

Problem Set 6

1. In cylindrical polar coordinates (r, θ, z) with corresponding velocities (u, v, w) , the vorticity vector is $\vec{\omega} = \omega_r \hat{r} + \omega_\theta \hat{\theta} + \omega_z \hat{z}$ where $\hat{r}$, $\hat{\theta}$, and $\hat{z}$ are unit vectors in the radial, azimuthal and vertical directions respectively. The vorticity components, in terms of the velocity are:

$$\omega_r = \left[\frac{1}{r} \frac{\partial w}{\partial \theta} - \frac{\partial v}{\partial z} \right], \quad \omega_\theta = \left[\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r} \right], \quad \omega_z = \left[\frac{1}{r} \frac{\partial (rv)}{\partial r} - \frac{1}{r} \frac{\partial u}{\partial \theta} \right]$$

(a) A point vortex is an idealization of a vortex in which the vorticity is concentrated at its centre. In polar coordinates its velocity is given as $(u, v, w) = (0, \frac{\gamma}{2\pi r}, 0)$ in the annular interval $0 < r < \infty$ (we exclude the “hole” at $r = 0$).

(i) Calculate the vorticity of the flow.

(ii) Calculate the circulation of the flow on any contour that encircles the origin. *Hint: use an easy contour such as a circle about the origin and use the result of part i) to generalize the result to an arbitrary contour.*

(iii) Calculate the circulation on any contour that does not encircle the origin.

Hints: First, compute the vorticity of this point vortex flow by computing the gradients of u , v and w in the radial (r), azimuthal (θ), and vertical (w) directions and combining them into the 3 components of the vorticity vector as defined above. Because $u=w=0$ and v is a function of r only, you should find the vorticity is zero for all $r > 0$.

Next, compute the circulation of this flow: $\Gamma = \oint_C \vec{u} \cdot d\vec{s}$. For convenience, choose a unit circle centred on the origin as the contour. Then, $\vec{u}$ tangent to the contour is the component v everywhere and $d\vec{s} = r\theta$ (an incremental arc length of the unit circle). For this contour choice, $\Gamma = \oint_C \vec{u} \cdot d\vec{s} = \int_0^{2\pi} v r d\theta$. You should find that the circulation for this unit circle about the origin is equal to γ .

To generalize the result to an arbitrary contour, exploit the result from (i) as the hint suggests. We can construct an arbitrary contour around the origin as the sum of a unit circle around the contour (which we know has circulation γ) and any arbitrary contour that encloses the origin but connects to the unit circle (such that the area enclosed is a ‘donut’ with the unit circle as its hole). From (i) we know that the vorticity in this area

(which doesn't include the origin) is 0. What does that imply about the circulation associated with the arbitrary contour/area inside the donut?

The result from (i) also allows us to know the circulation on any contour that does not encircle the origin. We know from (i) that the vorticity inside any area enclosed by a contour that does not include the origin must be zero. What does that mean for the circulation on this contour?

(b) Now consider the flow:

$$(u, v, w) = \left(0, \gamma \frac{1 - e^{-\frac{r^2}{r_o^2}}}{2\pi r}, 0 \right)$$

in the region $r \geq 0$ (It may help your understanding if you sketch the velocity profile as a function of r , and ultimately the profile of the vertical vorticity as a function of r).

(i) Calculate the vorticity of the flow. Interpret this in terms of the form of the velocity for $r \leq r_o$.

(ii) Calculate the circulation on a circular contour that encircles the origin.

(iii) Calculate the circulation on the contour in (ii) for $r \gg r_o$ and $r \ll r_o$ and discuss your result.

Hints: Here, repeat the calculations above for this new functional form of the azimuthal velocity v . Note that in this idealization of the vortex, there is no longer a singularity at the origin where all the vorticity is concentrated; instead there is a localized source of vorticity at the origin that has a continuous distribution that decays exponentially with radial distance (this idealized representation of a vortex is called a "Rankine vortex"). You should find that in this case, the vorticity of the flow is no longer singular at the origin and further that its z -component decays exponentially with r . Now, the circulation is not a constant but instead it too is a function of r (this makes sense because the vorticity enclosed by the circle now varies with radius). You should find that in the limit of $r \ll r_o$ (close to the origin), circulation increases as r^2 , and in the limit of $r \gg r_o$ (in the far field), circulation is equal to the constant γ i.e. it is the same as the point vortex case.

(c) Discuss the relation between parts (a) and (b) of the problem.

Hints: The point vortex is an idealization of the Rankine vortex with all the vorticity concentrated at the origin. The Rankine vortex is a smooth representation of the point vortex with its vorticity smoothly distributed in a core of radius r_o . In the limits $r \gg r_o$ (far away from the origin) and $r_o \rightarrow 0$, the 2 models look the same.

2. The dispersion relation for a barotropic Rossby wave as we derived in class is:

$$\sigma = -\frac{\beta k}{k^2 + l^2}$$

where k and l are the components of the wave vector in the x and y directions respectively, i.e. the phase of the wave is $kx + ly - \sigma t$.

(a) What is the group velocity in the x direction? How does it vary as the scale of the wave (i.e. the wavelength) of the wave varies? How does this variation compare to that for the phase speed in the x direction?

(b) What is the phase velocity of the Rossby wave in the y direction? What happens to the phase velocity in the y direction as $l \rightarrow 0$? Why? What does it mean?

Hints: This question explore properties of the phase and group velocities of the barotropic Rossby wave, properties of the wave propagation that depend on the wave's dispersion relation.

Recall, the phase velocity of a wave is the speed at which a single point of constant phase (like a wave crest) travels. It represents the speed of the “carrier wave” itself. It is given by the angular frequency of the wave divided by the angular wavenumber. Thus, for a wave with wavenumber (k, l, m) and angular frequency σ , the phase speed in the x , y and z directions is $(c_x, c_y, c_z) = (\frac{\sigma}{k}, \frac{\sigma}{l}, \frac{\sigma}{m})$.

In contrast, the group speed is the speed at which the overall shape or “envelope” of the wave packet travels. It represents the speed at which energy and information are transported by the wave. It is given by the rate of change of the angular frequency of the wave with respect to the components of the angular wavenumber. For the wave with phase $kx + ly + mz - \sigma t$, the group speed in the x , y and z directions is $(c_{gx}, c_{gy}, c_{gz}) = (\frac{\partial \sigma}{\partial k}, \frac{\partial \sigma}{\partial l}, \frac{\partial \sigma}{\partial m})$.

In (a), you are asked to derive the group speed in the x direction for the barotropic Rossby wave. To do so, express the wave frequency in terms of the wave wavenumber using the dispersion relation we derived in class, and compute the derivative of σ with respect to k . You should arrive at an expression for c_{gx} that is a function of k and l . Compare this to the phase speed in the x direction we derived in class/in the notes: $c_x = \frac{\sigma}{k} = \frac{-\beta}{k^2 + l^2}$.

Think about what these 2 expressions tell you about how the zonal phase speed and zonal group speed as the zonal (x) and meridional (y) spatial scales/wavenumbers of the wave vary. It can be helpful to plot the variation of c_x and c_{gx} as a function of k for a fixed value of l . How does the magnitude and sign of these speeds vary and what does this mean for the propagation of zonally-oriented waves (i.e. waves with a zonal wavelength $\gg$)

their meridional wavelength) vs. meridionally-oriented waves (i.e. waves with a zonal wavelength $\ll$ their meridional wavelength). Discuss.

In (b), you are asked to compute the phase speed in the y direction for this wave. The speed is computed as $c_y = \frac{\sigma}{l}$ where σ is expressed in terms of the wavenumber components using the dispersion relation. Plot the variation of c_y as a function of l for a fixed value of k and think about what this relationship implies for the meridional phase speed for long meridional waves (i.e. waves with small values of l /large meridional wavelengths) vs. short meridional waves (i.e. waves with large values of l /small meridional wavelengths). Consider what happens to c_y as the magnitude of l goes to zero. Does this singularity violate physical laws?

3. In our discussion of Kelvin's theorem for a rotating fluid, we derived the vorticity equation for a thin, homogeneous layer of fluid on a sphere using the beta plane approximation as:

$$\frac{\partial}{\partial t} \nabla^2 \psi + \psi_x (\nabla^2 \psi)_y - \psi_y (\nabla^2 \psi)_x + \beta \psi_x = 0$$

where ψ is the two-dimensional streamfunction such that $u_H = \hat{k} \times \nabla \psi$. Subscripts denote partial differentiation with respect to the variable indicated. $\beta = \frac{2\Omega \cos \theta}{R}$ is considered constant. We showed in class that this equation described the dynamics of a wave solution of the form $\psi = A \cos(kx + ly - \omega t)$, the so-called barotropic Rossby wave.

(a) Now consider a wave solution of the above equation in the presence of a uniform flow in the x direction, U , of the form:

$$\psi = -Uy + A \cos(kx + ly - \omega t)$$

Find the wave dispersion relation, $\omega = \omega(k, l, U, \beta)$, by substituting the assumed wave solution into the governing vorticity equation. You may assume that the wave amplitude A and its horizontal wavenumbers k and l are constant.

(b) For what values of U will the wave be stationary (i.e. have no time dependence)? What physically is going on to give rise to this time-independent flow field?

Hints: To find the dispersion relation in this case, follow the procedure we used in class to derive the dispersion relation for the barotropic Rossby wave but now look for the condition(s) necessary for the streamfunction $\psi = -Uy + A\cos(kx + ly - \omega t)$ to be a solution of the governing equation for ψ derived from Kelvin's circulation theorem. You should find that the dispersion relation you derived is modified from that we derived in class by the inclusion of an additional term that is a function of U .

For (b), for the wave to be stationary, ω must be zero (this eliminates the time dependence from the streamfunction solution). Use the dispersion relation derived above to find the condition on U that makes ω zero. Describe what physically is happening that leads to this steady solution.