u(z)$ *and* $v(z)$ *evaluated at z=0 into the expressions for the stress. This defines the stress (both magnitude and direction) exerted on the boundary by the fluid. Finally, use Newton's third law (action-reaction) to know the stress exerted on the fluid by its interaction with the boundary.*

2. For this same Ekman layer problem, derive an equation that governs the balance between the generation and dissipation of kinetic energy inside the boundary layer. Carefully discuss the sign of each term and interpret each term physically in terms of work done and viscous dissipation.

Hints: To form a kinetic energy equation, we take the dot product of the momentum equations with the velocity. In this case, we consider the Ekman layer horizontal momentum equations:

$$fu = fU + \nu v_{zz}$$

$$-fv = \nu u_{zz}$$

and multiply the first equation by v and the second equation by u and add them together.

To consider the kinetic energy balance for the boundary layer as a whole, we integrate in z from z=0 to $z \to \infty$. With some manipulation, applying what you know about the horizontal pressure gradients, vertical velocity and the boundary conditions in this problem, you should obtain:

$$0 = -\int_0^\infty \frac{1}{\rho}\vec{u}\cdot\nabla p\, dz - \nu\int_0^\infty \left(\frac{\partial\vec{u}}{\partial z}\right)\cdot\left(\frac{\partial\vec{u}}{\partial z}\right) dz$$

Argue that the frictional term is always positive. This term represents the rate of dissipation of kinetic energy due to stresses in the boundary layer. It is always a drain of kinetic energy.

Next consider the sign of the term that depends on the pressure gradient. This term could be positive or negative depending on the sign of the velocity relative to the pressure gradient, but in the case of the Ekman layer, the sign of this term is defined. You should find that in this case, this rate of pressure work term is always positive. This is the generation of kinetic energy by the frictional flow down the pressure gradient that is required in this steady state problem to balance the loss of kinetic energy due to viscous dissipation in the boundary layer.

3. Consider the problem that Ekman solved for Nansen for his doctoral thesis to explain why icebergs drifted at an angle to the wind rather than directly downwind. Specifically show that for the case of an applied stress $\vec{\tau}$ generated by the wind on the sea surface, the motion driven by the stress at the sea surface (*i.e.* at $z = 0$) is given by:

$$\vec{u} = \delta \frac{\left[\vec{\tau} - \hat{k} \times \vec{\tau}\right]}{2\rho\nu} = \frac{\left[\vec{\tau} - \hat{k} \times \vec{\tau}\right]}{2\rho\sqrt{\Omega\nu}}$$

You can assume that the water occupies the infinite region $z < 0$ and that for large negative z the velocities must go to zero (implying that there is no pressure gradient for large negative z, although this could be added on later.)

Hints: The set-up of the governing equations in this problem is the same as that for the Ekman layer over a solid surface and the general solution to these governing equations is the same. The different dynamics derive from different boundary conditions: here the stress at $z = 0$ is equal to the applied wind stress, τ=(τ_x,τ_y),0) based on the dynamic boundary condition at the air-sea interface of the continuity of tangential stress across the fluid interface. The velocity goes to zero at depth ($z \to \infty$). Solve for the constants C_1, C_2, C_3, C_4 of the general solution to the Ekman layer horizontal momentum equations to find the solution that satisfies this new boundary conditions (you should get a coupled system of equations for a set of 2 constants).

To get the required form, note:

$$\vec{\tau} - \hat{k} \times \vec{\tau} = (\tau_x, \tau_y) - (-\tau_y, \tau_x) = (\tau_x + \tau_y, \tau_y - \tau_x)$$

$$\vec{\tau} + \hat{k} \times \vec{\tau} = (\tau_x, \tau_y) + (-\tau_y, \tau_x) = (\tau_x - \tau_y, \tau_y + \tau_x)$$